Lecture-09, 23.12.2022

Recall: M^n smooth manifold, $p \in M$, the tangent space TM_p of M in p,

$$TM_p = \mathrm{Der}_{\mathbb{R}}(\mathcal{E}_p(M), \mathbb{R}) \ni \xi$$

$\xi : \mathcal{E}_p(M) \to \mathbb{R}$ linear satisfying

$$\xi(f_p \cdot g_p) = \xi(f_p) \cdot g_p(p) + f_p(p) \cdot \xi(g_p)$$

If $\varphi : \mathcal{U} \to V \subseteq \mathbb{R}^n$ is a chart. Then for $j = 1, \dots, n$:

$$\frac{\partial}{\partial x^j}\Big|_p : \mathcal{E}_p(M) \to \mathbb{R}$$

$$\frac{\partial}{\partial x^j}\Big|_p (f_p) := \frac{\partial}{\partial x^j}(f \circ \varphi^{-1})(x)\Big|_{x_0} \quad , \; x_0 = \varphi(p)$$

j-th coordinate vector w.r.t. the chart. $\lrcorner$

(10.10) Proposition. If M^n is smooth, $p \in M$ and φ is a chart around p, then $\left(\frac{\partial}{\partial x^1}\Big|_p, \dots, \frac{\partial}{\partial x^n}\Big|_p\right)$ is linearly independent.

Proof. Let $x^j : V \to \mathbb{R}$ the j-th coordinate function on V, i.e., $x \mapsto x^j$ ($j \in \{1, \dots, n\}$). Then we consider the smooth fctn's $\pi^j := x^j \circ \varphi \in \mathcal{E}(U)$. Then we calculate for each $i, j = 1, \dots, n$:

$$\frac{\partial}{\partial x^i}\Big|_p (\pi^j_p) = \frac{\partial}{\partial x^i}\Big|_{x_0} (\pi^j \circ \varphi^{-1})(x) = \frac{\partial x^j}{\partial x^i}(x_0) = \delta^j_i$$

(with $x_0 = \varphi(p) \in V$). Therefore, if $\lambda^1, \dots, \lambda^n \in \mathbb{R}$ with

$$\lambda^1 \frac{\partial}{\partial x^1}\Big|_p + \cdots + \lambda^n \frac{\partial}{\partial x^n}\Big|_p = 0,$$

then (in particular)

$$0 = 0(\pi_p^j) = \sum_{i=1}^{n} \lambda^i \underbrace{\frac{\partial}{\partial x^i}\Big|_p (\pi_p^j)}_{=\delta_i^j} = \sum_{i=1}^{n} \lambda^i \delta_i^j = \lambda^j$$

for all $j = 1, \ldots, n$, i.e., $(\partial/\partial x^1|_p, \ldots, \partial/\partial x^n|_p)$ is linearly independent.

$\boxed{\text{VI}}$

(10.11) Convention. Let us agree from here on the following Einstein sum convention, meaning, that if in a formula a summation index appears twice, one as a subscript and one as a superscript, then we sum over this (from 1 to $n = \dim M$).

(10.12) Problem. If $TM_p = \mathrm{Der}(\mathcal{E}_p(M), \mathbb{R})$ is really an adequate concept for the tangent space of a smooth mfd. M at some point $p \in M$, then $\left(\frac{\partial}{\partial x^1}\Big|_p, \dots, \frac{\partial}{\partial x^n}\Big|_p\right)$ should also be a generating system, i.e., $\dim_{\mathbb{R}} TM_p = n$.

On the first view this should be surprising, may be, but observe that the derivation property yields the following:

(i) If $\lambda \in \mathbb{R}$ and $\lambda_p \in \mathcal{E}_p(M)$ is the germ of the constant function $q \longmapsto \lambda$, $M \to \mathbb{R}$, then for any derivation $\xi : \mathcal{E}_p(M) \to \mathbb{R}$ we have

$$\xi(\lambda_p) = \xi(\lambda \cdot 1_p) = \lambda \cdot \xi(1_p)$$

by the $\mathbb{R}$-linearity of ξ and moreover

$$\xi(1_p) = \xi(1_p \cdot 1_p) = \xi(1_p)\cdot \underbrace{1_p(p)}_{=1} + 1_p(p)\cdot \xi(1_p) = 2\,\xi(1_p),$$

