Lecture 10 – 13.01.2023

[Recall: M^n smooth mfd., tangent space of M in $p \in M$ by

$$TM_p = \mathrm{Der}(\mathcal{E}_p(M), \mathbb{R}).$$

Already proved: TM_p is an n-dim.'l real vector space. If $\varphi : U \to V \subseteq \mathbb{R}^n$ is a chart around p, then

$$\frac{\partial}{\partial x^i}\Big|_p : \mathcal{E}_p(M) \longrightarrow \mathbb{R}, \quad f_p \longmapsto \frac{\partial}{\partial x^i}(f \circ \varphi^{-1})(x_0), \quad x_0 = \varphi(p).$$

In fact we proved: $\left(\frac{\partial}{\partial x^1}\Big|_p, \dots, \frac{\partial}{\partial x^n}\Big|_p\right)$ is a basis of TM_p.

$$TM_p = \{ \dot{\alpha}(0) \in \mathrm{Der}(\mathcal{E}_p(M), \mathbb{R}) : \exists \varepsilon > 0,\ \alpha : (-\varepsilon, \varepsilon) \to M \text{ smooth}, \alpha(0) = p \}. \quad \lrcorner$$

(10.17) Agreement (a) If $\varphi : U \to V \subseteq \mathbb{R}^n$ is a chart and if we denote by x the points in V, let us agree that we use x also for the chart (by abuse of notation).

(b) If $x : U \to V \subseteq \mathbb{R}^n$ and $y : \tilde{U} \to \tilde{V} \subseteq \mathbb{R}^n$ are two charts (with $U \cap \hat{U} \neq \emptyset$), we write for the transition function

$y \circ x^{-1} : x(U \cap \tilde{U}) \longrightarrow y(U \cap \hat{U})$ also only by y (or $y(x)$).

(c) If for example $f : M \to \mathbb{R}$ is a smooth fctn. and $x : U \to V$ is a chart on M, we'll write for the local description $f \circ x^{-1} : V \to \mathbb{R}$ often also just (or $f(x)$) and don't use a new symbol for it.

(10.18) Proposition. Let M^n be a smooth mfd., $p \in M$ and x, y charts around p. Then for all $j = 1, \dots, n$ (and $x_0 = x(p)$):

$$\frac{\partial}{\partial x^i}\Big|_p = \frac{\partial y^j}{\partial x^i}(x_0) \cdot \frac{\partial}{\partial y^j}\Big|_p .$$

Proof. For a last time we write $\varphi = x$, $\psi = y$ and $\tau = \psi \circ \varphi^{-1} = y \circ x^{-1}$ for the transition. If $s \in \mathcal{E}_p(M)$ is arbitrary let $f \in \mathcal{E}(U)$ with $s = f_p$ (and assume that x and y are defined on U). Then by the chain rule we get

$$\frac{\partial}{\partial x^i}\Big|_p (f_p) = \frac{\partial}{\partial x^i}\Big|_{x_0} (f \circ \varphi^{-1}) = \frac{\partial}{\partial x^i}\Big|_{x_0} (f \circ \psi^{-1} \circ \tau)$$

$$= \frac{\partial}{\partial y^j}\Big|_{y_0} (f \circ \psi^{-1}) \cdot \frac{\partial \tau^j}{\partial x^i}\Big|_{x_0} = \frac{\partial y^j}{\partial x^i}\Big|_{x_0} \cdot \frac{\partial}{\partial y^j}\Big|_p (f_p),$$

i.e.,

$$\frac{\partial}{\partial x^i}\Big|_p = \frac{\partial y^j}{\partial x^i}\Big|_{x_0} \cdot \frac{\partial}{\partial y^j}\Big|_p .$$

$\boxtimes$

(10.11) Remark. If M^n is smooth, $p \in M$ and $\xi \in TM_p$ has the coordinate description $\lambda = (\lambda^1, \dots, \lambda^n) \in \mathbb{R}^n$ w.r.t. chart x, and if it hat the coordinate description $\mu = (\mu^1, \dots, \mu^n)$ w.r.t. a chart y, then λ and μ transform in this way:

$$\lambda^i = \frac{\partial x^i}{\partial y^j}(y_0) \cdot \mu^j ,$$

since

$$\lambda^i \frac{\partial}{\partial x^i}\Big|_p = \xi = \mu^j \frac{\partial}{\partial y^j}\Big|_p = \mu^j \frac{\partial x^i}{\partial y^j}(y_0) \cdot \frac{\partial}{\partial x^i}\Big|_p$$

thus by coordinate comparison

$$\lambda^i = \frac{\partial x^i}{\partial y^j}(y_p) \cdot \mu^j.$$

(10.20) Definition. Let M^n be a smooth mfd. and $p \in M$.

