Vorlesung 03, 4.11.2022

(iv) Compactness

Definition. A Hausdorff space M is called *compact*, if every open covering $(U_\alpha)_{\alpha\in I}$ of M (i.e., $U_\alpha \subseteq M$ is open and $\bigcup_\alpha U_\alpha = M$) has a finite subcovering (i.e., $M = U_{\alpha_1} \cup \cdots \cup U_{\alpha_n}$, for some $\alpha_1, \ldots, \alpha_n \in I$).

Remarks: (a) Closed subsets $A \subseteq M$ of a compact space M are compact. (A is closed, if $CA = M \setminus A$ is open.)

(b) If M is hausdorf and $K \subseteq M$ is compact, then M is closed.

(c) If $f: M \to N$, M compact $\Rightarrow f(M)$ is compact.

Proposition. If M compact, N is hausdorf and $f: M \to N$ is continuous and bijective. Then f^{-1} is also continuous.

Pf.: It suffices to show: $A \subseteq M$ closed $\Rightarrow f(A) \subseteq N$ is closed. But if $A \subseteq M$ is closed, then A is is compact. Therefore $f(A)$ is compact. Therefore $f(A) \subseteq N$ is closed.

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§6. Homotopy

Definition. Let M and N be topological spaces. A continuous map $f: M \to N$ is homotopic to a continuous map $g: M \to N$, $f \simeq g$, if there exists a continuous map $H: M \times I \longrightarrow N$ (a homotopy from f to g), s.th.:

$$H(p,0) = f(p), \quad H(p,1) = g(p), \quad \forall p \in M.$$

Remarks. (a) If we consider $t \in I = [0,1]$ as a time parameter we could imagine that $f(p) \in N$ is deformed via the curve $\alpha_p : I \to N$ in N to $g(p) \in N$

