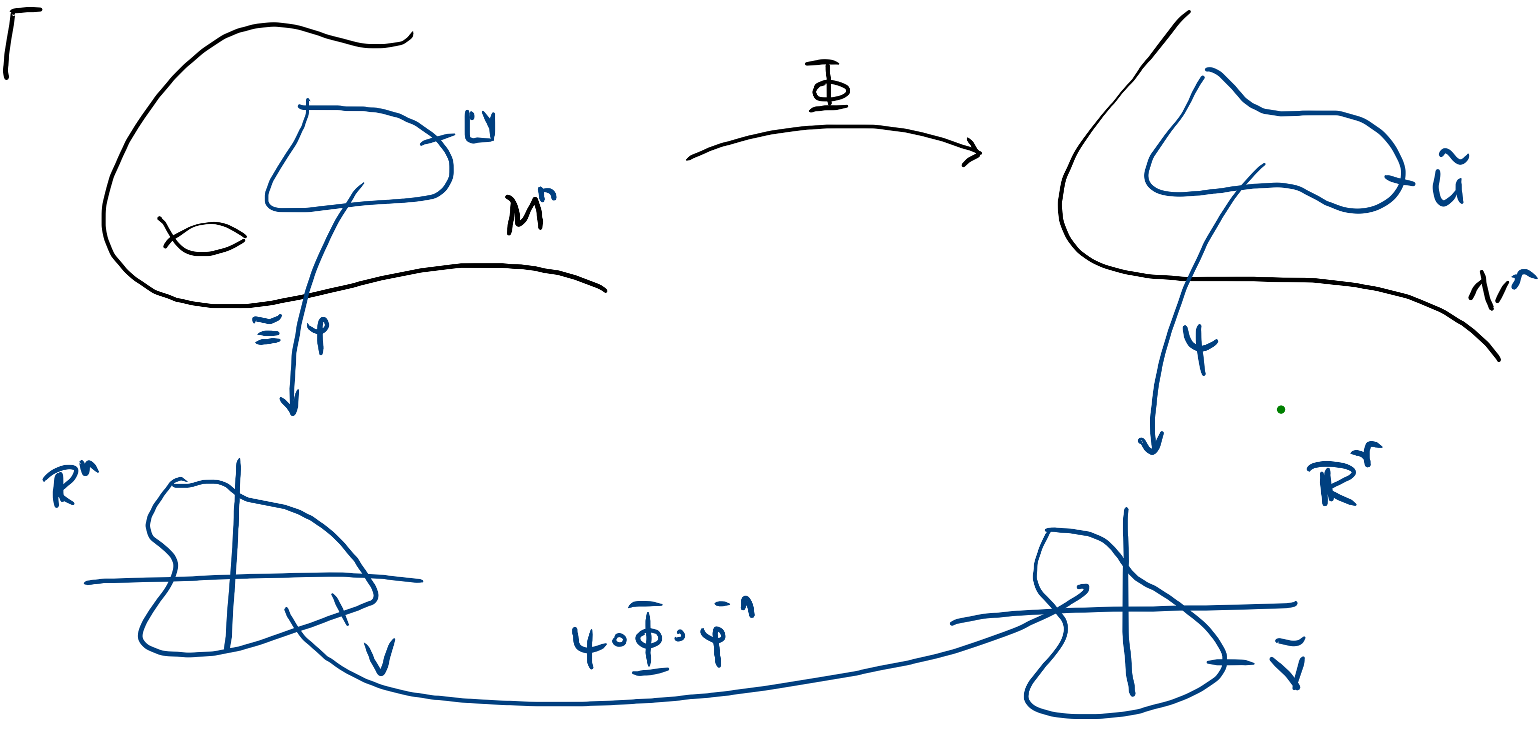
Lecture - 08, 16.12.2022



Φ differentiable $\Longleftrightarrow$ $\psi \circ \Phi \circ \varphi^{-1}$ is differentiable in the usual sense, for all charts φ of M and ψ of N

(9.10)

(c) If Φ is differentiable and bijective so that also $\Phi^{-1}: N \to M$ is differentiable, then Φ is called a diffeomorphism, $M \cong N$. M and N are called diffeomorphic, if there exists a diffeomorphism.

