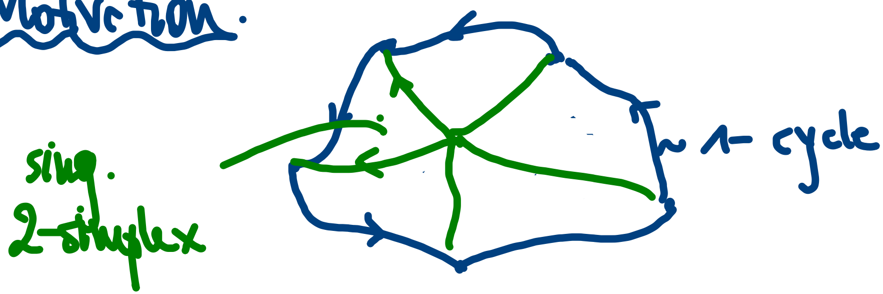
Vorlesung 4, 11.11.2022

Motivation.



(7.2) **Definition.** A (non-negative) chain complex $(C_k)_{k\in\mathbb{Z}}$ is a family of abelian groups C_k together with a family $(\partial_k)_{k\in\mathbb{Z}}$ of homomorphisms (called boundary operators), so that

- $C_k = (0)$, $\forall k < 0$
- $\partial_k : C_k \longrightarrow C_{k-1}$ with

$$\partial_{k-1} \circ \partial_k = 0, \quad \forall k \in \mathbb{Z}.$$

Remark. Sometimes we build the direct sum

$$C := \bigoplus_k C_k$$

and

$$\partial := \bigoplus_{k \in \mathbb{Z}} \partial_k : C \longrightarrow C,$$

call this a graded abelian group and a homomorphism of degree -1 [i.e., $\partial(C_k) \subseteq C_{k-1}$]. The condition for the chain complex then looks like simply:

$$\partial^2 = 0.$$

We write also a sequence

$\ldots$

$$(*)\quad \begin{tikzcd} \cdots \arrow[r] & C_{k+1} \arrow[r, "\partial_{k+1}"] \arrow[rr, bend right, "0"'] & C_k \arrow[r, "\partial_k"] & C_{k-1} \arrow[r] & \cdots \end{tikzcd}$$

(7.3) Definition. Let $C = ((C_k, \partial_k)_{k\in\mathbb{Z}}$ be a chain complex. Then for any $k \in \mathbb{Z}$

$$H_k(C) := \ker(\partial_k)/\operatorname{im}(\partial_{k+1})$$

is called the k. homology group of C.

Remarks. (a) Observe that due to $\partial_k \circ \partial_{k+1} = 0$ we always have

$$\mathrm{im}(\partial_{k+1}) \subseteq \ker(\partial_k)$$

as a subgroup of abelian.

(b) Elements of

$$Z_k(C) := \ker(\partial_k)$$

.

are called (k-) *cycles of C* and elements of $\mathrm{im}(\partial_{k+1})$ are called *(k-) boundaries of C*.

(c) A sequence of abelian groups

$$\cdots \longrightarrow G_{k+1} \xrightarrow{\alpha_{k+1}} G_k \xrightarrow{\alpha_k} G_{k-1} \longrightarrow \cdots$$

is called *exact*, if

$$\ker(\alpha_k) = \mathrm{im}(\alpha_{k+1}), \quad \forall k \in \mathbb{Z}.$$

The homology of a chain complex (C_k, ∂_k) measures how far the sequence (*) is away from being exact.

(7.4) Definition. Let $C = (C_k, \partial_k)$ and $D = (D_k, \partial_k)$ be chain complexes. A family $f = (f_k : C_k \to D_k)_{k \in \mathbb{Z}}$ of homomorphisms is called a chain map if

$$\partial_k \circ f_k = f_{k-1} \circ \partial_k, \quad \forall k \in \mathbb{Z},$$

$$\begin{tikzcd}
\vdots \arrow[d] & \vdots & \vdots \arrow[d] \\
C_{k+1} \arrow[d] & & D_{k+1} \arrow[d] \\
C_k \arrow[rr, "f_k"] \arrow[d, "\partial_k"'] \arrow[drr, phantom, "\#"] & & D_k \arrow[d, "\partial_k"] \\
C_{k-1} \arrow[rr, "f_{k-1}"'] \arrow[d] & & D_{k-1} \arrow[d] \\
\vdots & \vdots & \vdots
\end{tikzcd}$$

.

