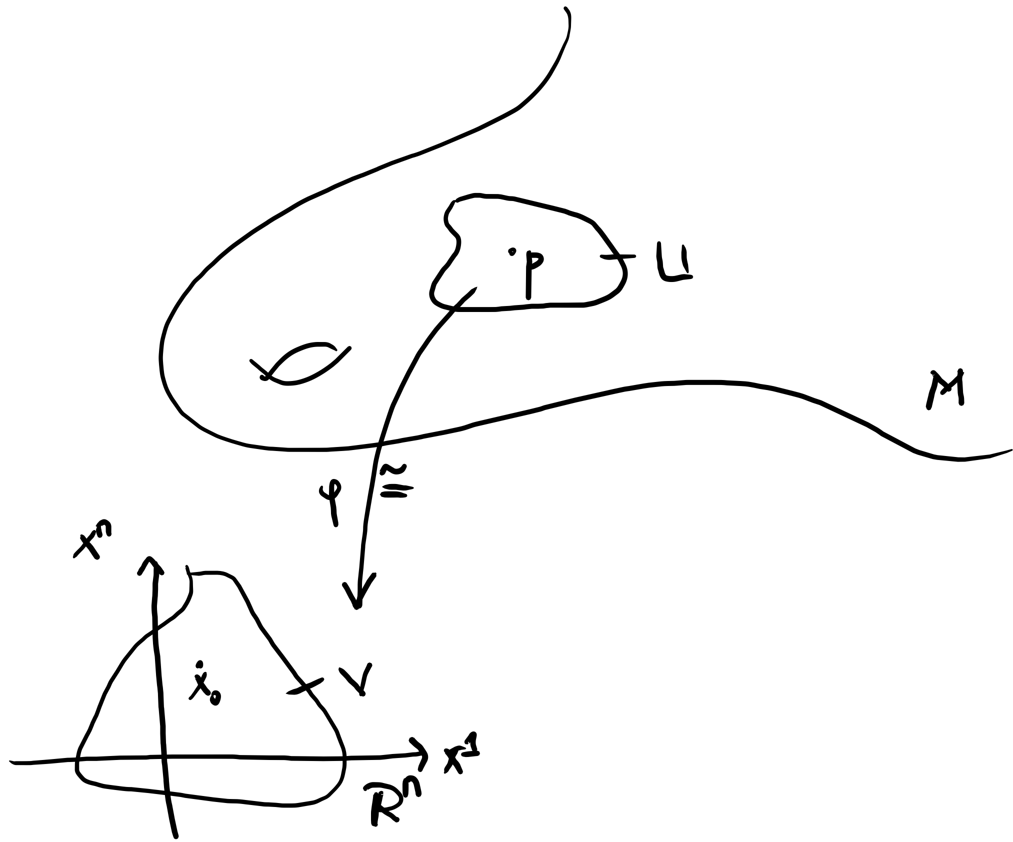
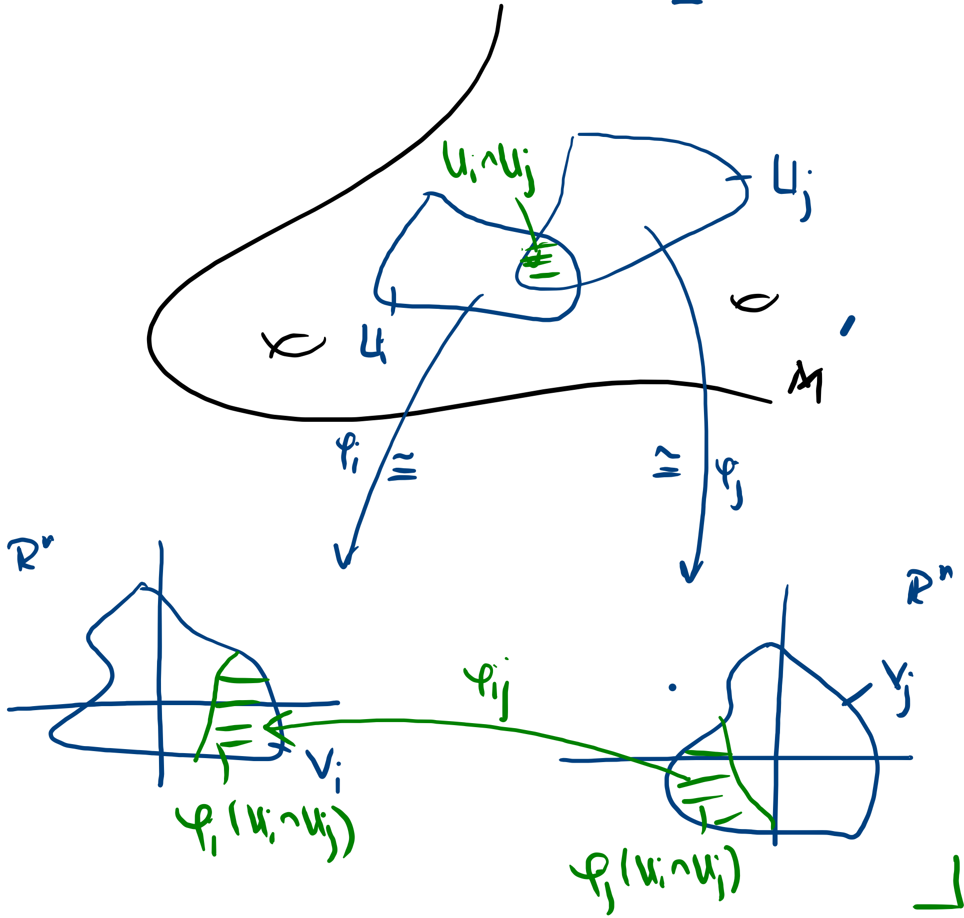
Lecture 07, 09.12.2022