(9.11) Examples. Let $M = \mathbb{R}$ and $c_1 = [(\varphi)]$, $c_2 =$

$= [(\psi)]$ be the diff. structures on $\mathbb{R}$ given by $\varphi = \mathrm{id}_{\mathbb{R}}$ and $\psi = \mathrm{pot}_3$. We know already that $c_1 \neq c_2$. However, observe that the continuous $\Phi : (\mathbb{R}, c_1) \longrightarrow (\mathbb{R}, c_2)$, given by

$$p \longmapsto \sqrt[3]{p},$$

is a diffeomorphism, since in local coordinates we have that

$$\psi \circ \Phi \circ \varphi^{-1}(x) = \psi \circ \Phi(x) = \mathrm{pot}_3(\sqrt[3]{x}) = x = \mathrm{id}(x)$$

as well as $\varphi \circ \Phi^{-1} \circ \Psi^{-1} = \mathrm{id}$, which are differentiable.

(9.12) Remarks. (a) If $f : M \to \mathbb{R}$ is a diff. function in the sense of Definition (9.8), then f is obviously a diff. map in the sense of Definition (9.10), if we give $\mathbb{R}$ its standard structure $[\varphi = \mathrm{id}]$.

(b) By the way, it is possible that a top. mfd. M^n carries no diff. structure at all (only for $n \geq 4$) and, quite surprisingly, it is also possible that there exists pairwise non-diffeomorphic structure on certain top.

manifolds. (Not for $n \leq 3$.) A spectacular result of John Milnor was that on the S^7 there exist, besides the standard structure, exactly 27 other pairwise non-diffeomorphic (so called exotic) structures.

§ 10. The tangent Bundle

(10.1) Motivation. Remember from Calculus: A submanifold of $\mathbb{R}^{n+k}$ of dimension $n \in \mathbb{N}$ is a closed subset $M \subseteq \mathbb{R}^{n+k}$ so that for each $p \in M$ there exists an open nbhd. $U \subseteq \mathbb{R}^{n+k}$ and a diff. map $F : U \to \mathbb{R}^k$ so that:

(i) $F^{-1}(0) = U \cap M$

(ii) $\operatorname{rank}(DF_p : \mathbb{R}^{n+k} \to \mathbb{R}^k) = k$

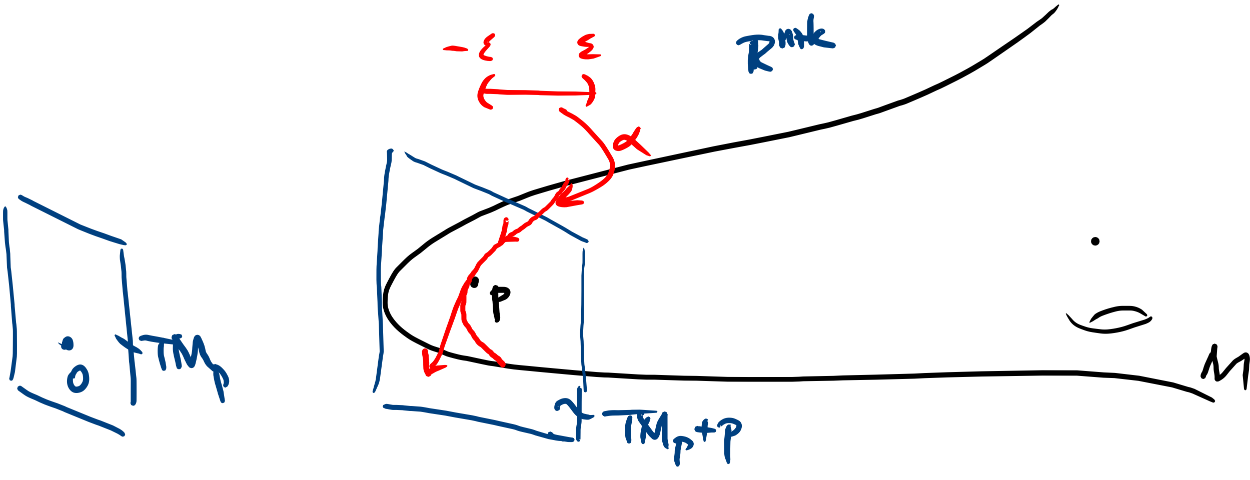
Then:

$$TM_p := \ker(DF_p) \subseteq \mathbb{R}^{n+k}$$

was called the tangent space of M in p and it was

$$TM_p = \{ \dot{\alpha}(0) \in \mathbb{R}^{n+k} : \exists \varepsilon > 0,\ \alpha : (-\varepsilon, \varepsilon) \longrightarrow M \text{ diff. with } \alpha(0) = p \}$$

the space of tangent vectors of curves through p lying on M.