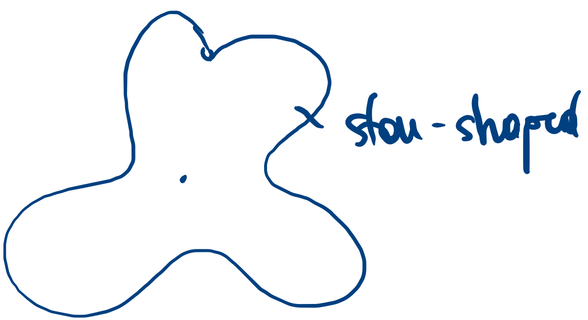
i.e., $\xi(1_p) = 0$, by the derivation property. So $\xi(\lambda_p) = 0$, $\forall \lambda \in \mathbb{R}$, $\forall \xi \in \mathrm{Der}(\mathcal{E}_p(M), \mathbb{R})$.

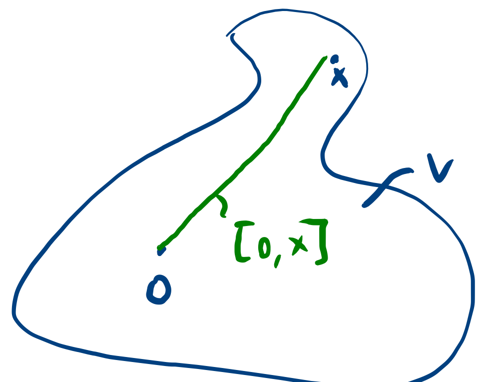
(ii) If $f_p, g_p \in \mathcal{E}_p(M)$ with $f_p(p) = 0 = g_p(p)$, then

$$\xi(f_p \cdot g_p) = \xi(f_p)\cdot \underbrace{g_p(p)}_{=0} + \underbrace{f_p(p)}_{=0}\cdot \xi(g_p) = 0$$

This a derivation measures in a sense what a germ s in p does in „the first order"

(10.13) Lemma. Let $V \subseteq \mathbb{R}^n$ be a star shaped region w.r.t. $0 \in V$ (i.e., for all $x \in V$ the line segment $[0,x] := \{ tx \in \mathbb{R}^n : t \in [0,1] \}$ is in V).





Let $g : V \to \mathbb{R}$ be smooth. Then there exists smooth functions $h_1, \dots, h_n : V \to \mathbb{R}$ so that

$$g(x) = g(0) + x^j h_j(x).$$

(10.14) Remark. Observe that by the product rule we get immediately that in this case

$$\frac{\partial g}{\partial x^j}(0) = h_j(0).$$

Proof. We define $h_j : V \to \mathbb{R}$ by

$$h_j(x) := \int_0^1 \frac{\partial g}{\partial x^j}(tx)\,dt.$$

Then, by the fundamental theorem of calculus we see that for every $x \in V$:

$$g(x) - g(0) \overset{F.T.}{=} \int_0^1 \frac{d}{dt}\big(g(tx)\big)\,dt \overset{C.R.}{=} \int_0^1 \sum_{j=1}^{n} \frac{\partial g}{\partial x^j}(tx)\cdot x^j\,dt$$

$$= \sum_{j=1}^{n} h_j(x)\cdot x^j.$$

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Break up to 9.30.

(10.15) Proposition. Let M^n be a smooth mfd. of dimension $n \in \mathbb{N}$ and $p \in M$. Then the tangent space TM_p of M in p is a real vector space of dimension n.

If $\varphi : U \to V \subseteq \mathbb{R}^n$ is a chart around p, then the coordinate vectors w.r.t. φ $\left(\frac{\partial}{\partial x^1}\big|_p, \dots, \frac{\partial}{\partial x^n}\big|_p\right)$ form a basis of TM_p.

Proof. We just have to prove that $\left(\frac{\partial}{\partial x^1}\big|_p, \dots, \frac{\partial}{\partial x^n}\big|_p\right)$

is generating.

