Welcome to „Morse Theory"

Literature: J. Milnor: Morse Theory. Annals of Math. Studies 51, Princeton Univ. Press, 1963

§1. Motivation and Introduction

(1.1) **Aim**. A (topological) manifold of dimension $n \in \mathbb{N}$ is a topological space which locally looks like an open subset of $\mathbb{R}^n$. A smooth manifold .

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has enough structure in order to define and use <u>local</u> concepts of analysis, e.g., smooth functions on it.
You may think, e.g., at a „surface" (here $n=2$) in $\mathbb{R}^3$ which is given by the zero set of a smooth function $f: \mathbb{R}^3 \to \mathbb{R}$,

$$M = \{ x \in \mathbb{R}^3 : f(x) = 0 \}$$

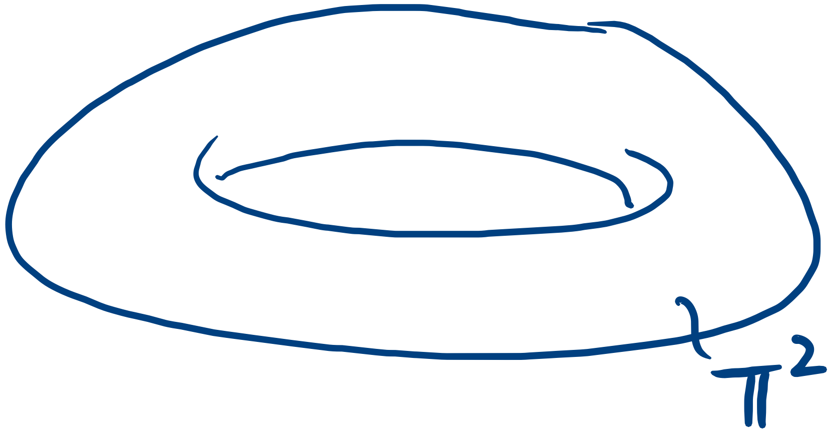
under the condition that

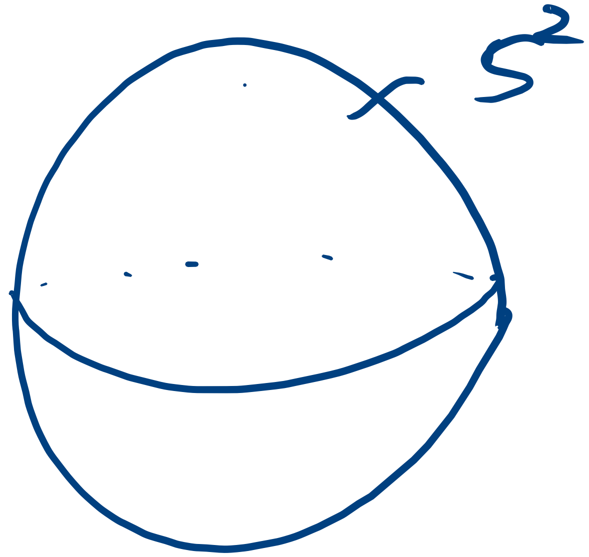
$$\operatorname{grad}(f)(p) = \left(\frac{\partial f}{\partial x_1}(p), \dots, \frac{\partial f}{\partial x_n}(p) \right) \neq 0, \quad \forall p \in M.$$

Example 1. The 2-sphere S^2 is given by

$$S^2 = \{ x \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^2 - 1 = 0 \}$$

Example 2. The 2-torus $\mathbb{T}^2$ (a doughnut) looks like





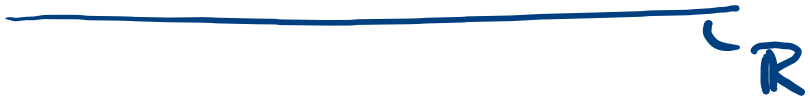
(It is not too hard to find an equation for T^2.)

Now, one is interested in the topology of M, e.g.:

- Are S^2 and T^2 „homeomorphic" or even „diffeomorphic"?
- How many different homeomorphy resp. diffeomorphy types exist?