⌈ Remember:

- top. mfd. of dimension $n \in \mathbb{N}$



· differentiable structure on a top. mfd.



· $\varphi_{ij} = \varphi_i \circ \varphi_j^{-1}$

transition functions of an atlas $\mathfrak{A} = (\varphi_i)$

If all φ_{ij} are diff., $\mathfrak{A}$ is called diff.

$c = [\mathfrak{A}]$ equiv. class

(9.5) Remarks. (a) On a given topological n-fold it could be that there are many different differentiable structure. (It is also possible, that there exists no one at all). Trivial example: Consider $\mathbb{R} = M$ which is a top. mfd. of dimension 1, of course. Now consider the following atlasses consisting of a single chart:

$$\mathfrak{A} = (\varphi = \mathrm{id} : \overset{M}{\overset{\shortparallel}{\mathbb{R}}} \longrightarrow \mathbb{R})$$
$$\mathfrak{L} = (\psi = \mathrm{pot}_3 : \mathbb{R} \to \mathbb{R},\ x \mapsto x^3)$$

These are both differentiable since the only transition function of a single-chart-atlas is $\varphi \circ \varphi^{-1} = id$, which is, of course, differentiable. However, $\mathfrak{A}$ and $\mathfrak{L}$ are not equivalent, so $c_1 = [\mathfrak{A}]$ and $c_2 = [\mathfrak{L}]$ define different structures on $M = \mathbb{R}$. Indeed, the transition $\varphi \circ \psi^{-1} : \mathbb{R} \to \mathbb{R}$ of the atlas $\mathfrak{A} + \mathfrak{L} = (\varphi, \psi)$ of $\mathbb{R}$ is not differentiable (in $x = 0$), since

$$\varphi \circ \psi^{-1}(x) = \sqrt[3]{x}.$$

(However, we will see in a moment, that $(\mathbb{R}, c_1)$ and $(\mathbb{R}, c_2)$ are diffeomorphic, $(\mathbb{R}, c_1) \cong (\mathbb{R}, c_2)$. Namely via $\Phi : \mathbb{R} \to \mathbb{R}$, $p \longmapsto \sqrt[3]{p}$.

(b) Among all differentiable atlasses in a given differentiable structure, there exists a „maximal" one, taking simply all charts of all these atlasses (without repetion) together. Another possibility of defining a diff. str. would be therefore to define it as a „maximal differentiable atlas". However, in general, a concrete diff. structure will be given usually in terms of a rather poor atlas consisting of only a few charts.

(9.6) Examples. (a) $\mathbb{R}^n$ has a standard structure $\mathfrak{A} = (\varphi)$ given by $\varphi = \mathrm{id}$.