Observe, that $\frac{\partial}{\partial x^j}\Big|_p$ do not change, if we shrink our chart or if we translate the chart. Therefore, by shrinking and translating $\varphi: U \to V$ we may assume that $\varphi(p) = 0$ and V is star shaped w.r.t. $0 \in V$. (e.g. $V = B_r(0)$ for some $r>0$). We consider again the coordinate fctn's $\pi^j \in \mathcal{E}(U)$ w.r.t. φ, i.e., $\pi^j = x^j \circ \varphi$ $(j = 1, \dots, n)$. Now let $\xi \in TM_p$ be arbitrary. We set $\lambda^j := \xi(\pi^j_p) \in \mathbb{R}$ and claim that

$$\xi = \sum_{j=1}^{n} \lambda^j \frac{\partial}{\partial x^j}\Big|_p .$$

So let $s \in \mathcal{E}_p(M)$ and $f \in \mathcal{E}(U)$ a representative of s, $s = f_p$. By shrinking U we may assume that the domain of our chart is the domain of f. We let $g := f \circ \varphi^{-1} \in \mathcal{E}(V)$. Then, by the lemma, we find $h_1, \dots, h_n \in \mathcal{E}(V)$ with

$$g(x) = g(0) + h_j(x) \cdot x^j .$$

Therefore

$$f = g \circ \varphi = g(0) + h_j \circ \varphi \cdot x^j \circ \varphi$$
$$= f(p) + H_j \cdot \pi^j$$

with $H_j := h_j \circ \varphi \in \mathcal{E}(U)$ (and $\pi^j := x^j \circ \varphi$). Since $\pi^j(p) = x^j(0) = 0$, $\forall j$, we conclude

$$\xi(f_p) = \underbrace{\xi(f(p)_p)}_{=0} + \xi((H_j)_p) \cdot \underbrace{\pi^j(p)}_{=0} + H_j(p) \cdot \overbrace{\xi(\pi^j_p)}^{=\lambda^j}$$
$$= \lambda^j \cdot H_j(p).$$

(10.16) Remark. Observe that every tangent space occurs as the tangent vector of a suitable curve through p. In fact, if $\xi \in TM_p$, and $\xi = \lambda^i \frac{\partial}{\partial x^i}\big|_p$ w.r.t. a chart $\varphi : U \to V \subseteq \mathbb{R}^n$, with $\varphi(p) = 0$, we consider the curve $\alpha : (-\varepsilon, \varepsilon) \to M$ given by

$$\alpha(t) = \varphi^{-1}(t \cdot \lambda)$$

(with $\lambda = (\lambda^1, \dots, \lambda^n) \in \mathbb{R}^n$), and $\varepsilon > 0$ small enough. Then we compute for all $f \in \mathcal{E}_p(M)$:

$$\dot{\alpha}(0)(f_p) = \frac{d}{dt}\Big|_0 (f\circ\alpha)(t) = \frac{d}{dt}\Big|_0 f\circ\varphi^{-1}(t\lambda)$$

$$= \frac{\partial}{\partial x^j}\Big|_0 (f\circ\varphi^{-1}) \cdot \frac{d}{dt}\Big|_{t=0} (t\lambda^j) = \lambda^j \frac{\partial}{\partial x^j}\Big|_p (f_p),$$

i.e.,

$$\dot{\alpha}(0) = \lambda^j \cdot \frac{\partial}{\partial x^j}\Big|_p = \xi.$$

Therefore indeed

$$TM_p = \left\{ \dot{\alpha}(0) \in \mathrm{Der}_{\mathbb{R}}(\mathcal{E}_p(M), \mathbb{R}) : \exists\, \varepsilon > 0 : \alpha : (-\varepsilon, \varepsilon) \to M \text{ smooth with } \alpha(0) = p \right\}.$$

On the other hand

$$H_j(p) = h_j(\varphi(p)) = h_j(0) = \frac{\partial g}{\partial x^j}(0) = \frac{\partial}{\partial x^j}\Big|_p (f_p)$$

And that for all $f_p \in \mathcal{E}_p(M)$, i.e.:

$$\xi = \lambda^j \frac{\partial}{\partial x^j}\Big|_p .$$

$\square$