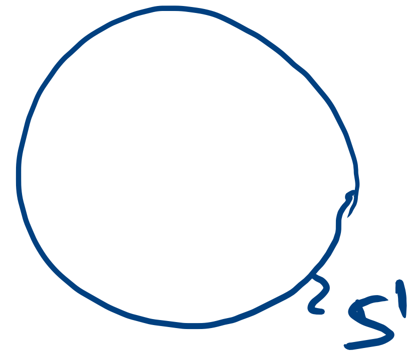
Results. (1) $n=1$ and if M is a connected 1-dim. mfd. (top. or smooth), then M is homeom. and even diffeomorphic to one of the following:

(i) $\mathbb{R}$ (the line)



(ii)

$$S^1 = \{x \in \mathbb{R}^2 : x_1^2 + x_2^2 = 1\}$$



(the „circle").

(2) If $n=2$ and M is a connected compact and „orientable" smooth 2-dim.'l mfd., then M is diff-om. to one of

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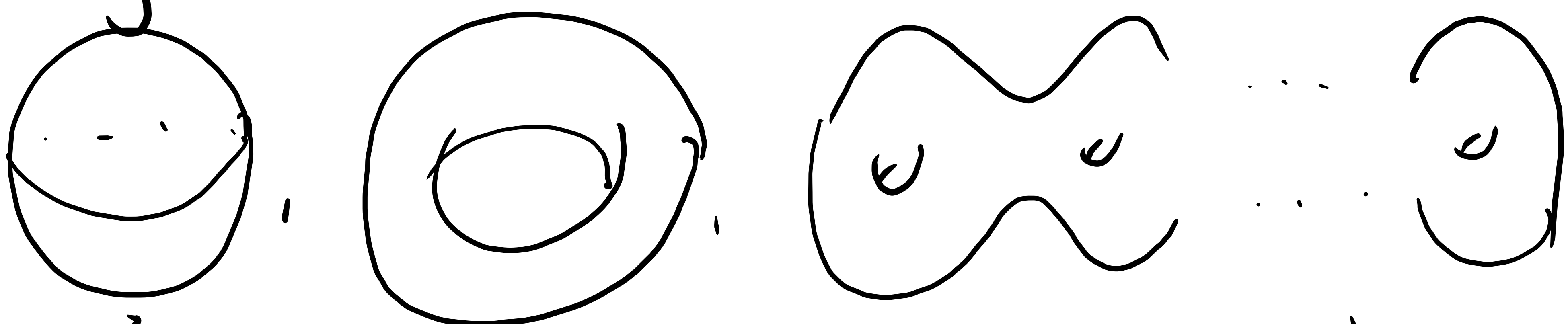
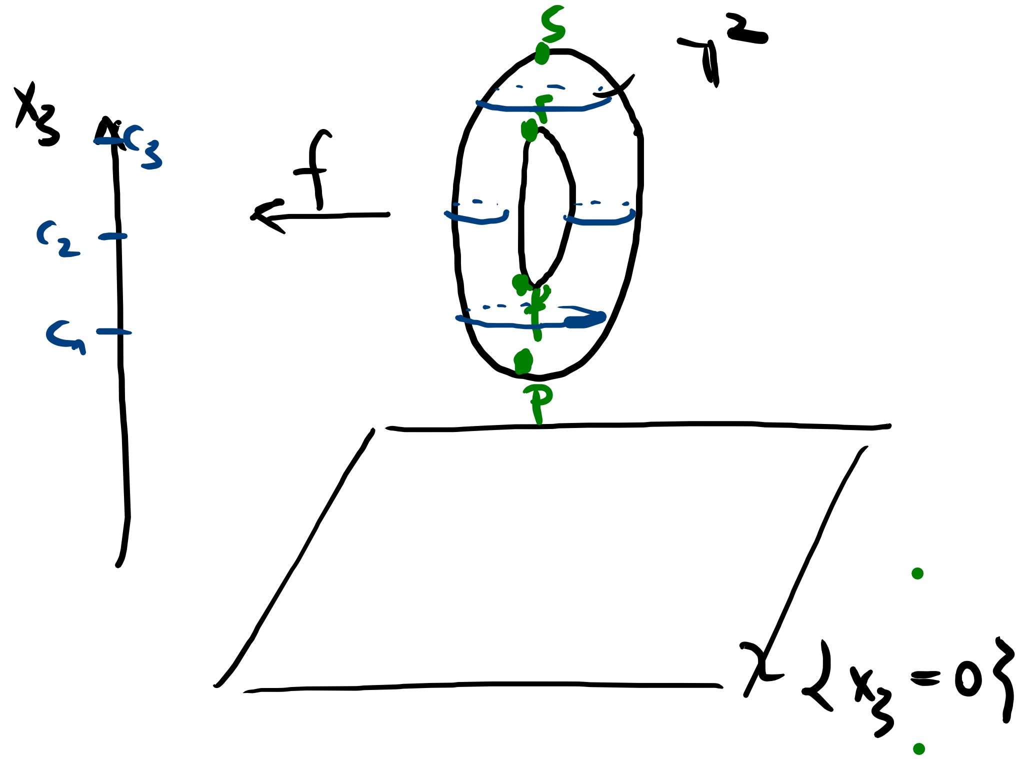
following:



S^2 $\quad$ T^2 $\quad$ Σ_g („g holes") ($g \in \mathbb{N}$)

All of this can be proven by using Morse Theory

(1.2) Basic Idea. Turn the torus up and look for the „height function" $f : T^2 \to \mathbb{R}$, $f(x) = x_3$:



Then it turns out that f has exactly 4 critical points $p \in \mathbb{T}^2$, i.e., in local coordinates (y_1, y_2) around that points we have

$$\frac{\partial f}{\partial y_1}(p) = \frac{\partial f}{\partial y_2}(p) = 0$$

Let us call these: $p, q, r, s \in \mathbb{T}^2$.

Now observe that the level set

$$N_c = \{ x \in \mathbb{T}^2 : f(x) = c \}$$

does not change its type between critical values:

(i) $f(p) < c < f(q) \implies N_c \cong S^1$

(ii) $f(q) < c < f(r) \implies N_c \cong S^1 + S^1$ (non connected)

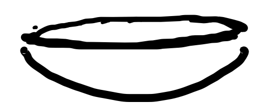
(iii) $f(r) < c < f(s) \implies N_c \cong S^1$

Morse theory now investigates the sublevel sets

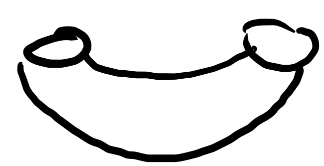
$$M_c = \{ x \in M : f(x) \leq c \},$$

more exactly its so called homotopy type, which is a weaker notion of homeomorphy type. Observe:

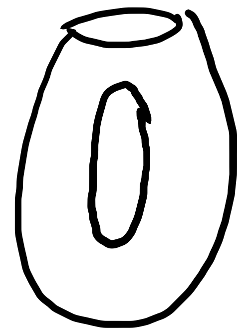
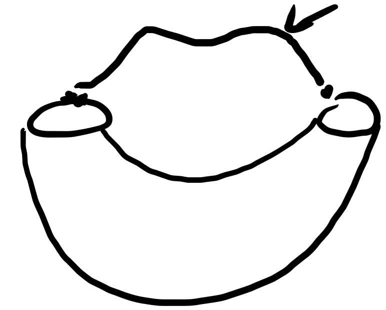
(i) $c < f(p)$: $M_c = \emptyset$

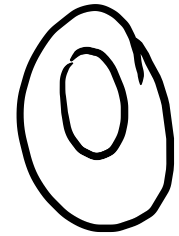
(ii) $f(p) < c < f(q) \Rightarrow$ $M_c \cong$  $\simeq$ $\cdot$

homotopy type

(iii) $f(q) < c < f(r) \Rightarrow$ $M_c \cong$  $\simeq$ 

„1-cell attached"

(iv) $f(r) < c < f(s) \Rightarrow M_c \cong$  $\simeq$  "1-cell attached"

