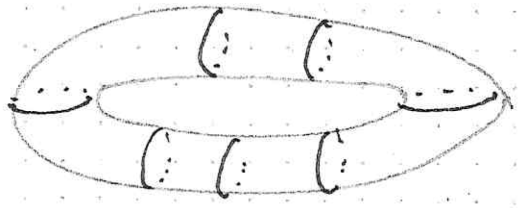
Example. For $n \in \mathbb{N}$ we call

$$\mathbb{T}^n := \mathbb{S}^1 \times \dots \times \mathbb{S}^1 \quad (n \text{ times})$$

the n-dimensional torus (together with its product topology)



Example (a) For $n \in \mathbb{N}$ we call

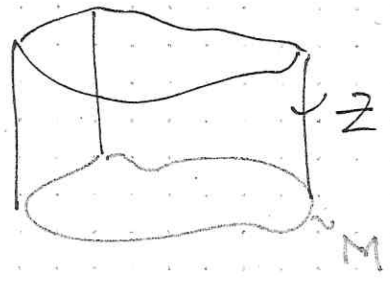
$$\mathbb{B}^n := \{ x \in \mathbb{R}^n : \|x\| \leq 1 \}$$

the n-dimensional Ball (together with its subspace topology).

(b) If M is an arbitrary top. space, we call with $I := [0,1] \subseteq \mathbb{R}$ (with its subspace topology)

$$Z := M \times I$$

the cylinder over M



(iii) Quotient topology

Definition. Let M be a top. space and $R \subseteq M \times M$ an equivalence relation. (We write $p \sim q$ for $(p,q) \in R$.) Let

$$Q = \{ [p] \in \mathcal{P}(M) : p \in M \}$$

the corresponding quotient space, where

$$[p] = \{ q \in M : q \sim p \}$$

denotes the equivalence class of p. Finally let $\pi : M \to Q$, $p \mapsto [p]$ the canonical projection. We equip Q with the following quotient topology:

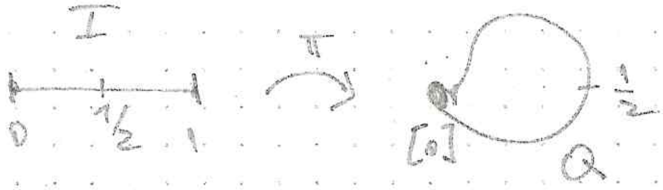
$$U \subseteq Q \text{ open } :\Leftrightarrow \pi^{-1}(U) \subseteq M \text{ is open.}$$

Examples. (a) Let $I = [0,1]$ (with its standard topology). Consider the equivalence relation $\sim$ on I which is generated by $0 \sim 1$ (i.e., the smallest which contains $(0,1) \in I \times I$;

$R = \{ (0,1), (1,0), (x,x) : x \in I \}$. Then $Q = I/\sim$ is homeomorphic to $S^1 \subseteq \mathbb{C}$ via

$$f : Q \to S^1, [t] \mapsto e^{2\pi i \cdot t},$$

as we will see in a moment.



(b) Let M be a topological space and $A \subseteq M$ a subspace. We consider the equivalence relation ~~generated by~~

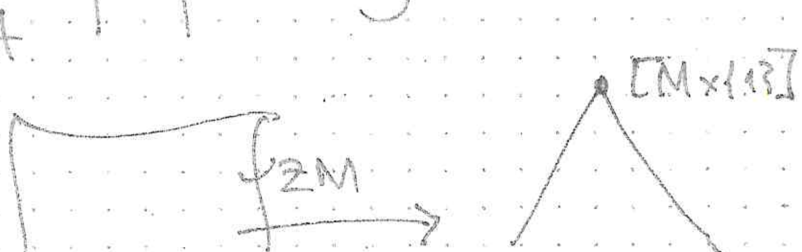
$$p \sim q \;:\Longleftrightarrow\; \begin{cases} p = q \\ p, q \in A \end{cases},$$

and call $Q =: M/A$ the space, where A is identified to a point.