<u>Remark</u>. (a) Chain complexes together with chain maps build a <u>category</u> $\underline{CC}$

(b) If $f: C \to D$ is a chain map it induces a homomorphism

$$f_* = H_k(f) : H_k(C) \longrightarrow H_k(D)$$

by

$$f_*([z]) := [f_k(z)],$$

which is well defined. In this way $H := (H_k)$ is a functor from $\underline{\mathcal{CC}}$ to the category of graded abelian groups $\underline{gAb}$.

(c) Observe that the concept up to here is purely algebraic. No topology is inside.

(7.6) **Definition.** Two chain maps $f, g : C \to C'$ between chain complexes are called chain homotopic, $f \simeq g$, if there exists a *chain homotopy* D *from* f *to* g, i.e., a family $(D_k : C_k \to C'_{k+1})_{k \in \mathbb{Z}}$ s.th.

$$\partial'_{k+1} \circ D_k + D_{k-1} \circ \partial_k = g_k - f_k, \quad \forall k \in \mathbb{Z},$$

$$
\begin{tikzcd}
\arrow[d] & & \arrow[d] \\
C_{k+1} \arrow[d] \arrow[rr, phantom] & & C'_{k+1} \arrow[d, "\partial'_{k+1}"] \\
C_k \arrow[d, "\partial_k"'] \arrow[rru, "D_k"] \arrow[rr, shift left, "f_k"] \arrow[rr, shift right, "g_k"'] & & C'_k \arrow[d] \\
C_{k-1} \arrow[d] \arrow[rru, "D_{k-1}"'] & & C'_{k-1} \arrow[d] \\
\vdots & & \vdots
\end{tikzcd}
$$

Objective. If $f \simeq g$, then

$$H_k(f) = H_k(g).$$

Pf.: For $z \in Z_k(C)$

$$g_k(z) - f_k(z) =$$

$$= \partial'_{k+1} \circ D_k z + D_{k-1} \circ \underbrace{\partial_k z}_{=0}$$

$$\in B_k(C')$$

$$\Longrightarrow g_*([z]) = f_*([z]).$$

$\blacksquare$

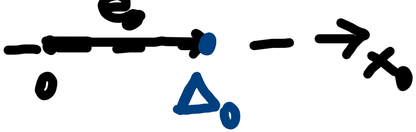
(7.7) **Definition.** For every $k \in \mathbb{N}_0$ we call

$$\Delta_k := \left\{ x \in \mathbb{R}^{k+1} : x = \sum_{j=0}^{k} \lambda_j e_j \,,\ 0 \leq \lambda_j \leq 1 \,, \sum_{j=0}^{k} \lambda_j = 1 \,,\ j = 0, \dots, k \right\}$$

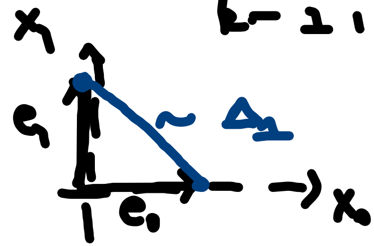
the *k-dimensional standard simplex*.

B

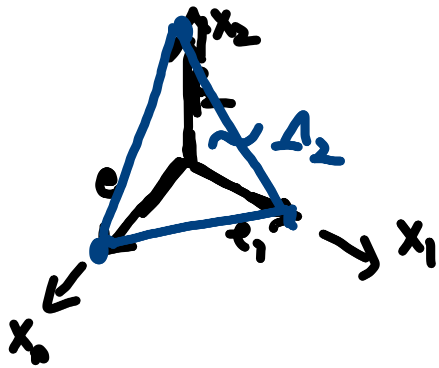
k = 0



k = 1,



k = 2,



Here $(e_0, \dots, e_k)$ is the canonical basis of the vector space $\mathbb{R}^{k+1}$.