(iv) $f(s) < c \Rightarrow M_c = M -$  $\simeq$  "2-cell attached"

All together: The homotopy type of the sublevel set does

- not change within two critical values
- changes by attaching a k-cell, where $0 \le k \le n$ is the index of the critical point to be defined, in our case:

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$\mathrm{ind}(p) = 0$

$\mathrm{ind}(q) = \mathrm{ind}(r) = 1$

$\mathrm{ind}(s) = 2$

(1.3) Methods. During the course we review (or learn) basic concepts of

- topology (set topology, algebraic topology)
- differential topology (smooth mfds and sm. functions)
- dynamical system on smooth mfds, i.e., the qualitative

behaviours of solution curves for vector fields on manifolds.

§2. Basic set theoretic topology

(2.1) Definition. Let M be a set. A topology on M is a subset $\tau \subseteq \mathcal{P}(M)$, the set of all subsets of M, so that

(i) $\emptyset, M \in \tau$

(ii) $U_1, U_2 \in \tau \Rightarrow U_1 \cap U_2 \in \tau$

(iii) $U_i \in \tau,\ i \in I \Rightarrow \bigcup_{i \in I} U_i \in \tau$.

(M, τ) is called a topological space, the elements of τ are called open sets in M.

Basic example is the metric topology of a metric space (M,d). Recall that $d: M\times M \longrightarrow [0,\infty)$ on a set M, if

(i) $\forall x,y \in M: \quad d(x,y)=0 \Longleftrightarrow x=y$

(ii) $\forall x,y \in M: \quad d(y,x) = d(x,y)$

(iii) $\forall x,y,z \in M:$

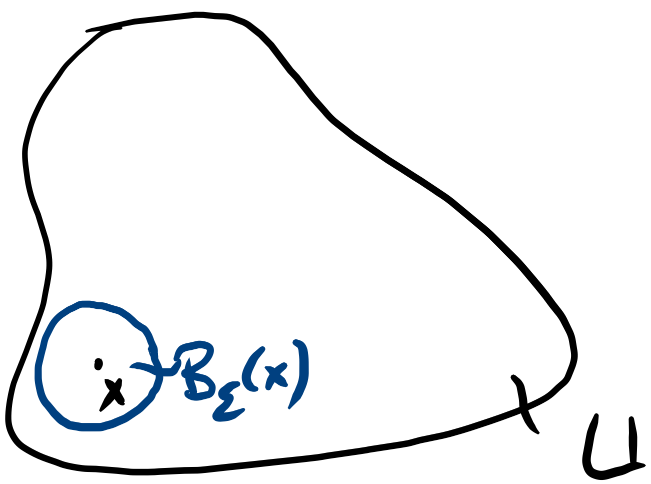
$$d(x,z) \leq d(x,y) + d(y,z).$$

Here on defines $\mathcal{U} \subseteq M$ as open, if

$$\forall x \in M \; \exists \varepsilon > 0 : B_\varepsilon(x) \subseteq \mathcal{U}$$

The (open) balls $B_r(x) \subseteq M$, for $x \in M$ and $r > 0$, are defined as

$$B_r(x) = \{ y \in M : d(y,x) < r \}$$



It is not hard to see that these open sets together build a topology on M, the metric topology corresponding to d.

(2.2) Basic example. Consider n-space ($n\in\mathbb{N}$)

$$\mathbb{R}^n = \{(x_1,\dots,x_n) : x_j \in \mathbb{R},\ j=1,\dots,n\}$$

together with its euclidean metric $d : \mathbb{R}^n \times \mathbb{R}^n \to [0,\infty)$,

$$d(x,y) = \left[\sum_{j=1}^{n} (y_j - x_j)^2\right]^{1/2}$$

Then $(\mathbb{R}^n, d)$ is called the n-dim.'l euclidean space and if we write $M=\mathbb{R}^n$, then we mean $\mathbb{R}^n$ together with its induced metric topology.

(2.3) Basic constructions

(i) Subspace topology.