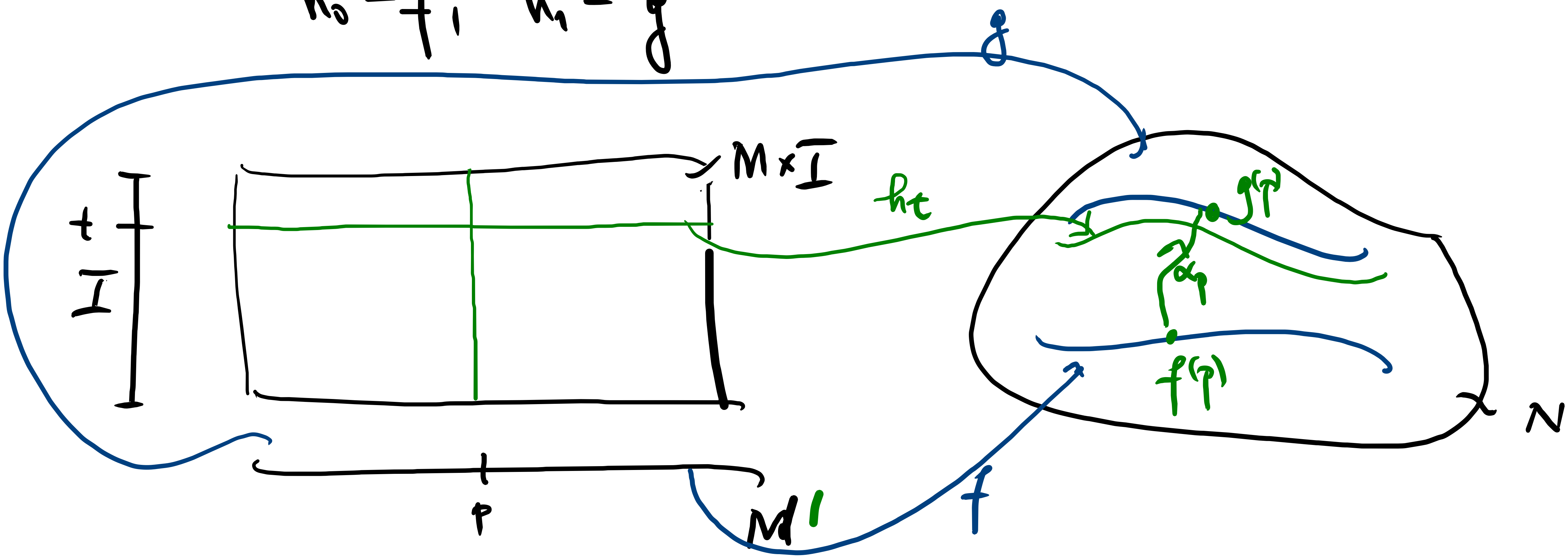
(b) If we denote $h_t : M \to N$ by

$$h_t(p) = H(p,t),$$

we sometimes denote the homotopy H by the family

(h_t). The condition is then simply:

$$h_0 = f, \quad h_1 = g$$



<u>Remarks</u>. (a) $\simeq$ is an equivalence relation on the set of all continuous maps $\mathcal{C}(M,N)$. We denote a <u>homotopy class</u> of a map f by $[f]$

(b) $\simeq$ is compatible with the composition of maps, i.e.: if $f_1 \simeq f_2$ and $g_1 \simeq g_2 \Rightarrow g_1 \circ f_1 \simeq g_2 \circ f_2$. Thus the composition of classes

$$[g] \circ [f] := [g \circ f]$$

is well defined. This makes top. spaces together with homotopy classes of maps a <u>category</u> <u>HTop</u>, where objects are the topological spaces and the morphisms are the homotopy classes of continuous maps.)

(c) If $f : M \to N$ is homotopic to a constant map $c_q : M \to N$ ($q \in N$), then we call f null homotopic.

(d) A space M is called path connected, if for every $p, q \in M$ there exists a path, i.e., a continuous map $\alpha : I \to M$, from p to q, i.e., $\alpha(0) = p, \alpha(1) = q$. A path connected space is connected (Exercise).

Definition. A top. space M is called contractible, if $\mathrm{id}_M \simeq c_p$ (for some $p \in M$)

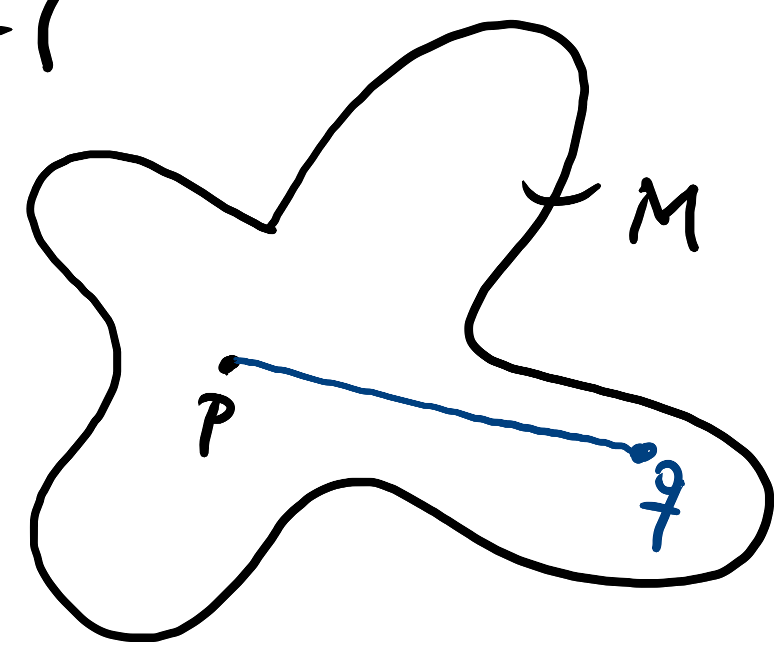
Remarks. (a) Contractible spaces are necessarily path connected

(b) $M = \mathbb{R}^n$, or more generally every star shaped region $M \subset \mathbb{R}^n$, (i.e., there exists a point $p \in M$ (a star point), s.th. for all $q \in M$ the line segment

$$[q,p] = \{(1-t)q + tp \in M : t \in I\}$$

is in M) is contractible. In fact,

$H : M \times I \to M$,



$$H(p,t) = (1-t)p + tq$$

Definition. A homotopy equivalence between to spaces M and N is a continuous map $f: M \to N$, s.th. there exits a continuous map $g: N \to M$ with

$$g \circ f \simeq id_M, \quad f \circ g \simeq id_N \quad \left(\text{i.e. } [g] \circ [f] = [id_M], \; [f] \circ [g] = [id_N] \right)$$

(It is an isomorphism in the category $\underline{HTop}$.)