(7.8) Remarks. (a) For every point $x \in \Delta_k$ one has a unique representation as $x = \sum_0^k \lambda_j e_j$ with $\sum \lambda_j = 1$.

(b) For every $0 \le i \le k$ we let

$$\Delta_{k-1}^{i} := \{ x \in \Delta_k : x = \sum_{\substack{j=0 \\ j \neq i}}^{k} \lambda_j e_j$$

the side of Δ_k opposite to the corner $e_i \in \Delta$.

Furthermore let

$$\delta_{k-1}^{i} : \Delta_{k-1} \longrightarrow \Delta_{k-1}^{i}$$

the restriction of the linear map $\mathbb{R}^k \to \mathbb{R}^{k+1}$ sending

$$e_j^{(k)} \longmapsto \begin{cases} e_j^{(k+1)} & \text{for } j < i \\ e_{j+1}^{(k+1)} & \text{for } j \geq i \end{cases}$$

to the standard simplex Δ_{k-1}. Then $f_{k-1}^i : \Delta_{k-1} \to \Delta_{k-1}^i$ is obviously bijective, even a homeomorphism:

(d) For every set A we can build the free abelian group generated by A given by

$$F(A) := \bigoplus_{a \in A} \mathbb{Z} \quad \left(= \{ s : A \to \mathbb{Z},\ s(a) = 0 \text{ almost everywhere} \} \right),$$

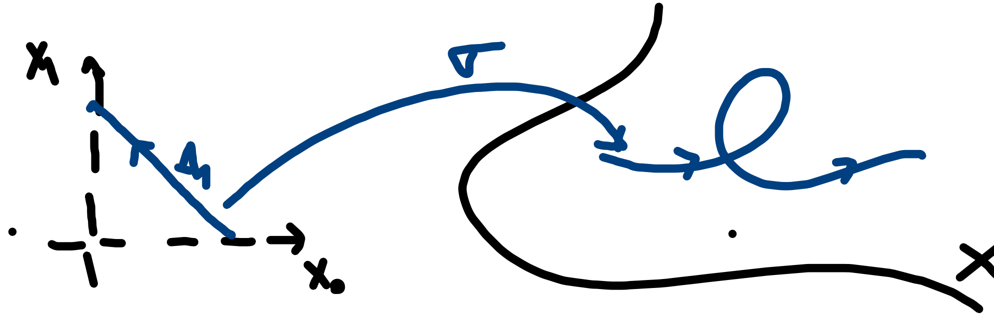
i.e., every element $g \in F(A)$ can be (uniquely) written by

$$g = \sum_{a \in A} n_a a \quad , \quad \text{with } n_a \in \mathbb{Z},$$

and all but a finite of the coefficients will be zero.

(7.9) Definition. Let X be a top. space and $k \in \mathbb{N}_0$.

(a) A singular k-simplex in X is a continuous map $\sigma : \Delta_k \longrightarrow X$



(b) Let $\Sigma_k(X)$ be the set of all k-simplices in X,

$$\Sigma_k(X) = \mathcal{C}(\Delta_k, X).$$

Then we call the free abelian group $S_k(X)$ generated by $\Sigma_k(X)$,

$$S_k(X) := F(\Sigma_k(X)),$$

the *k. singular chain group of X*. Its elements

$$c = \sum_{\sigma \in \Sigma_k(X)} n_\sigma \sigma \quad (n_\sigma \in \mathbb{Z})$$

