Vorlesung 05, 18.11.2022

[Remember: X top. space, $k \in \mathbb{N}_0$; $\Delta_k \subseteq \mathbb{R}^{k+1}$ k-dim.'l standard simplex, $\Sigma_k(X) = C(\Delta_k, X)$, $S_k(X) = F(\Sigma_k(X))$, $\partial_k : S_k(X) \to S_{k-1}(X) \rightsquigarrow S(X) = (S_k(X), \partial_k)$ is the so-called singular chain complex; $H_k(X) := H_k(S(X)) = \ker(\partial_k)/\operatorname{im}(\partial_{k+1})$ k. homology group of X.]

Remarks. (b) While $S_k(X)$, $Z_k(X)$ and $B_k(X)$ are in general huge groups, the quotient group $H_k(X)$ is mostly not so large (e.g. it is finitely generated for a compact differentiable manifold).

(c) It is a little bit surprising that one does not need any further properties than just the continuity of the simplices (e.g. no injectivity, so $\sigma \in \Sigma_k(X)$ could be constant, for example), and the theory gives indeed what one was motivated for, namely it detects the holes of X. E.g. it comes down to

$$H_k(S^n) = \begin{cases} \mathbb{Z} & \text{for } k=0,\ k=n \\ (0) & \text{otherwise} \end{cases} \qquad n \in \mathbb{N},\ k \in \mathbb{N}_0 .$$

§8. A little bit about category theory

(8.1) **Definition.** A *category* $\mathcal{C}$ consists of the following three data:

(a) A class of *objects* $X, Y, Z, \cdots$; $Ob(\mathcal{C})$;

(b) for any two objects X and Y in $\mathcal{C}$ a set of *morphisms* $Mor(X, Y) = \{f, g, h, \cdots\}$;

(c) for any three objects X, Y and Z in $\mathcal{C}$ a map

$$Mor(X, Y) \times Mor(Y, Z) \longrightarrow Mor(X, Z), \quad (f, g) \longmapsto gf, .$$

called the composition in the category. These data must fulfill the following axioms:

(i) For any choice of X, Y, Z and W and $f \in \mathrm{Mor}(X,Y)$, $g \in \mathrm{Mor}(Y,Z)$ and $h \in \mathrm{Mor}(Z,W)$:

$$h(gf) = (hg)f \qquad \text{(Associatity)}$$

(ii) for every object X there exists an element $id_X \in \mathrm{Mor}(X,X)$ satisfying: For all objects Y and all $g \in \mathrm{Mor}(X,Y)$, $h \in \mathrm{Mor}(Y,X)$ it holds:

$$g\, id_X = g, \quad id_X h = h.$$

(8.2) **Remarks.** (a) The so called identity $\mathrm{id}_X \in \mathrm{Mor}(X,X)$
is uniquely determined.

(b) Let $f \in \mathrm{Mor}(X,Y)$. If there exists a $g \in \mathrm{Mor}(Y,X)$ such that

$$gf = \mathrm{id}_X, \quad fg = \mathrm{id}_Y,$$

then g is uniquely determined (and is called *the inverse of f*). In this case f (and g) is called an *isomorphism in $\mathcal{C}$* (and X and Y are called *isomorphic in $\mathcal{C}$*, $X \cong Y$).

(8.3) **Examples:** (a) Set, the category of sets has as objects all sets as morphisms the maps between sets,

$\mathrm{Mor}_{\mathcal{C}}(X,Y) = \mathrm{Maps}(X,Y)$, and the composition in $\mathcal{C}$ is simply
the composition of maps. The isomorphisms in $\mathcal{C}$ are just the bijective maps. So $X \cong Y$ in $\mathcal{C}$, if and only if X and Y have the same cardinality.

(b) The category $\mathcal{C} = \underline{\mathrm{Top}}$ of topological spaces has as objects all topological spaces, as morphisms the continuous maps between spaces, $\mathrm{Mor}(X,Y) = \mathcal{C}(X,Y)$, and the composition is again the composition of maps. The isomorphisms in $\mathcal{C}$ are the homeomorphisms and X is isomorphic to Y iff X and Y are homeomorphic, $X \cong Y$.

(c) The homotopy category of topological spaces $\mathcal{C} = \underline{\mathrm{HTop}}$ has as objects again the topological spaces. But as morphisms the homotopy classes of continuous maps, $\mathrm{Mor}_{\mathcal{C}}(X,Y) = [X,Y] = \{[f] : f \in \mathcal{C}(X,Y)\}$. The composition is defined via representatives by

$$[g][f] := [g \circ f]$$

(which is well defined). The isomorphisms in $\mathcal{C}$ are the *homotopy equivalences* and X and Y are isomorphic in $\mathcal{C}$ iff X and Y are *homotopy equivalent*, $X \simeq Y$.

(d) A variant of $\underline{\mathrm{Top}}$ is given by the category of *pointed topological spaces* $\mathcal{C} = \underline{\mathrm{Top}}_0$, where the objects are pairs (X, x_0) and X is a top. space and $x_0 \in X$. A morphism between (X, x_0) and (Y, y_0) is a continuous map $f : X \to Y$

sending x_0 to y_0. The composition in $\mathcal{C}$ is obvious now and the isomorphisms of Top have no special name

(e) The category of groups $\mathcal{C} = \underline{\text{Grp}}$ has as objects the groups, as morphisms the homomorphisms between groups and an isomorphism in $\mathcal{C}$ is usually called a group isomorphism.

