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Geometric Inequalities 14. February 2023

Part 1: Introduction to Geometric Inequalities

Let us consider a triangle ABC with:

- **Sides** labeled by $a = BC$, $b = CA$ and $c = AB$.
- **Angles** denoted by $\alpha = \angle A = \angle BAC$, $\beta = \angle B = \angle ABC$, $\gamma = \angle C = \angle ACB$.
- **Midpoints of the sides** denoted by A_1 (for BC), B_1 (for CA) and C_1 (for AB).
- **Feet of the altitudes** from A, B, C to the opposite sides labeled by A', B', C', respectively.
- **Points of intersection of the internal bisectors and the sides** denoted by A'', B'', C''.
- **Length of the medians** AA_1, BB_1, CC_1 denoted by m_a, m_b, m_c.
- **Length of the altitudes** AA', BB', CC' denoted by h_a, h_b, h_c.
- **Length of the segments of internal bisectors** AA'', BB'', CC'' denoted by l_a, l_b, l_c.
- **Semi-perimeter** of the triangle denoted by $p = \frac{a+b+c}{2}$.
- **Circumradius** of the triangle labeled by R.
- **Inradius** of the triangle labeled by r.
- **Radii of the excircles** (three circles tangent to one side and the extensions of the other two sides) labeled by r_a, r_b, r_c.
- **Area** of the triangle S_{ABC}.

Theorem 1. Triangle inequality. If ABC is a triangle, then the following two statements hold:

1. The lengths of the sides are related by $a < b + c, b < a + c, c < a + b$.

   Conversely, if a, b, c are positive real numbers such that each is smaller than the sum of the other two, then there exists a triangle of sides a, b, c.

2. $AB < BC$ if, and only if, $\angle ACB < \angle BAC$.

Using only these inequalities and the notation above, we can solve the following problems:

Problem 1. Prove that, for an arbitrary triangle ABC, the following inequalities hold:

$$p < m_a + m_b + m_c < 2p \,. \tag{1}$$

Problem 2. Prove that, for an arbitrary triangle ABC, the sum of its medians is greater than 3/4 of the sum of its sides.

Theorem 2. Ptolemy For any four points A, B, C, D in the plane, the following inequality holds:

$$AC \cdot BD \leq AB \cdot CD + AD \cdot BC\,. \tag{2}$$

Equality holds if, and only if

- $ABCD$ is cyclic with diagonals AC and BD;
- or A, B, C, D are collinear and exactly one of B, D is between A and C.

Theorem 3. Parallelogram Inequality For every fours points A, B, C, D in the space we have

$$AB^2 + BC^2 + CD^2 + DA^2 \geq AC^2 + BD^2\,. \tag{3}$$

Equality holds if, and only if, $ABCD$ is a parallelogram (or degenerated parallelogram).

Using these theorems, we can solve the following problems.

Problem 3. Let ABC be an acute-angled triangle. Using a straight-edge and compass, construct a point M inside the triangle ABC for which the sum $MA + MB + MC$ is minimal.

Such a point is called *Toricelli point*. Note that, if Q_A, Q_B and Q_C are exterior points to the triangle such that $\triangle BAQ_C$, $\triangle ACQ_B$, $\triangle CBQ_A$ are equilateral, then M is the intersection of AQ_A, BQ_B and CQ_C.

Problem 4. Prove that $h_a \leq l_a \leq m_a$.

Part 2: Geometric Substitution

This is a trick used to solve some inequalities constructed from the sides a, b, c of a triangle. At first sight, the only relation we have between these three numbers is

$$a < b + c, b < a + c, c < a + b\,. \tag{4}$$

However, since A'', B'', C'' are the points of intersection of the internal bisectors and the sides, we have $AB'' = AC''$, which we denote by x. Similarly, we write $y = BA'' = BC''$ and $z = CA'' = CB''$. Then, it is clear that

$$a = y + z, b = z + x, c = x + y\,. \tag{5}$$

Therefore, we have the following result: The following two facts are equivalent for positive real numbers a, b, c

- They are the sides of a triangle.
- There are positive real numbers x, y, z such that $a = y + z, b = z + x, c = x + y$.