(b) Problem. How can one define „TM_p" of an abstract manifold M^n (as a vector space of dimension n)?

Idea. Interprete any tangent vector $\xi = \dot{\alpha}(0) \in TM_p \subseteq \mathbb{R}^{n+k}$ as a directional derivative on differentiable functions $f: W \to \mathbb{R}$, defined on an open nhd. $W \subseteq M$ of $p \in M$.

$$\xi(f) := \frac{d}{dt}\Big|_{t=0} (f \circ \alpha)(t)$$

Observe: ξ is an $\mathbb{R}$-linear derivation, i.e.:

$$\xi(f\cdot g) = \xi(f)\cdot g(p) + f(p)\cdot \xi(g)$$

since with the Leibniz rule

$$\frac{d}{dt}(f\cdot g\circ\alpha)(t) = \frac{d}{dt}(f\circ\alpha\cdot g\circ\alpha)(t) = (f\circ\alpha)^{\cdot}(0)\cdot g\circ\alpha(0) + f\circ\alpha(0)\cdot(g\circ\alpha)^{\cdot}(0)$$

$$= \xi(f)\cdot g(p) + f(p)\cdot\xi(g).$$

(10.2) Definition. Let M^n be a top. mfd. of dim. n

A topological atlas $\mathfrak{A} = (\varphi_i : U_i \to V_i)$ is called a $\mathcal{C}^\infty$-atlas, if the transition functions φ_{ij} are $\mathcal{C}^\infty$-maps. Two $\mathcal{C}^\infty$ atlas $\mathfrak{A}$ and $\mathfrak{B}$ are called equivalent, if $\mathfrak{A} + \mathfrak{B}$ is a $\mathcal{C}^\infty$-atlas as well. An equivalence class $c = [\mathfrak{A}]$ of $\mathcal{C}^\infty$-atlas' is called a $\mathcal{C}^\infty$-structure on M. A $\mathcal{C}^\infty$-manifold, or smooth manifold, of dimension $n \in \mathbb{N}$ is a pair (M, c).

(10.3) Remark. (a) In an obvious way we define smooth functions $f : M \to \mathbb{R}$ and smooth maps $\Phi : M \to N$ between smooth mfds. M and N.

(b) For every open subset $\mathcal{U} \subseteq M$, M a smooth mfd., we denote

$$\mathcal{E}_M(\mathcal{U}) := \{ f : \mathcal{U} \to \mathbb{R} : f \text{ is smooth} \} \quad (\text{or } \mathcal{C}^\infty(\mathcal{U})).$$

Then $\mathcal{E}(\mathcal{U})$ has the structure of an $\mathbb{R}$-algebra via

$$(f+g)(p) := f(p) + g(p)$$
$$(\lambda \cdot f)(p) := \lambda \cdot f(p)$$
$$(f \cdot g)(p) := f(p) \cdot g(p)$$

for $f, g \in \mathcal{E}(U)$, $\lambda \in \mathbb{R}$. [$\mathcal{E}_M$ becomes a sheaf over M.]

(10.4) Definition. Let M be a smooth mfd. and $p \in M$. On the set theoretic sum

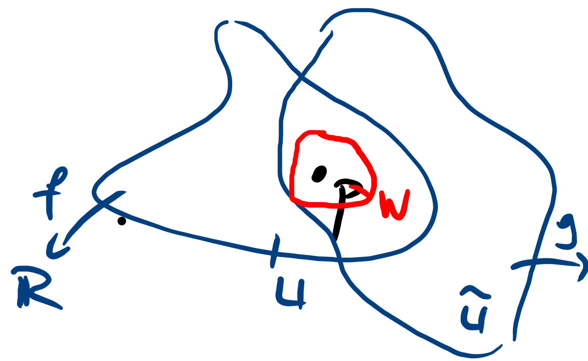
$$\sum_{U \in \mathfrak{U}(p) \text{ open}} \mathcal{E}(U)$$

(where $\mathfrak{U}(p)$ denotes the set of all nhd's of p).
we define the following equivalence relation:

$f \in \mathcal{E}(U)$ and $g \in \mathcal{E}(\tilde{U})$ ($U, \tilde{U} \in \mathcal{U}(p)$ open) are called <u>equivalent</u>, $f \sim g$, if there exists an open $W \in \mathcal{U}(p)$ with $W \subseteq U \cap \tilde{U}$ such that $f|W = g|W$.