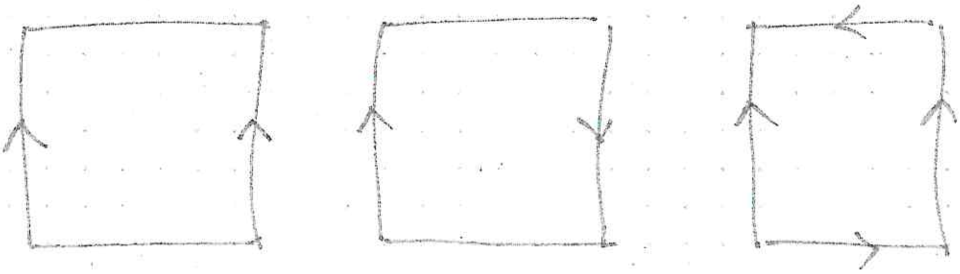
The cone over a space M is e.g.

$$CM := M \times I / M \times \{1\} = ZM / M \times \{1\},$$

where the top of the cylinder is clatched to a point.



(c) Let $M = I \times I$ and $\sim$ generated by identifying boundary points in the following way:



(a) (b) (c)

Then (a) is homeom. to $S^1 \times I$ (cylinder), (b) is called the Möbius strip and (c) the Klein bottle.

(d) Let $M = S^n$ and

$$p \sim q :\Leftrightarrow p = \pm q$$

We identify antipodal points on the sphere. The quotient space

$$\mathbb{RP}^n := S^n/\sim$$

is called the n-dimensional real projective space.

(iv) Topological sum

Let M_α be topological spaces ($\alpha \in I$). On the set-theoretic sum $M := \sum M_\alpha$ (also called the disjoint union) together with its canonical inclusions $i_\alpha : M_\alpha \hookrightarrow M$ we define $U \subseteq M$ as open, if $i_\alpha^{-1}(U) \subseteq M_\alpha$ is open, $\forall \alpha \in I$.

Example. Let (M_1, p_1) and (M_2, p_2) be pointed topological spaces (i.e. $p_j \in M_j$, $j = 1,2$). The one-point union of (M_1, p_1) and (M_2, p_2) is the pointed space

$$M := M_1 \vee M_2 := (M_1 + M_2)/\sim,$$

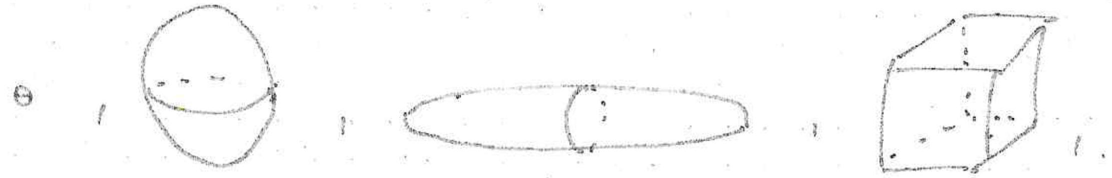
where $\sim$ identifies $i_1(p_1)$ and $i_2(p_2)$. $[i_1(p_1)]$ is then the distinguished point in M. E.g.

$S^1 \vee S^1 \cong$  , the „figure eight".

(2.4) Continuous maps and homeomorphisms

Definition. Let M and N be topological spaces.
(a) A map $f: M \to N$ is called continuous if $f^{-1}(V) \subseteq M$ is open, for all open $V \subseteq N$.
(b) A map $f: M \to N$ is a homeomorphism if f is continuous, bijective and $f^{-1}: N \to M$ is also continuous. M and N are homeomorphic, $M \cong N$, if there exists a homeomorphism $f: M \to N$.

Remark. (a) If M and N are homeomorphic, they have so-to-say the same shape. E.g. the following surfaces in $\mathbb{R}^3$ are all homeom.:



(b) Usually, by the constructions of (2.5) the new spaces come together with maps. Typically these maps turn out to be continuous (with some extremal property):

(i) product top: $\pi_\alpha : \prod_{\beta \in I} M_\beta \longrightarrow M_\alpha$ is cont.

(ii) subspace top.: The inclusion $\iota : N \hookrightarrow M$ is cont.

(iii) quotient top.: the canonical proj. $\pi : M \to M/\sim$ is continuous

(iv) sum topology: the inclusions ι_α: $M_\alpha \hookrightarrow \sum_{\beta \in I} M_\beta$ is continuous.

