# Lecture 06, 28.11.2022

[Recall: $X$ top. space, then $\Sigma_k(X) = \mathcal{C}(\Delta_k, X)$ is part of a functor $\Sigma_k : \underline{\mathrm{Top}} \longrightarrow \underline{\mathrm{Set}}$. In fact: If $f: X \to Y$ is a cont. map, then $f$ induces

$$\Sigma_k f : \Sigma_k X \longrightarrow \Sigma_k Y, \; \sigma \longmapsto f \circ \sigma .$$

Then, by the universal property of the free abelian group over a set the map $\Sigma_k f : \Sigma_k X \to \Sigma_k Y$

$$\begin{tikzcd} \Delta_k \arrow[r, "\sigma"] \arrow[dr, "\Sigma_k f(\sigma)"'] & X \arrow[d, "f"] \arrow[dl, phantom, "?"] \\ & Y \end{tikzcd}$$

induces a homomorphism (a morphism in $\underline{Ab}$)

$$S_k f := F(\Sigma_k f) : S_k(X) \to S_k(Y)$$

given by

$$S_k f\Big(\sum_{\sigma \in \Sigma_k X} n_\sigma \sigma\Big) = \sum_{\sigma \in \Sigma_k X} n_\sigma \Sigma_k f(\sigma),$$

so $S = F \circ \Sigma_k : \underline{Top} \to \underline{CC}$ is a functor, checking before that $Sf : S(X) \to S(Y)$ is in fact a chain map.

(e) Finally associating to every chain complex $C = ((C_k, \partial_k)$ and every $k \in \mathbb{Z}$ the abelian group $H_k(C)$ is part of a functor $H_k : \underline{CC} \to \underline{Ab}$, since for every chain map $f: C \to C'$ we have the induced homomorphism

$$f_*([z]) = [f(z)], \quad f_* = H_k(f) : H_k(C) \to H_k(C')$$

This makes $H_k$ to a functor. Usually the composition functor $H_k \circ S : \underline{Top} \to \underline{Ab}$ associating to a top. space $X$ its $k$. homology group is also denoted by $H_k$.

(8.6) Proposition. If $T: \mathcal{C}_1 \to \mathcal{C}_2$ is a functor and $f \in Mor(X, Y)$ is an isomorphism in $\mathcal{C}_1$, then $f_* = T(f) \in Mor(TX, TY)$ is an isomorphism as well.

Proof. So let $g \in \mathrm{Mor}(Y, X)$ be the inverse of $f$ in $\mathcal{C}_1$.

$$gf = \mathrm{id}_X, \quad fg = \mathrm{id}_Y.$$

Then $T(g) \in \mathrm{Mor}(TY, TX)$ is an inverse of $Tf \in \mathrm{Mor}(TX, TY)$, since

$$T(g)\,T(f) = T(gf) = T(\mathrm{id}_X) = \mathrm{id}_{T(X)}$$

$$T(f)\,T(g) = T(fg) = T(\mathrm{id}_Y) = \mathrm{id}_{T(Y)}.$$

(8.7) Remarks. (a) In some sense Algebraic Topology constructs functors from topological categories, e.g.

**Top** or **HTop**, to algebraic categories, e.g. **Ab** or $\mathbf{Vect}_K$ (for some field $K$). Functors are able to conduct that objects are **not** isomorphic, namely if one has that for two objects $X$ and $Y$ we find that $TX$ and $TY$ are not isomorphic, then also $X \not\cong Y$.

(b) If for two topological spaces, say $X = S^2$, $Y = T^2$, we find $H_1(X) \not\cong H_1(Y)$, as is the case here, since

$$H_1(S^2) = (0), \quad H_1(T^2) = \mathbb{Z}^2,$$

we see that $X$ and $Y$ cannot be homeomorphic.

$f \simeq g$

## (8.8) The homotopy theorem in homology theory.

A basic theorem in homology theory states: If $X$ and $Y$ are topological spaces and $f, g : X \to Y$ are homotopic, then the induced homomorphisms on the homology groups $f_*, g_* : H_k(X) \longrightarrow H_k(Y)$ agree, $f_* = g_*$.