The quotient set

$$\mathcal{E}_p(M) := \Big(\sum_{U} \mathcal{E}(U) \Big) \Big/ \sim$$



is called the space of <u>smooth function germs in p</u>.

(10.5) Remarks. (a) $\mathcal{E}_p(M)$ inherits from all $\mathcal{E}(U)$ ($U \in \mathfrak{V}(p)$ open), the structure of an $\mathbb{R}$-algebra since we can define addition, scalar multiplication and inner multiplication via representatives: If we write $s = f_p := [f] \in \mathcal{E}_p(M)$, $t = g_p \in \mathcal{E}_p(M)$, $\lambda \in \mathbb{R}$, then

$$s + t := f_p + g_p := (f+g)_p$$
$$\lambda \cdot s := \lambda \cdot f_p := (\lambda f)_p$$
$$s \cdot t := f_p \cdot g_p := (fg)_p$$

This is well defined and makes $\mathcal{E}_p(M)$ into a (local) algebra.

(6) The $\mathbb{R}$-algebra $A = \mathcal{E}_p(M)$ comes along with a so called evaluation homomorphism $\varrho : A \to \mathbb{R}$, namely

$$\varrho(f_p) := f(p) =: f_p(p),$$

which is obviously well defined. Observe, that you cannot evaluate a germ $s \in \mathcal{E}_p(M)$ in another point $q \in M$ different from p.

(10.6) Definition. Let M^n be a smooth mfd. and $p \in M$. Let $\mathcal{E}_p(M)$ the $\mathbb{R}$-algebra of smooth germs of M in p.

(a) A tangent vector of M in p is a derivation on $\mathcal{E}_p(M)$, i.e., a linear map $\xi : \mathcal{E}_p(M) \to \mathbb{R}$ satisfying

$$\xi(f_p \cdot g_p) = \xi(f_p) \cdot g_p(p) + f_p(p) \cdot \xi(g_p).$$

(b) The tangent space of M in p is the $\mathbb{R}$-vector space

$$TM_p := \mathrm{Der}_{\mathbb{R}}(\mathcal{E}_p(M), \mathbb{R}).$$

(It has in a natural way the structure of an $\mathbb{R}$-vector space as a subspace of $\mathrm{Hom}_{\mathbb{R}}(\mathcal{E}_p(M), \mathbb{R})$.)

(10.7) Example. Let $\varepsilon > 0$ and $\alpha : (-\varepsilon, \varepsilon) \to M$ a smooth curve on M through p. We define $\xi : \mathcal{E}_p(M) \to \mathbb{R}$ by

$$\xi(f_p) := \frac{d}{dt}\Big|_{t=0} (f \circ \alpha)(t)$$

and check immediately that this is well defined, it is obviously linear and indeed a derivation (cf. (10.1)). We will write

$$\dot{\alpha}(0) := \xi \in TM_p$$

for this derivation and call it the tangent vector of the curve α in p.

(10.8) Example. Let M^n be a smooth mfd., $p \in M$, and $\varphi : U \to V \subseteq \mathbb{R}^n$ a chart around p, and $x_0 = \varphi(p) \in V$. For every $j = 1, \dots, n$ we define

$$\frac{\partial}{\partial x^j}\Big|_p : \mathcal{E}_p(M) \longrightarrow \mathbb{R},\ f_p \longmapsto \frac{\partial}{\partial x^j}\Big|_{x_0}(f\circ\varphi^{-1})$$

(10.9) Remarks. (a) Then $\frac{\partial}{\partial x^j}\Big|_p$ is well defined, it is obviously linear (since $(f+g)\circ\varphi^{-1} = f\circ\varphi^{-1} + g\circ\varphi^{-1}$ and $(\lambda f)\circ\varphi^{-1} = \lambda\cdot(f\circ\varphi^{-1})$) and it is also a derivation, since with the Leibniz rule we have:

$$\frac{\partial}{\partial x^j}\Big|_p(f_p\cdot g_p) = \frac{\partial}{\partial x^j}\Big|_{x_0}(f\cdot g\circ\varphi^{-1}) = \frac{\partial}{\partial x^j}\Big|_{x_0}(f\circ\varphi^{-1}\cdot g\circ\varphi^{-1})$$

$$= \frac{\partial}{\partial x^j}\Big|_{x_0}(f\circ\varphi^{-1})\cdot(g\circ\varphi^{-1})(x_0) + (f\circ\varphi^{-1})(x_0)\cdot\frac{\partial}{\partial x^j}\Big|_{x_0}(g\circ\varphi^{-1})(x)$$