(b) If M^n is a diff. mfd. given by a diff. atlas $\mathfrak{A} = (\varphi_i)_{i \in I}$, any open subset $N \subseteq M$ inherits the structure of a diff. mfd. of the same dimension. Of course, N is a top. n-fold, and the induced diff. structure is simply given by the restrictions $(\varphi_i : U_i \cap N \longrightarrow W_i)_{i \in I}$ (with $W_i := \varphi_i(U_i \cap N) \subseteq V_i \subseteq \mathbb{R}^n$) is independent of the ...

representative.

(c) In particular, every open subset $V \subseteq \mathbb{R}^n$ is a diff. mfd. of dim. n via the identity chart.

(d) Let V be a real vector space of dimension $n \in \mathbb{N}$. Then choosing a basis $(v_1, \dots, v_n)$ of V yields a coordinate isomorphism $\varphi : V \to \mathbb{R}^n$. We give V the topology via φ. Then V becomes a top. mfd. of dimension n and φ will be a homeomorphism. Another basis would give the same topology on V, since the change $\varphi^{-1} \circ \psi : \mathbb{R}^n \to \mathbb{R}^n$ is a linear isomorphism of $\mathbb{R}^n$, so a homeo-

homeomorphism. We can give now V also a diff. structure via φ and, of course, also this is independent of the chosen basis, since $\bar{\varphi}^{1} \circ \psi$ is even a diffeomorphism. This gives the abstract vector space V in a natural way the structure of an n-dimensional diff. manifold.

(9.7) The n-sphere The most prominent example of a „true" manifold (meaning that you cannot find an atlas with a single chart) will be perhaps the following:

$$S^n = \{ p \in \mathbb{R}^{n+1} : \|p\|^2 = 1 \} \qquad (n \in \mathbb{N}_0)$$

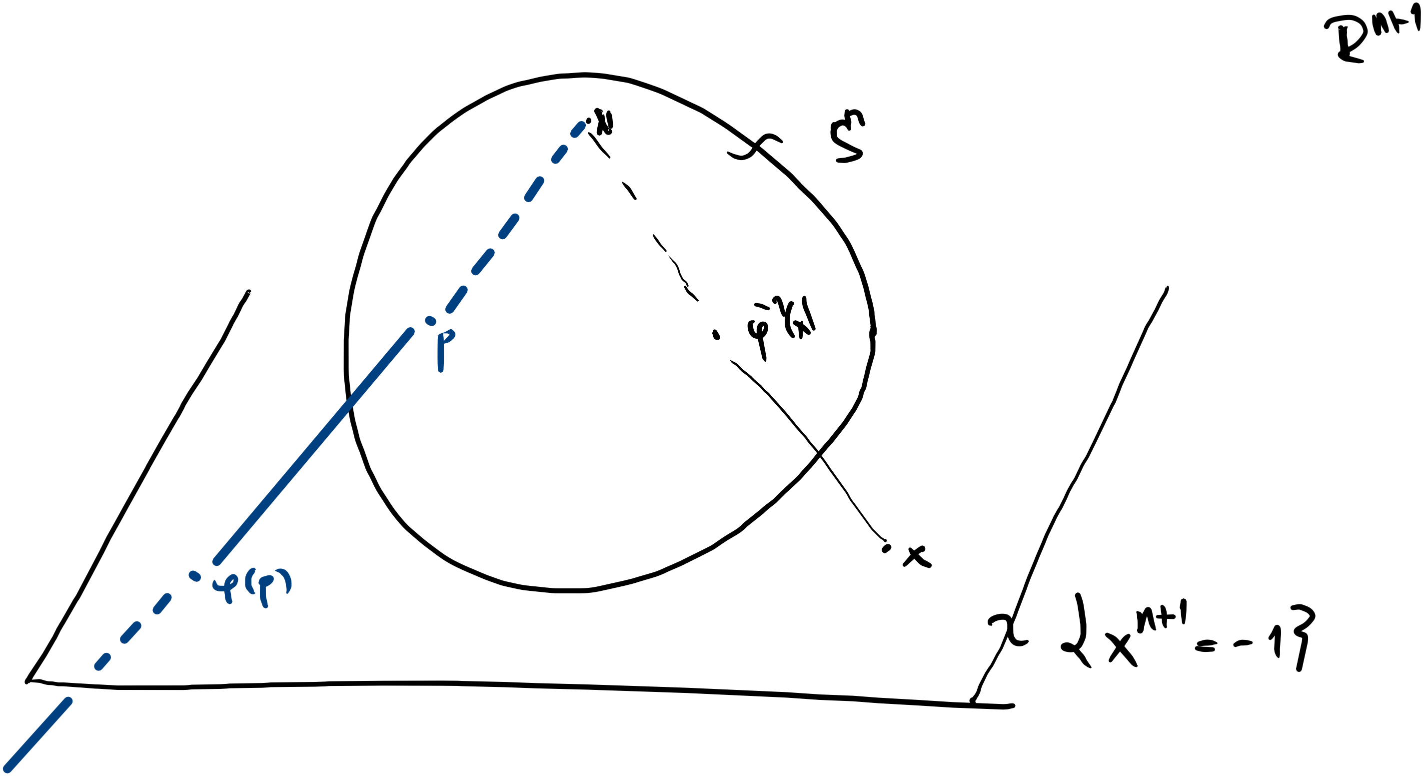
with its subspace topology of $\mathbb{R}^{n+1}$ is hausdorff and of countable topology since $\mathbb{R}^{n+1}$ is. It is connected and compact (by Heine-Borel). So it is not possible to cover S^n with a single chart (since an open non-empty $V \subseteq \mathbb{R}^n$ is not compact). Consider now for