In fact, one proves more precisely, that a homotopy $H : X \times I \to Y$ from $f$ to $g$ induces a chain homotopy $D : SX \to SY$ between the induced chain maps $Sf, Sg : SX \longrightarrow SY$. Therefore (cf. (7.6))

$$f_* = H(Sf) = H(Sg) = g_*.$$

**Corollary.** The homology functor $H : \underline{Top} \to \underline{gAb}$ descends to the homotopy category $\underline{HTop}$, i.e., there exists a functor $\tilde{H} : \underline{HTop} \to \underline{gAb}$, so that $\tilde{H} \circ \pi = H$, where $\pi : \underline{Top} \to \underline{HTop}$ is the canonical reduction sending $X$ to $X$ and $f \mapsto [f]$.

$$
\begin{tikzcd}
\underline{Top} \arrow[r, "H"] \arrow[d] & \underline{gAb} \\
\underline{HTop} \arrow[ur, dashed, "\tilde{H}"'] &
\end{tikzcd}
$$

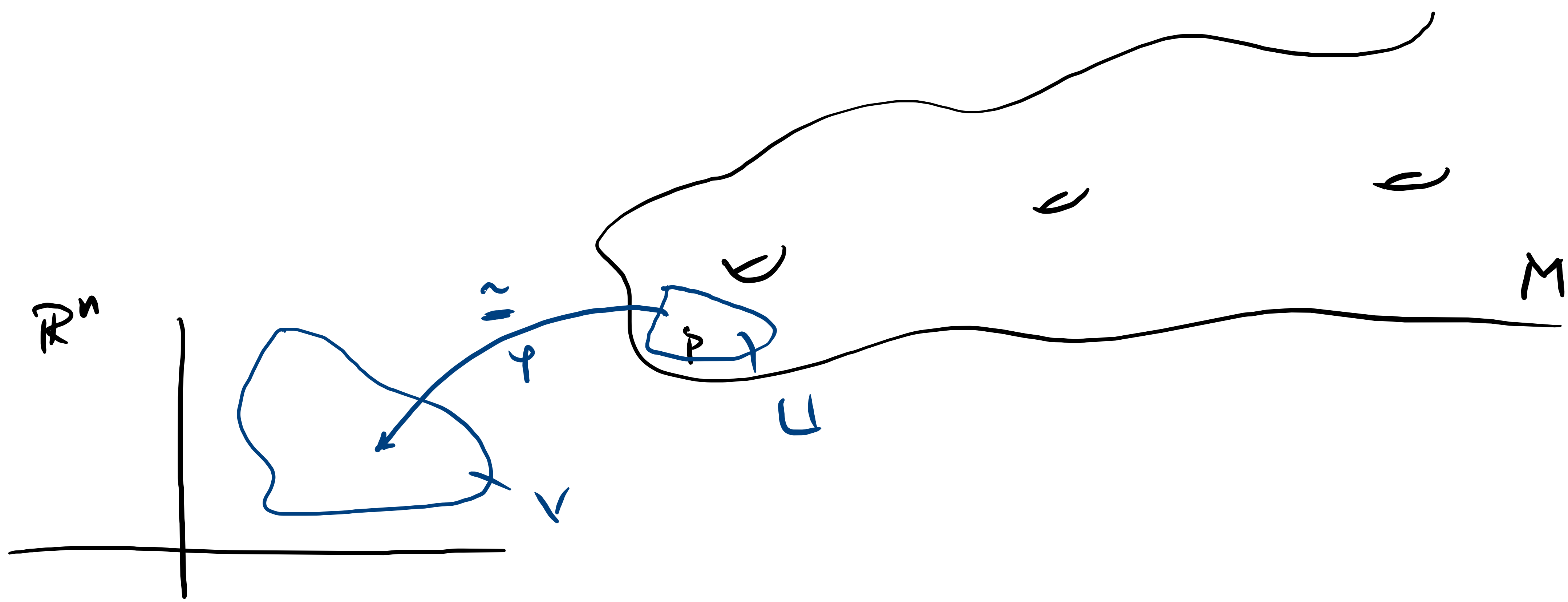
**Remark.** So the homology functor cannot distinguish non-homeomorphic spaces which are homotopy equivalent.