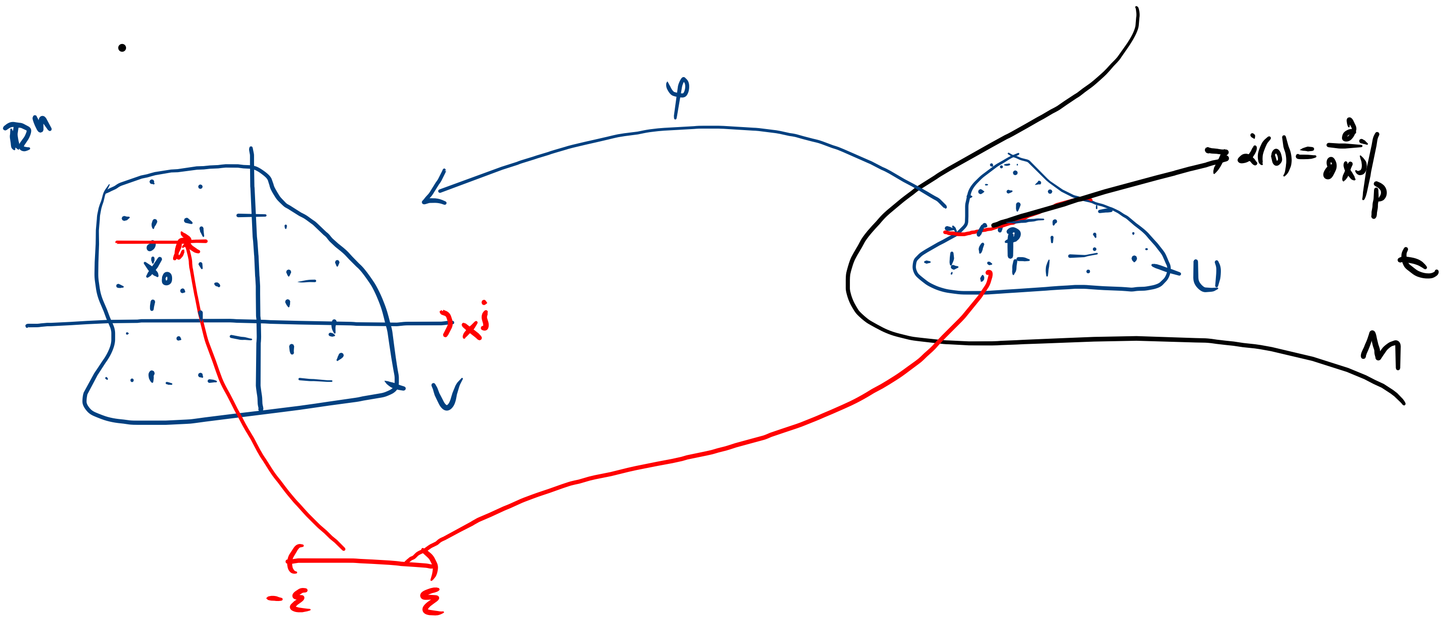
$$= \frac{\partial}{\partial x^j}\Big|_p (f_p) \cdot g_p(p) + f_p(p) \cdot \frac{\partial}{\partial x^j}\Big|_p (g_p).$$

We call $\frac{\partial}{\partial x^j}\Big|_p \in TM_p$ the j coordinate vector w.r.t. the chart φ.

(b) Observe: If $\alpha : (-\varepsilon, \varepsilon) \to M$ is the curve

$$\alpha(t) = \varphi^{-1}(x_0 + t \cdot e_j)$$

(for $\varepsilon > 0$ small enough), then: $\quad \dot{\alpha}(0) = \frac{\partial}{\partial x^j}\Big|_p.$



φ
$\mathbb{R}^n$
x_0
x^j
V
U
p
$\dot\alpha(0) = \frac{\partial}{\partial x^j}\Big|_p$
t
M
$-\varepsilon$
ε