$$N = (0, \dots, 0, 1) \text{ and } S = (0, \dots, 0, -1)$$

the so-called stereographic projections

$$\varphi : S^n \setminus \{N\} \longrightarrow \mathbb{R}^n, \; p \longmapsto \frac{1}{1-p^{n+1}} (p^1, \dots, p^n).$$

(For reasons that become clear later I denote the components of a vector $x \in \mathbb{R}^n$ by x^i $(i = 1, \dots, n)$ with a superscript.)



resp.

$$\psi : S^n \setminus \{s\} \longrightarrow \mathbb{R}^n, \quad p \longmapsto \frac{1}{1+p^{n+1}}(p^1, \dots, p^n)$$

Then φ and ψ are continuous (since restrictions of cont. maps from $\mathbb{R}^{n+1} \setminus \{x^{n+1} = 1\}$ to $\mathbb{R}^n$ resp. $\mathbb{R}^{n+1} \setminus \{x^{n+1} = -1\}$ and bijective, since

$$x = (x^1, \dots, x^n) \longmapsto \frac{1}{1+\|x\|^2}(2x^1, \dots, 2x^n, -1+\|x\|^2)$$

$$\mathbb{R}^n \longrightarrow S^n \setminus \{N\}$$

resp.

$$x = (x^1, \dots, x^n) \longmapsto \frac{1}{1+\|x\|^2}\left(2x^1, \dots, 2x^n, 1-\|x\|^2\right),$$

$$\mathbb{R}^n \longrightarrow S^n \setminus \{S\},$$

are inverses for φ and ψ. These are obviously also cont.,
so φ and ψ are homeomorphisms and, of course

$$S^n = \left(S^n \setminus \{N\}\right) \cup \left(S^n \setminus \{S\}\right).$$

Therefore S^n is a topological mfd. of dimension n.

Next observe that the atlas $\mathcal{O}\!\iota = (\varphi, \psi)$ is differentiable since for the transitions $\psi \circ \varphi^{-1}$ and $\varphi \circ \psi^{-1}$ from $\mathbb{R}^n - \{0\} \to \mathbb{R}^n \setminus \{0\}$ we have (exercise)