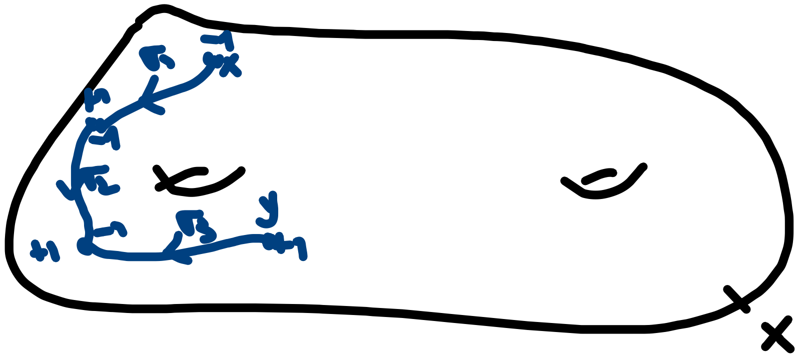
are called *singular k-chains in X*.

(c) For any $k \in \mathbb{N}_0$ one defines the k. boundary operator

$$\partial_k : S_k(X) \longrightarrow S_{k-1}(X)$$

on the basis $\{ \sigma : \sigma \in \Sigma_k(X) \}$ by

$$\partial_k \sigma := \sum_{i=0}^{k} (-1)^i (\sigma \circ \delta^i_{k-1})$$



$$c = \sigma_1 + \sigma_2 + \sigma_3 \to \partial c = y - x \in S_0(X)$$

(7.10) <u>Proposition</u> The family $S(X) = (S_k(X), \partial_k)$ is a chain complex.

Pf. Using (*) one computes for $\sigma \in \Sigma_k(X)$:

$$\partial_{k-1} \circ \partial_k \sigma = \cdots = 0 .$$

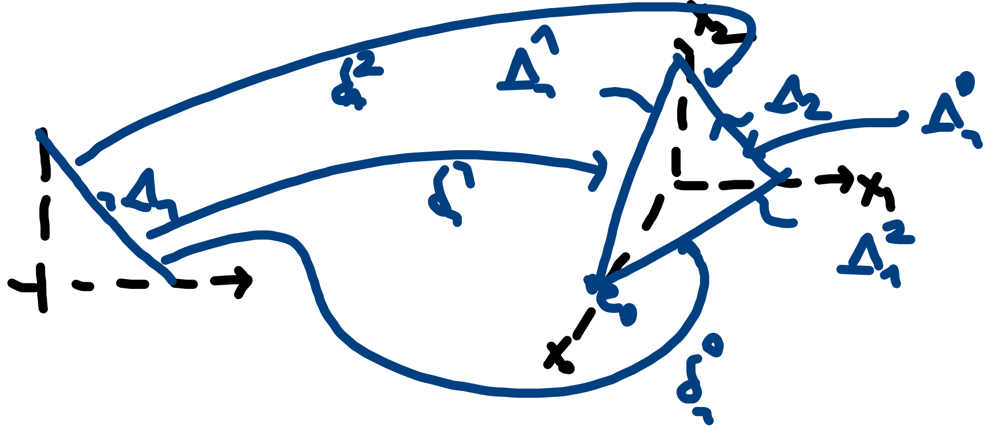
$\square$

(7.11) Remarks. (a) $S(X)$ is called the singular chain complex of X. The elements of $Z_k(X) := Z_k(S(X)) = \ker \partial_k$ are called singular k-cycles of X and those of $B_k(X) := B_k(S(X)) = \operatorname{im} \partial_{k+1}$ singular k-boundaries of X. Finally

$$H_k(X) := H_k(S(X))$$

is called the k. (singular) homology group of X.
$\mathcal{H}$ is an abelian group for every $k \in \mathbb{N}_0$.

(b) While $S_k(X)$, $Z_k(X)$ and $B_k(X)$ are in general huge, the quotient group $H_k(X)$ is often not so large (e.g., H is finitely generated for a compact manifold).



(c) Then it is easy to see that for all $k \in \mathbb{N}_{\geq 2}$ and $0 \leq i < j \leq k$ we have

$$\delta^j_{k-1} \circ \delta^i_{k-2} = \delta^i_{k-1} \circ \delta^{j-1}_{k-2} \qquad (*)$$