We write

$$M \simeq N$$

and say that M and N have the same homotopy type.

<u>Example</u>: If M is contractible, then $M \simeq \{p\}$. (Since $i : \{p\} \hookrightarrow M$ and $r : M \to \{p\}$, $r = c_p$ are homotopic inverses:

$$r \circ i = \mathrm{id}_{\{p\}}, \quad i \circ r \simeq \mathrm{id}_M .$$

Definition. Let M be a topological space.

(a) A subspace $A \subseteq M$ is called a *retract of M*, if there exists a cont. map $r : M \to A$ with $r|A = \mathrm{id}_A$ (a *retraction*).

(b) $A \subseteq M$ is called a *deformation retract*, if there exist a retraction $r : M \to A$ with $i \circ r \simeq \mathrm{id}_M$, where $i : A \hookrightarrow M$ is the inclusion.

Remark. (a) A retraction $r : M \to A$ is a left inverse of the inclusion $i : A \hookrightarrow M$: $r \circ i = \mathrm{id}_A$.

(b) A deformation retraction is a retraction which is also a right inverse of the inclusion, if we turn over to homotopy classes:

$$r \circ i = \mathrm{id}_A \,, \quad i \circ r \simeq \mathrm{id}_M$$

Therefore r is in particular a homotopy equivalence from M to A, $M \simeq A$.

Examples. (i) $S^n \subseteq \mathbb{R}^{n+1} \setminus \{0\}$ is a deformation retract via $r : \mathbb{R}^{n+1} \setminus \{0\} \to S^n$,