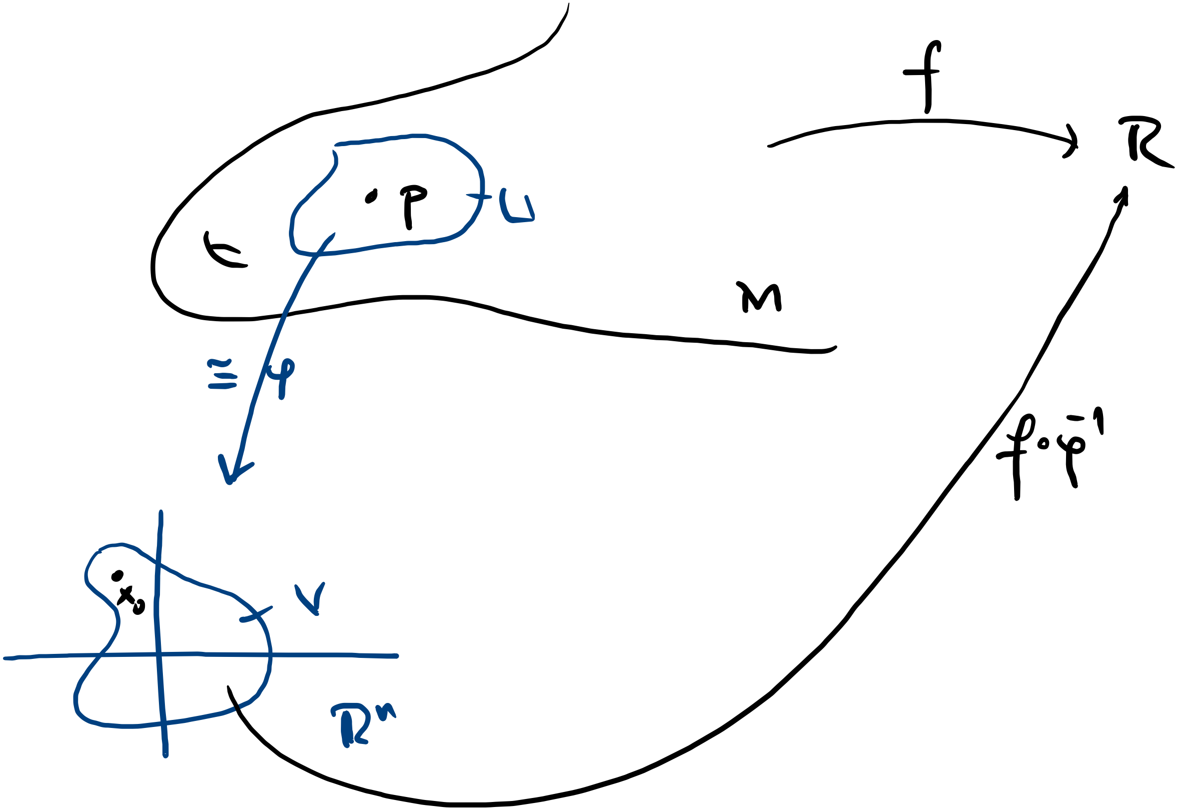
$$\psi \circ \varphi^{-1}(x) = \varphi \circ \psi^{-1}(x) = \frac{x}{\|x\|^2}.$$

There $(S^n, [(\varphi, \psi)])$ is a diff. mfd. of dimension n.

19.8) *Definition.* Let (M, c) be a differ. mfd. (written in the following simply by M) and $p \in M$. A continuous function $f: M \to \mathbb{R}$ is called *differentiable at p*, if for one (and then every) chart $\varphi: U \to V \subseteq \mathbb{R}^n$ (of some atlas $\alpha \in c$) around p (i.e. $p \in U$) the composition

$$f|U \circ \varphi^{-1} : V \longrightarrow \mathbb{R}$$

is differentiable in $x_0 = \varphi(p)$. f is called *differentiable* if it is differentiable in all points $p \in M$.


p
U
M
f
R
≅ φ
x_0
V
$\mathbb{R}^n$
$f \circ \varphi^{-1}$

(9.9) Remarks. As already motivated this definition is indeed independent of the chosen chart (and the chosen atlas of the structure $\mathcal{C}$). Namely, if $\psi : \tilde{U} \to \tilde{V} \subset \mathbb{R}^n$ is another chart, then

$$f|_{\tilde{U}} \circ \psi^{-1} = \left(f|_U \circ \varphi^{-1} \right) \circ \left(\varphi \circ \psi^{-1} \right) \quad \text{on } \psi(U \cap \tilde{U})$$

and therefore also differentiable in $\psi(p)$.