$$X \simeq Y \implies H(X) \cong H(Y).$$

VII

## §9. Differentiable manifolds

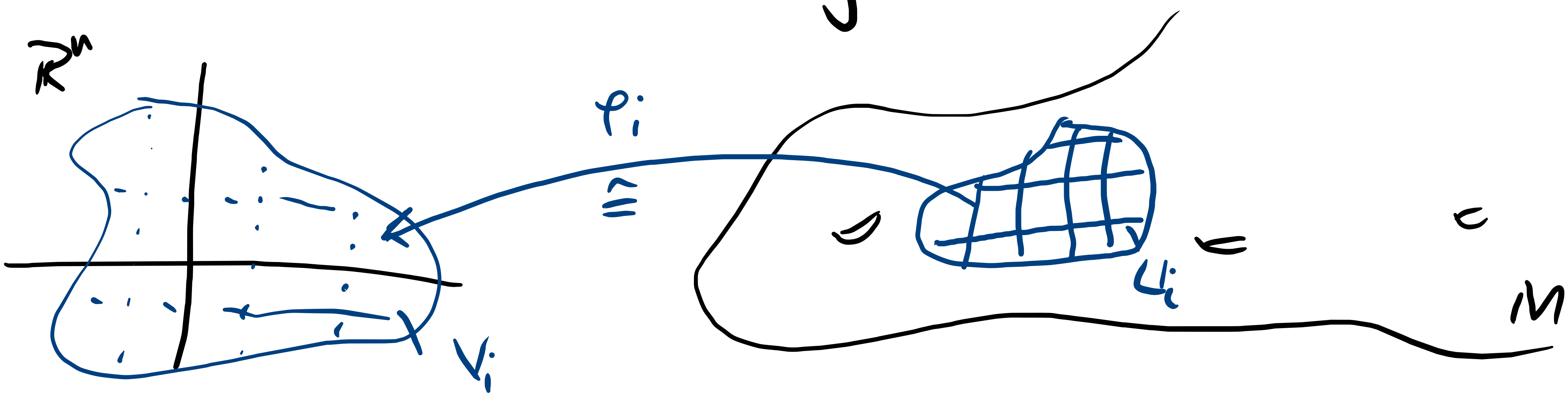
(9.1) Definition. Let $n \in \mathbb{N}$. A Hausdorff space with countable topology $M$ is called an n-dimensional topological manifold, if every point $p \in M$ has an open neighborhood $U \subseteq M$ being homeomorphic to an open set $V \subseteq \mathbb{R}^n$.




(9.2) Remarks. (a) So for a top. mfd. $M$ of dimension $n$, sometimes denoted a $M^n$, there exists an open covering $(U_i)_{i\in I}$ such that each $U_i \subseteq M$ $(i \in I)$ is homeomorphic to some open set $V_i \subseteq \mathbb{R}^n$,

$$M = \bigcup_{i\in I} U_i .$$

- A family of such homeomorphisms $(\varphi_i : \mathcal{U}_i \to V_i)_{i\in I}$ is called a (topological) atlas of $M$. The members $\varphi_i : \mathcal{U}_i \to V_i$ are called charts of the atlas and its inverses $\varphi_i^{-1} : V_i \to \mathcal{U}_i$ are called local coordinate systems on $M$.