$$r(x) = \frac{x}{\|x\|}.$$

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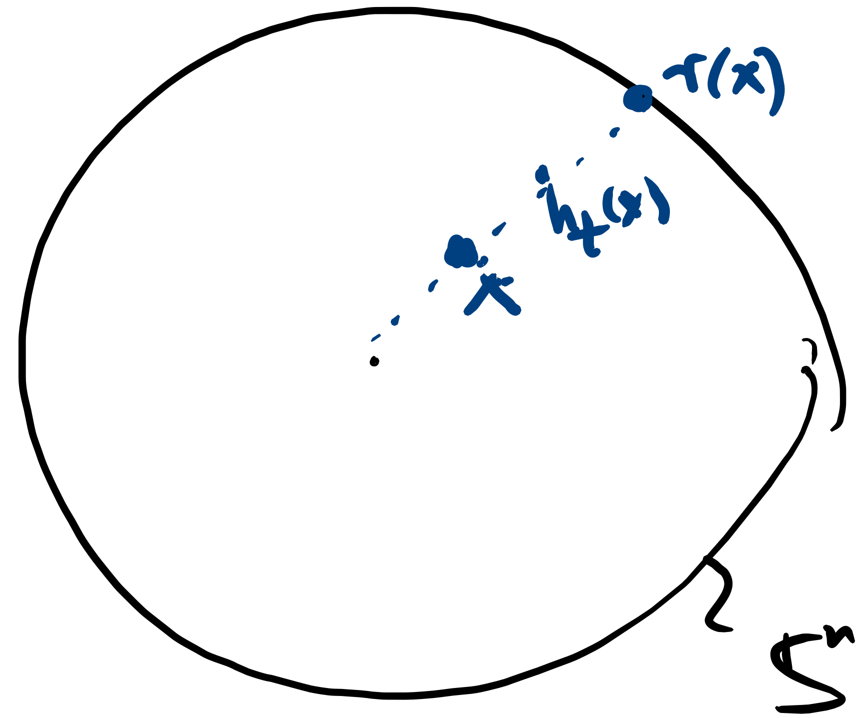
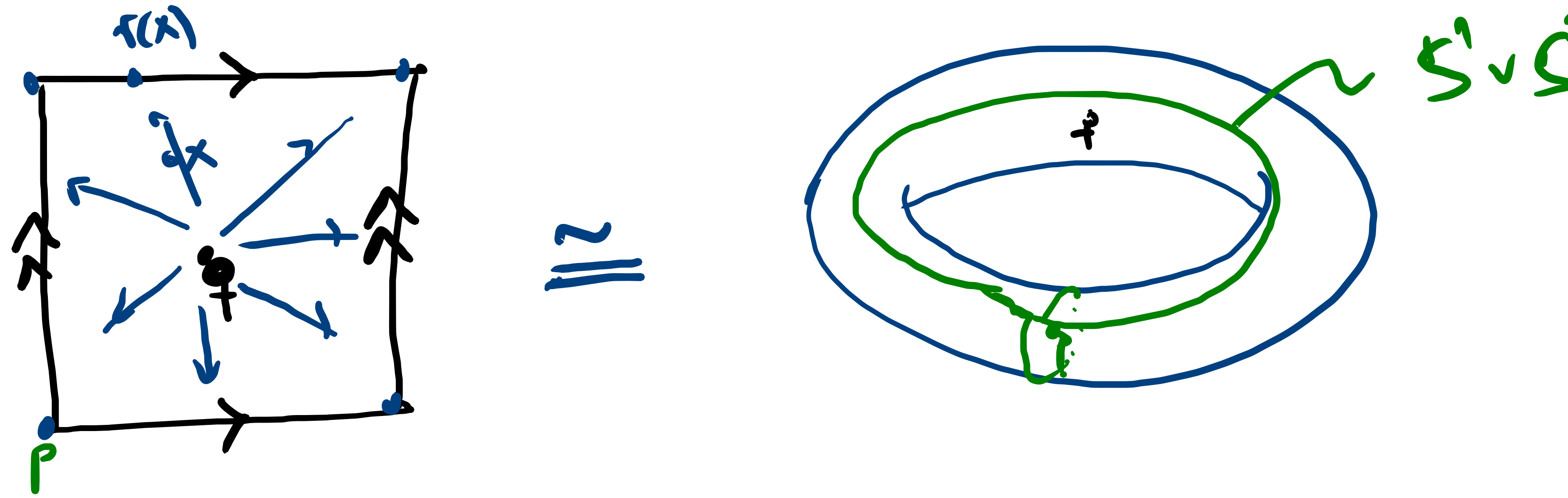
$$h_t(x) = (1-t)x + t\frac{x}{\|x\|}.$$



(ii) $S^1 \vee S^1$ is a subspace of T^2 via the equivalence classes of the boundary of $I \times I$:

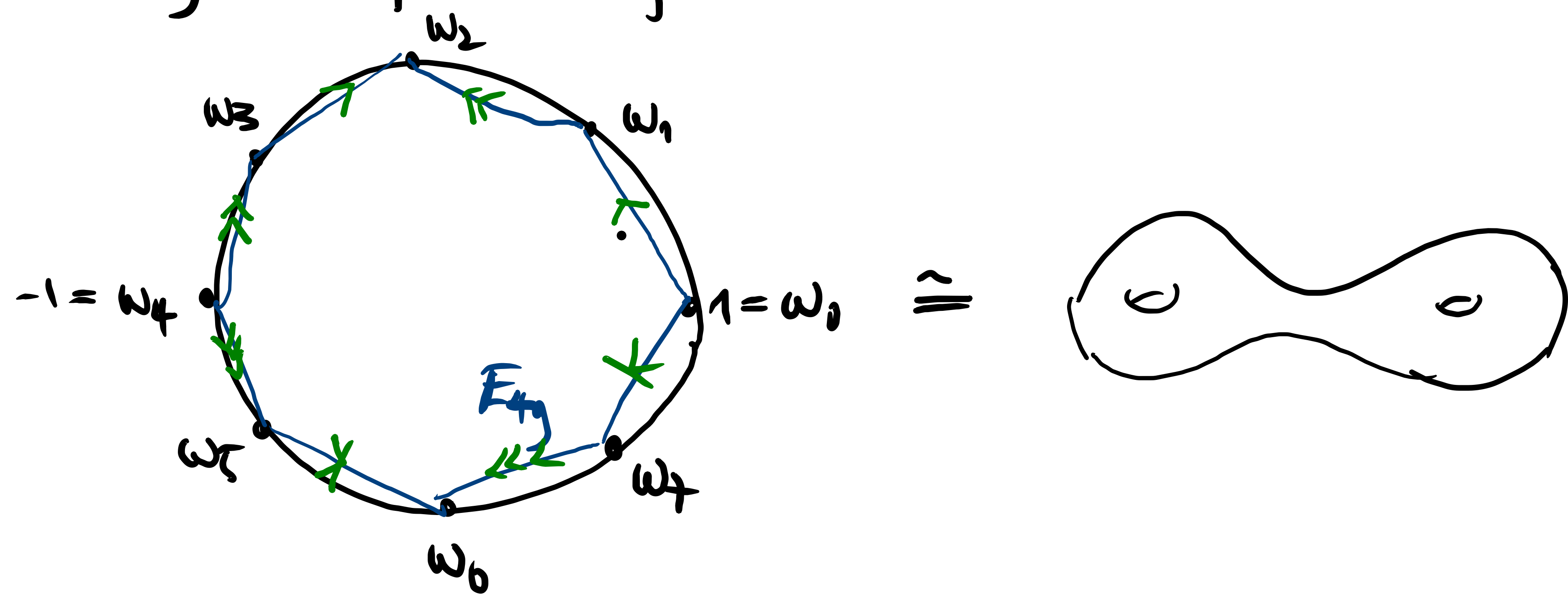


Then $S^1 \vee S^1$ is a deformation retract of $T^2 \setminus \{q\}$

(iii) Similarly the oriented surface of genus $g \in \mathbb{N}$ is defined by

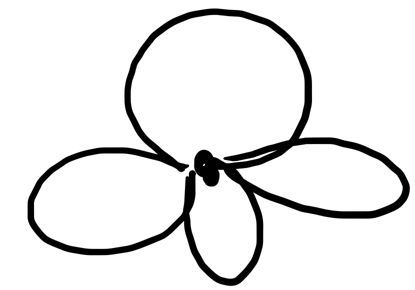
$$F_g = E_{4g}/\sim,$$

where $E_{4g} \subseteq \mathbb{R}^2$ is the (inside of the) regular $4g$-gon with its boundary identified as follows:



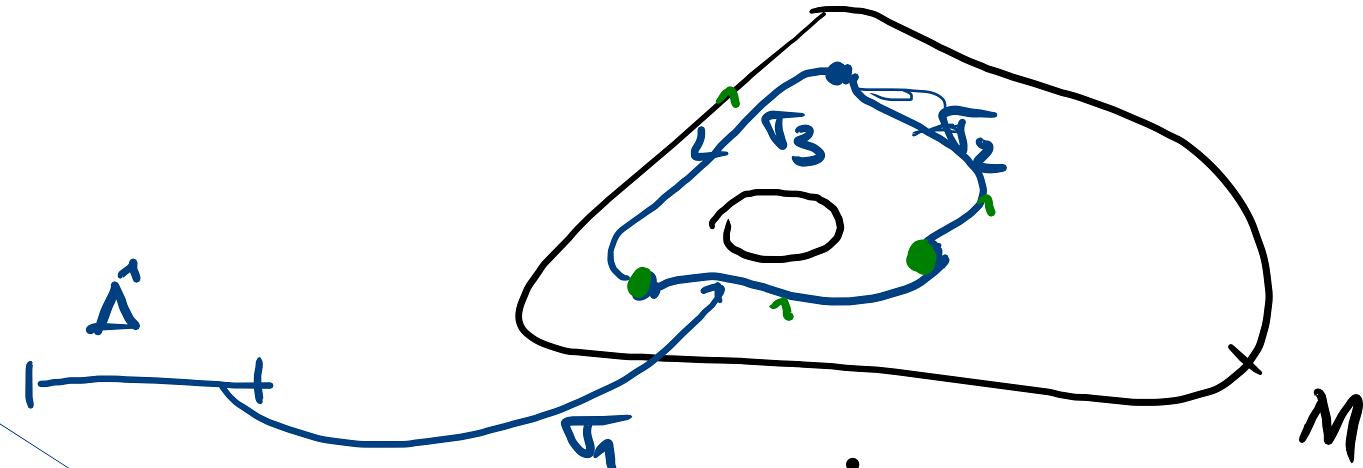
($w_0, \ldots, w_7 \in \mathbb{C}$ are the $4g$-roots of unity). Via the boundary we have included $S^1 \vee \cdots \vee S^1$ ($2g$ times). It is a deformation retract of $\mathbb{F}_g \setminus \{p\}$