(f) Similarly we have the category of abelian groups $\underline{\text{Ab}}$ and for a given field K the category of vector spaces over K $\mathcal{C} = \underline{\text{Vect}_K}$ and so forth

(g) The category $\mathcal{C} = \underline{\text{CC}}$ of chain complexes has as its objects the chain complexes (C, ∂) and as its morphisms its chain maps $f:(C,\partial) \to (C', \partial')$. The isomorphisms in $\mathcal{C}$ are the chain isomorphisms

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(h) There is also a homotopy category of chain complexes $\mathcal{C} = \underline{HCC}$ where the objects are the chain complexes and morphisms are the homotopy classes of chain maps

(i) The category of differentiable $\mathcal{C} = \underline{Diff}$ with objects diff. mfds. and morphisms differentiable maps $f: M \to N$.

(j) Let M be a diff. mfd. The category of $\mathbb{R}$- or $\mathbb{C}$-vector bundles over M with objects vector bundles $\pi: E \to M$ and morphisms so called bundle homomorphisms $f: E \to F$ (with $\pi_F \circ f = \pi_E$).

(8.4) Definition. Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be categories. A (covariant) functor $T: \mathcal{C}_1 \to \mathcal{C}_2$ from $\mathcal{C}_1$ to $\mathcal{C}_2$ associates to every object X_1 in $\mathcal{C}_1$ an object $T(X_1)$ in $\mathcal{C}_2$ and it associates to every morphism $f \in \mathrm{Mor}_{\mathcal{C}_1}(X_1, Y_1)$ in $\mathcal{C}_1$ a morphism $T(f)$ (sometimes simply written as f_* if the functor T is clear) $\in \mathrm{Mor}_{\mathcal{C}_2}(T(X_1), T(Y_1))$. Furthermore the following two rules must be fulfilled:

(i) for every object X in $\mathcal{C}_1$:

$$T(\mathrm{id}_X) = \mathrm{id}_{T(X)}$$

(ii) for all objects X_1, X_2, X_3 in $\mathcal{C}_1$ and morphisms $f \in \mathrm{Mor}(X_1, X_2)$, $g \in \mathrm{Mor}(X_2, X_3)$ we have:

$$T(gf) = T(g)\ T(f)$$

(or shortly :

$$(gf)_* = g_* f_*).$$

<u>Remark</u>. A <u>contravariant functor</u> $S: \mathcal{C}_1 \to \mathcal{C}_2$ associates also to every object X_1 in $\mathcal{C}_1$ an object $S(X_1)$ in $\mathcal{C}_2$, but for every morphism $f \in \mathrm{Mor}_{\mathcal{C}_1}(X_1, Y_1)$ a morphism $T(f) = \mathrm{Mor}_{\mathcal{C}_2}(T(Y_1), T(X_1))$, shortly $f^* = S(f)$. The rules are then:

(i) $(id_X)^* = id_{S(X)}$; (ii) $(gf)^* = f^* g^*$.

(8.5) Examples. (a) The cylinder functor $Z : \underline{Top} \to \underline{Top}$ associates to a topological space X (an object in $\underline{Top}$) the cylinder

$$ZX = X \times I \qquad (I = [0,1])$$

and to every continuous map $f : X \to Y$ (a morphism in $\underline{Top}$) the continuous map

$$Zf : ZX \to ZY, \quad (x,t) \longmapsto (f(x), t).$$

(b) The cone functor $C : \underline{Top} \to \underline{Top}$ is similarly defined as $CX = ZX / X \times \{1\}$

and since for $f: X \to Y$ obviously $Zf(X \times \{1\}) \subseteq Y \times \{1\}$ the map

$$Cf: CX \longrightarrow CY, \quad [x, t] \longmapsto [f(x), t]$$

is well defined (and it is continuous due to the universal property of the quotient topology).

(c) Taking the dual of a K-vector space V, i.e.

$$V^* = \mathrm{Hom}_K(V, K)$$

is the first part of a contravariant functor

$\mathrm{Hom}(-, k) : \underline{\mathrm{Vect}}_k \longrightarrow \underline{\mathrm{Vect}}_k$ Namely, if $f : V \to W$ is a k-linear map (a morphism in $\underline{\mathrm{Vect}}_k$), we define

$$f^* = \mathrm{Hom}(f, k) : W^* \longrightarrow V^*,$$

$$f^*(\lambda) = \lambda \circ f .$$

(d) For every topological space X recall the singular chain complex $S(X) = (S_k(X), \partial_k)$ given by

$$S_k(X) = F(\Sigma_k(X)).$$

(In fact, already $\Sigma_k : \underline{\mathrm{Top}} \longrightarrow \underline{\mathrm{Set}}$ is a functor and $F : \underline{\mathrm{Set}} \to \underline{\mathrm{Ab}}$ is a functor, so S_k is a composition of functors which is itself a functor.)