With this trick, we can solve the following problems:

Problem 5. If a, b, c are the lengths of the sides of a triangle, prove that

$$\frac{a}{b+c-a} + \frac{b}{a+c-b} + \frac{c}{a+b-c} \geq 3\,. \tag{6}$$

Problem 6. If a, b, c are the lengths of the sides of a triangle and s is the semi-perimeter of the triangle, prove that

$$a^2(p-a)+b^2(s-b)+c^2(s-c)\leq \frac{3}{2}abc\,. \tag{7}$$

Part 3: Some important theorems in Geometry

Theorem. Let M be a point on the side BC of the triangle ABC. Then,

$$\overrightarrow{AM}=\frac{\overrightarrow{MC}}{\overrightarrow{BC}}\cdot\overrightarrow{AB}+\frac{\overrightarrow{BM}}{\overrightarrow{BC}}\cdot\overrightarrow{AC}\,. \tag{8}$$

Theorem. Stewart Let M be a point on the side BC of the triangle ABC. Then,

$$AM^2=\frac{MC}{BC}\cdot AB^2+\frac{BM}{BC}\cdot AC^2-BM\cdot MC\,. \tag{9}$$

Problem 7. If a, b, c are the lengths of the sides of a triangle and m_a is the length of the median corresponding to the side a, prove that

$$m_a^2\leq\frac{2b^2+2c^2-a^2}{4}\,. \tag{10}$$

Problem 8. Let O be the circumcenter of the triangle ABC and G its centroid. Prove that

$$OG^2=R^2-\frac{1}{9}(a^2+b^2+c^2)\,. \tag{11}$$

Problem 9. Let a, b, c be the lengths of the sides of $\triangle ABC$ and R its circumradius. Prove that

$$9R^2\geq a^2+b^2+c^2\,. \tag{12}$$

Theorem. Incircle-excircle Let ABC be a triangle and O its circumcenter, I its incenter, I_a, I_b, I_c the centers of the excircles (k_a, k_b, k_c) corresponding to the sides BC, CA, AB, and G its centroid. Let a, b, c be the side lengths, R the circumradius and r the inradius. Let r_a, r_b, r_c be the exradii and s the semiperimeter of the triangle. Then, the following statements hold:

1. AI intersects the circumcircle at the midpoint Q of the arc BC. I_bI_c contains the point A and the midpoint P of the arc BC that contains A. The circumcenter O belongs to PQ.
2. If M and N are the points of tangency of k_b and k_c with BC, then P is the midpoint of I_aI_b, A_1 is the midpoint of MN and $PA_1=\frac{r_b+r_c}{2}$.
3. Denote by U the point of tangency of the incircle with BC and by V the point of tangency of k_a with BC. Then, A_1 is the midpoint of UV, Q is the midpoint of II_a and $QA_1=\frac{r_a-r}{2}$.
4. $S=\sqrt{s(s-a)(s-b)(s-c)}$.

Problem 10. Prove that

$$s^2 \leq m_a^2 + m_b^2 + m_c^2\,. \tag{13}$$

Theorem. Erdos-Mordell Let M be a point inside the triangle ABC. Denote by A_1, B_1, C_1 the feet of perpendiculars from M to BC, CA and AB. Then,

$$MA + MB + MC \geq 2(MA_1 + MB_1 + MC_1)\,. \tag{14}$$

Problem 11. Let M be a point inside the triangle ABC. Denote by A_1, B_1, C_1 the feet of perpendiculars from M to BC, CA and AB. Prove that

$$\frac{1}{MA} + \frac{1}{MB} + \frac{1}{MC} \leq \frac{1}{2}\left(\frac{1}{MA_1} + \frac{1}{MB_1} + \frac{1}{MC_1}\right)\,. \tag{15}$$

Part 4: Problems

Problem 12. Prove that

$$9r \leq h_a + h_b + h_c \leq l_a + l_b + l_c \leq m_a + m_b + m_c \leq \frac{9}{2}R\,. \tag{16}$$

Problem 13. Prove that

$$27r^2 \leq h_a^2 + h_b^2 + h_c^2 \leq l_a^2 + l_b^2 + l_c^2 \leq p^2 \leq m_a^2 + m_b^2 + m_c^2 \leq \frac{27}{4}R^2\,. \tag{17}$$

Problem 14. Prove that

$$r \leq \frac{\sqrt{\sqrt{3}S}}{3} \leq \frac{\sqrt{3}}{9}s \leq \frac{1}{2}R\,. \tag{18}$$