$$\mathbb{F}_g \setminus \{p\} \simeq \bigvee_{j=1}^{2g} S^1$$



§7. Homology

(2.1) Motivation. Let M be a top. space. Basic idea of homology of M is that $H_k(M)$ should be an abelian group whose rank gives the number of „k-dimensional holes" in M. In case $k=1$, e.g., one has singular 1-simplices, which are just paths in M.



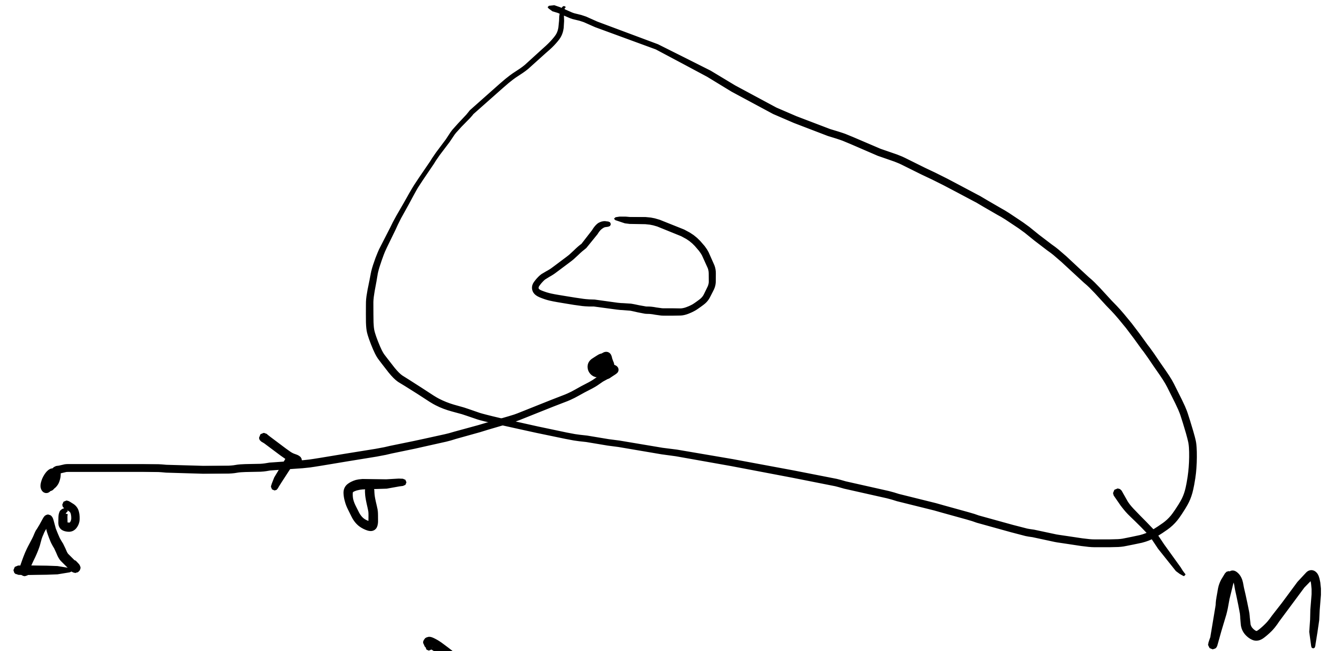
$\partial\sigma_1 = \sigma_1(1) - \sigma_1(0)$

A 1-chain in M is a formal sum of singular 1-simplices.

$$c = \sum_{j=1}^{r} m_j \sigma_j \qquad (m_j \in \mathbb{Z})$$

And the Boundary of a 1-chain $c = \sum m_j \sigma_j$ is the formal of 0-simplices, a 0-chain, where a