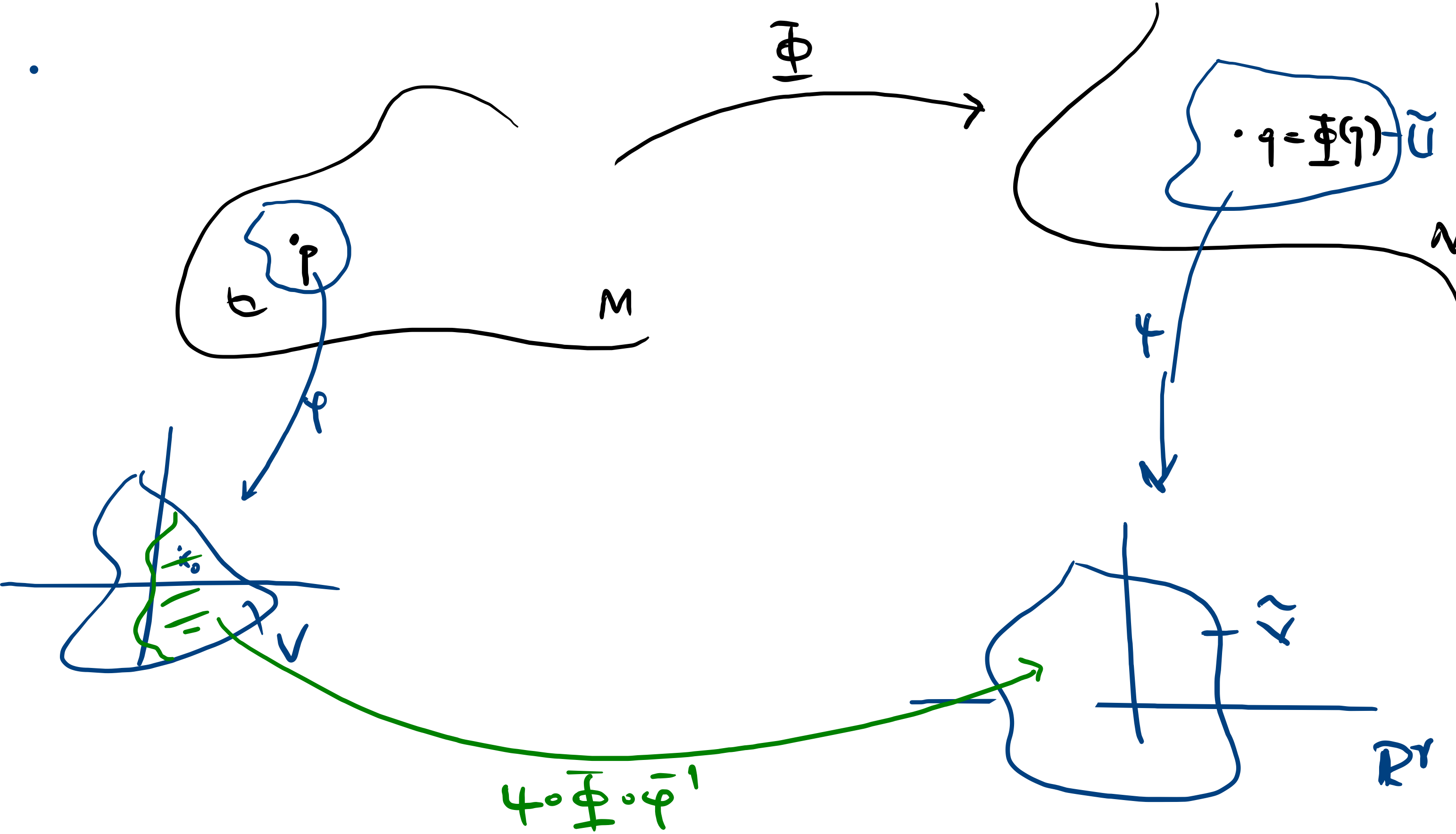
(9.10) Definition. Let M^n and N^r be diff. mfds.,

$p \in M$ and $\Phi : M \to N$ a continuous map.

(a) Then Φ is called *differentiable in p*, if for some (and then every) choice of charts $\varphi : U \to V \subseteq \mathbb{R}^n$ around p and $\psi : \tilde{U} \to \tilde{V} \subseteq \mathbb{R}^r$ around $q := \Phi(p) \in N$ the composition

$$\psi \circ \Phi \circ \varphi^{-1} : \varphi(U \cap \Phi^{-1}(\tilde{U})) \longrightarrow \tilde{V} \subseteq \mathbb{R}^r$$

is differentiable in $x_0 = \varphi(p)$.



(b) Φ is called diff., if Φ is diff. in all points $p \in M$.