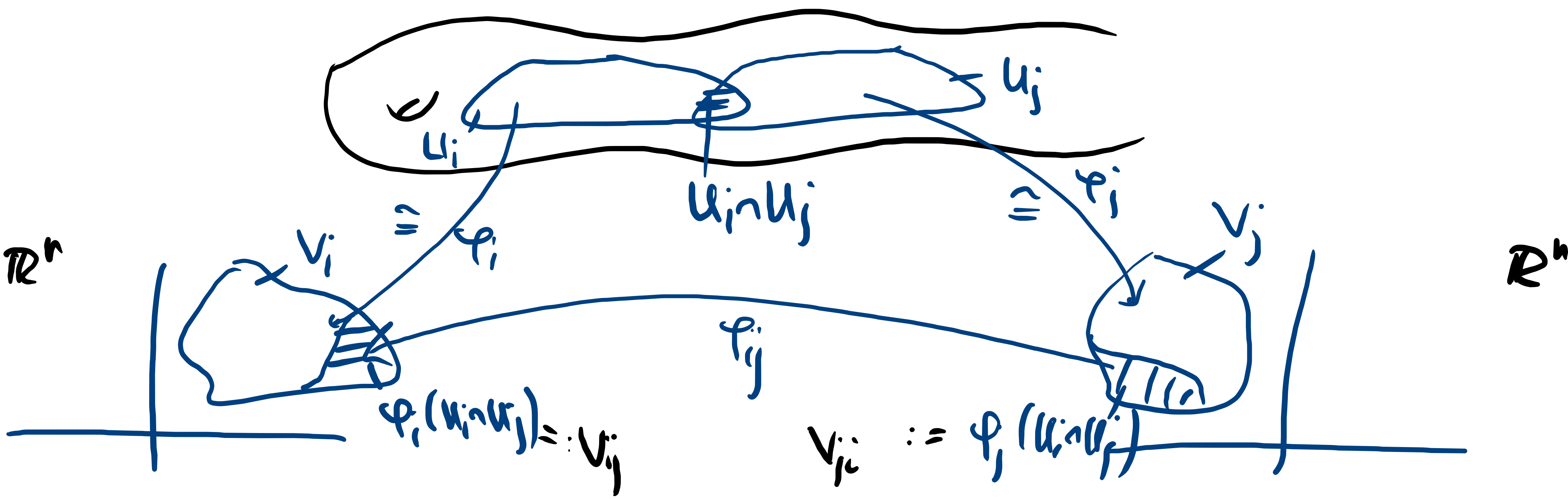



(6) If $M$ is a top. mfd. and $\mathfrak{A} = (\varphi_i : \mathcal{U}_i \to V_i)_{i\in I}$ is a topological atlas, we have for any pair $(i,j) \in I \times I$ the continuous map

$$\varphi_{ij} : \varphi_j(U_i \cap U_j) \longrightarrow \varphi_i(U_i \cap U_j)$$

$$\varphi_{ij}(x) = \varphi_i \circ \varphi_j^{-1}(x),$$

called a <u>transition function of $\mathcal{U}$</u>



**Observe** that for every $(i,j) \in I \times I$ the transition map $\varphi_{ij}$ is in fact a homeomorphism with $\varphi_{ij}^{-1} = \varphi_{ji}$.

(c) So locally a topological n-fold looks very well-known but globally it could be quite complicated. If $\mathfrak{A} = (\varphi_i : U_i \to V_i)$ is a topological atlas of M, you can imagine M as the topological sum $\sum_{i \in I} V_i$ (the „disjoint union") of the open sets $V_i \subseteq \mathbb{R}^n$ where you „patch together" $V_i$ and $V_j$ along the open subsets $V_{ij}$ and $V_{ji}$ via the transition function $\varphi_{ij}$:

$$\iota_i(x_i) \sim \iota_j(x_j) \;:\Longleftrightarrow\; x_i = \varphi_{ij}(x_j)$$

(where $\iota_i : V_i \longrightarrow \sum V_j$ are the canonical injection),

$$M \cong \sum V_i / \sim .$$

(9.3) Observation. (a) If $M^n$ is a top. mfd. and $p \in M$, then you can check, if a given function $f: M \to \mathbb{R}$ is continuous in $p$ via a chart $\varphi: U \to V \subseteq \mathbb{R}^n$ around $p$. Clearly

$$f \text{ cont. in } p \iff f|U \text{ cont. in } p \iff f|U \circ \varphi^{-1}: V \to \mathbb{R} \text{ is continuous in } x = \varphi(p).$$

(b) Now for $f|U \circ \varphi^{-1}: V \to \mathbb{R}$ it makes also sense to speak of differentiability of $f|U \circ \varphi^{-1}$ in $= \varphi(p)$. If you would check this property for all charts $\varphi_i : U_i \to V_i$ of a given atlas $\mathfrak{A} = (\varphi_i : U_i \to V_i)$, this would be independent of the chosen chart, if all the transition functions would not only be homeomorphic but moreover diffeomorphism (so differentiable as well as their inverses). This motivates:

(9.4) Definition. Let $M^n$ be a top. mfd.

(a) A topological atlas $\mathfrak{A} = (\varphi_i)$ of $M$ is called differentiable if all its transition functions $\varphi_{ij}$ are differentiable. (Since $\varphi_{ij}^{-1} = \varphi_{ji}$ they are then in fact diffeomorphisms.)

(b) Two differentiable atlasses $\mathfrak{A} = (\varphi_i)_{i \in I}$ and $\mathfrak{B} = (\psi_j)_{j \in J}$ are called equivalent, if also the atlas $\mathfrak{A} + \mathfrak{B} := (\varphi_i, \psi_j)_{i \in I\, j \in J}$

is differentiable.

(c) An equivalence class $c = [\mathfrak{A}]$ of differentiable atlasses on $M$ is called a <u>differentiable structure</u> on $M$. A pair $(M, c)$ of a top. $n$-fold and a diff. structure $c$ on it is called a <u>differentiable manifold of dimension $n$</u>.