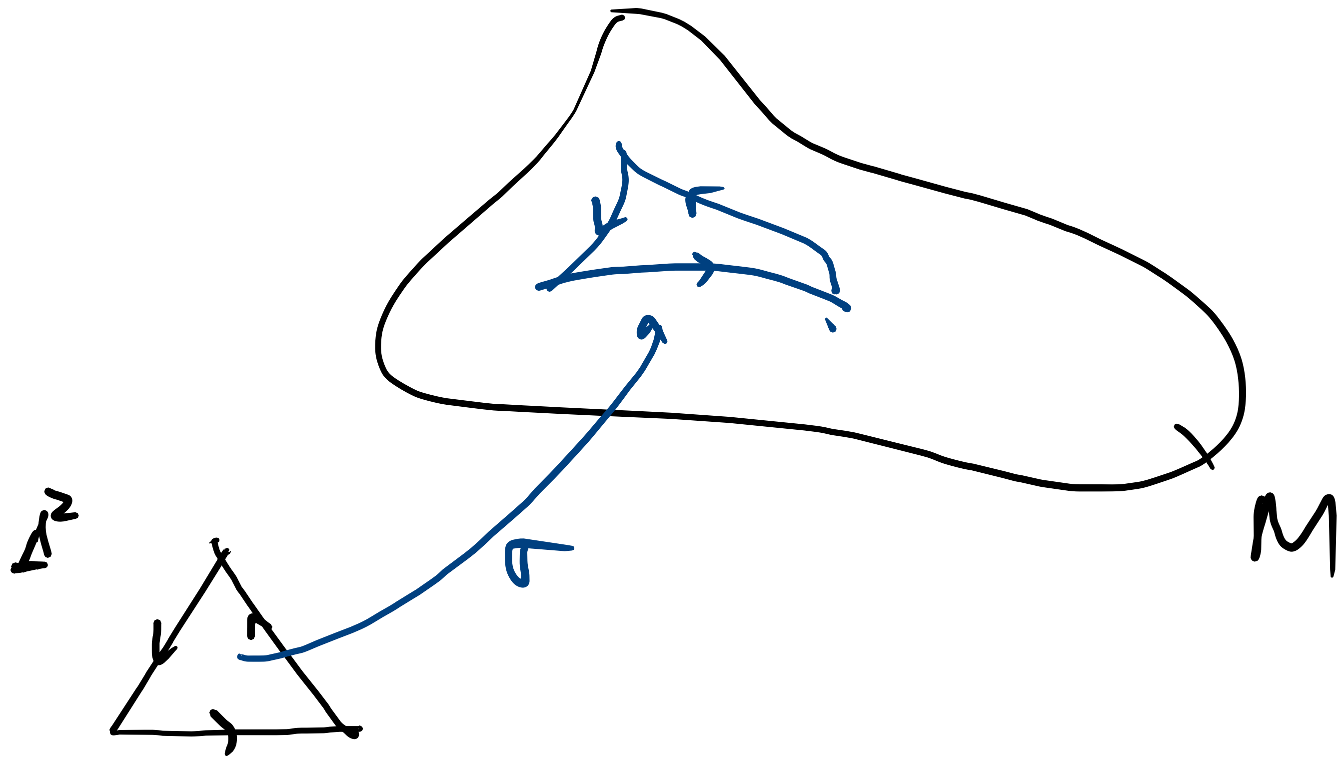
0- simplex is just a point



$$\partial c = \sum_{j=1}^{r} m_j \partial \sigma_j = \sum m_j \left(\sigma_j(1) - \sigma_j(0)\right)$$

A 1- cycle is a 1- chain c with zero Boundary. $\partial c = 0$

A 2- chain is a formal sum of sing. 2- simplices, where a sing. 2- simplex is a continuous map from the (standard) triangle Δ to M, $\sigma : \Delta^2 \longrightarrow M$,



The boundary of σ is defined in an expectable way.
The boundary is always a 1-cycle. If you find a 1-cycle which is not the boundary of a 2-chain, you have found a 1-dimensional hole.