Example for a homeomorphism: Observe $F : I \to S^1$, $F(t) = e^{2\pi i t}$ is continuous. Since $F(0) = F(1)$, there exists a unique map $f : I/\sim \longrightarrow S^1$ with $f \circ \pi = F$ (universal property of the quotient).

$$\begin{tikzcd} I \arrow[r, "F"] \arrow[d, "\pi"'] & S^1 \\ I/\sim \arrow[ur, dashed, "f"'] & \end{tikzcd}$$

By the universal property of the quotient top. f is continuous. Obviously f is bijective. Since $I/\sim$ is compact and S^1 is Hausdorff, f^{-1} is also continuous. Therefore: f is a homeomorphism.

2.5. Properties of topological spaces

(i) Countable topology
see above

(ii) Hausdorff property

Definition A top. space M is called Hausdorff (or separable), if for all $p, q \in M$, $p \neq q$, there exist open sets U, V in M with

- $U \cap V = \emptyset$
- $p \in U$, $q \in V$

(U and V separate p and q).

Remark (a) If a topology τ on a set M is metrizable, i.e., if there exists a metric d on M so that the topology is the metric topology corresponding d, then τ must be Hausdorff. In fact: Let $p, q \in M$ and $r := d(p,q) > 0$, if $p \neq q$.

Then set $U := B_{r/2}(p)$, $V := B_{r/2}(q)$ and observe: $p \in U$, $q \in V$ and $U \cap V = \emptyset$ using the Δ-inequality of d (if $z \in U \cap V \Rightarrow d(p,q) \leq d(p,z) + d(z,q) < \frac{r}{2} + \frac{r}{2} = r$).

(b) If M is hausdorff, $N \subseteq M$ a subspace, then N is hausdorff. If M_α is hausdorff, $\forall \alpha \in I$, then $\prod M_\alpha$ is hausdorff, and also $\sum M_\alpha$. Quotients are in general not hausdorff: The line $\mathbb{R} + \mathbb{R}/\sim$, with $i_1(x) \sim i_2(x)$ for $x \neq 0$,

is non

Hausdorff,
since you cannot separate $[i_1(0)]$ and $[i_2(0)]$.

(iii) Connectedness

Definition. A top. space M is called connected, if for all open sets U, V in M with
- $U \cap V = \emptyset$
- $U \cup V = M$

it must be: $U = \emptyset$ or $V = \emptyset$.

A non-connected space.

Remark (a) If $f: M \to N$ is continuous between top. spaces and ~~if~~ M is connected then so is $f(M) \subseteq N$.

(b) (i), (ii) and (iii) are invariants under homeomorphism, i.e.: if $M \cong N$ and M has the property, then N has it as well. The same is true for the following property:

(iv) Compactness

Definition A Hausdorff space M is called compact, if every open covering $(U_\alpha)_{\alpha \in I}$ of M (i.e.: $U_\alpha \subseteq M$ is open and $\bigcup U_\alpha = M$) has a finite sub covering (i.e.: $M = U_{\alpha_1} \cup \dots \cup U_{\alpha_n}$ for some $\alpha_1, \dots, \alpha_n \in I$).

Remarks (a) Closed subsets $A \subseteq M$ of a compact space M (A is closed, if $CA = M \setminus A$ is open) is compact.
(b) If M is Hausdorff and $K \subseteq M$ is compact, then M is closed. (c) $f: M \to N$ cont., M cpt. $\Rightarrow f(M)$ is compact.

Proposition: If M is compact, N Hausdorff and $f: M \to N$ is cont. and bijective. then f^{-1} is also continuous.

Pf.: It suffices to show: $A \subseteq M$ closed $\Rightarrow f(A) \subseteq N$ is closed. But if $A \subseteq M$ is closed, then A is

compact and therefore ~~∂A~~ $f(A)$ is compact.
Therefore $f(A) \subseteq N$ is closed. ☒

§6. Homotopy

Definition. Let M and N be topological spaces.
A continuous map $f: M \to N$ is homotopic to a continuous map $g: M \to N$, $f \simeq g$, if there exists a continuous map $H: M \times I \to N$ (a homotopy from f to g), s.th.

$$H(p,0) = f(p), \; H(p,1) = g(p), \; \forall p \in M.$$

Remark. (a) If we ~~imagine~~ consider $t \in I$ as a time parameter we could imagine that $f(p)$ is deformed via the curve $\alpha_p : I \to N$ in N to $g(p)$

(b) If we denote $h_t : M \to N$ by $h_t(p) := H(t,p)$, we sometimes denote the homotopy H by the family (h_t). The condition is then simply: $h_0 = f$, $h_1 = g$.