Let M be a top. space and $N \subseteq M$ a subset. We call a subset $V \subseteq N$ open in N (relatively open), if there exists an open set $U \subseteq M$, so that

$$V = U \cap N.$$

Then: $\{V \in \mathcal{P}(N) : V \text{ open}\}$ is a topology on N. It is called the subspace topology or induced topology on N.

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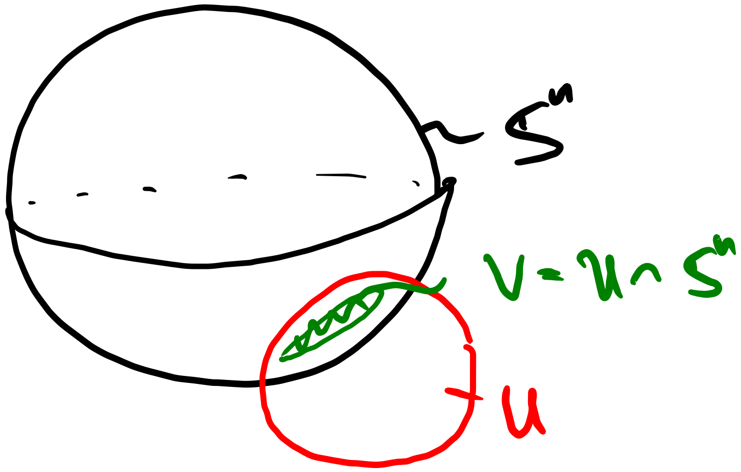
Example. $S^n \subseteq \mathbb{R}^{n+1}$ $(n \in \mathbb{N})$ inherits from the euclidean $\mathbb{R}^{n+1}$ its subspace topology. We call S^n together with this topology the *n-dimensional sphere*.

$$S^n = \{ x \in \mathbb{R}^{n+1} : d(x,0) = 1 \}$$



(ii) Product topology

"german V"

Definition. (a) A subbasis $\mathfrak{V} \subseteq \tau$ of a topology τ on a set M is a subset, so that each element of τ is a union of elements which are finite intersections of elements in $\mathfrak{V}$. $\mathfrak{V}$ is then also called a generator of the topology τ.

(b) A subset $\mathfrak{B} \subseteq \tau$ is called a basis of the topology, if each element $U \in \tau$ is a union of certain elements B_i $(i \in I)$ of $\mathfrak{B}$,

$$U = \bigcup_{i \in I} B_i .$$

(c) A top. space (M,τ) has *countable topology* (fulfills the 2^{nd} axiom of countability), if there exists a countable Basis $\mathfrak{B}$ of τ.

Example. (i) The set

$$\mathfrak{B} = \{ B_r(x) \subseteq \mathcal{P}(\mathbb{R}^n) : x \in \mathbb{Q}^n, r \in \mathbb{Q}_+ \}$$

is a countable Basis of the euclidean space $\mathbb{R}^n$. So $\mathbb{R}^n$ has countable topology.

(ii) If M has countable topology, and $N \subseteq M$ is a subspace, then N has countable topology as well.

<u>Definition</u>. Let $(M_\alpha)_{\alpha\in I}$ be a family of top. spaces. Then on the cartesian product

$$M = \prod_{\alpha\in I} M_\alpha$$

we consider the smallest topology which contains all preimages $\pi_\alpha^{-1}(U_\alpha) \subseteq M$, where $U_\alpha \subseteq M_\alpha$ is open and $\pi_\alpha : M \to M_\alpha$ is the α-th projection,

$$\pi_\alpha((x_\beta)) = x_\alpha .$$

In particular: On $M = M_1 \times M_2$ this product topology is generated by the the sets $U_1 \times U_2 \subseteq M$, $U_1 \subseteq M_1$, $U_2 \subseteq M$ are open.

<u>Remark</u> (i) On $\mathbb{R}^n$ this gives the same topology as this given by euclidean balls.
(ii) If M_1, M_2 have countable topology, so has $M_1 \times M_2$

<u>Example</u>. For $n \in \mathbb{N}$ we call

$$\mathbb{T}^n := S^1 \times \cdots \times S^1 \quad (n \text{ times})$$

the n-dimensional torus (together with its product topology).