(a) The dual space

$$TM_p^* = \mathrm{Hom}_{\mathbb{R}}(TM_p, \mathbb{R})$$

of the tangent space of M in p is called the cotangent space of M in p.

(b) If $x : U \to V \subseteq \mathbb{R}^n$ is a chart around p and

and $\left(\frac{\partial}{\partial x^1}\Big|_p, \dots, \frac{\partial}{\partial x^n}\Big|_p\right)$ is the associated basis of TM_p, we denote by $(dx^1_p, \dots, dx^n_p)$ the associated dual basis of TM_p^*, i.e.,

$$dx^j_p\left(\frac{\partial}{\partial x^i}\Big|_p\right) = \delta^j_i ,$$

for all $i, j = 1, \dots, n$.

(10.21) Remark. (a) In a similar way as in (10.15) one sees that for two charts x and y around p we have the transformation behaviour:

$$dy^j_p = \frac{\partial y^j}{\partial x^i}(x_0) \cdot dx^i_p$$

(b) As in (10.19) it follows that the transformation of the representation of a cotangent vector $\alpha \in TM_p^*$ as

$$\lambda_i \, dx^i_p = \alpha = \mu_j \, dy^j_p$$

looks this way:

$$\lambda_i = \frac{\partial y^j}{\partial x^i}(x_0) \cdot \mu_j .$$

(10.22) Definition Finally we define (first as sets) the tangent bundle resp. cotangent bundle of M as

$$TM := \sum_{p \in M} TM_p , \quad TM^* := \sum_{p \in M} TM_p^*$$

together with the projection maps

$$\pi : TM \to M , \quad \varrho : TM^* \to M ,$$

$$\pi(\xi) = p :\Leftrightarrow \xi \in TM_p , \quad \varrho(\alpha) = p :\Leftrightarrow \alpha \in TM_p^* .$$

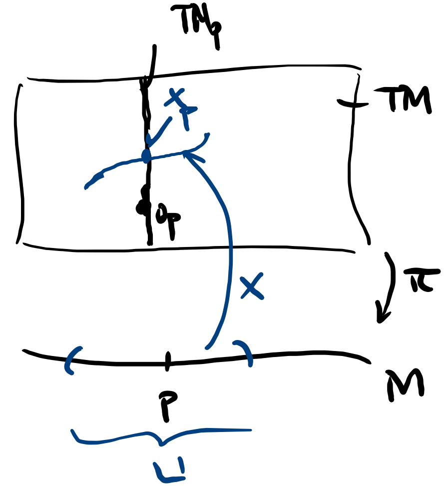
§11. Vector fields, differential forms, and Riemannian metrics

(11.1) Motivation. Up to here we made essentially a pointwise discussion on manifolds, e.g., considered single tangent vectors $\xi \in TM_p$ or cotangent vectors $\alpha \in TM_p^*$. Now we want to consider fields of these objects, i.e., we vary p (in a smooth way) to get a whole collection (in fact a map) of these objects.

(11.2) Definition. Let M be a smooth mfd. and $U \subseteq M$ be open.

(a) We call a map $X : U \to TM$ a *vector field on U*, if for all $p \in M$ we have $X(p) \in TM_p$, i.e., $\pi \circ X = \mathrm{id}_U$, if $\pi : TM \to M$ π the natural projection.

(b) A map $\omega : U \to TM^*$ is called a *differential form*, if $\omega(p) \in TM_p^*$, $\forall p \in M$, i.e., again, $\pi \circ \omega = \mathrm{id}_U$.



(11.3) Example. If $x : U \to V \subseteq \mathbb{R}^n$ is a chart on M, then we have the so-called coordinate vector fields w.r.t. x given by

$$\frac{\partial}{\partial x^j} : U \longrightarrow TM, \quad \frac{\partial}{\partial x^j}(p) := \frac{\partial}{\partial x^j}\Big|_p \qquad (j = 1, \dots, n)$$

and accordingly the coordinate differential forms

$$dx^j : U \longrightarrow TM^*, \quad dx^j(p) := dx^j_p \,.$$

(11.4) Remark. (a) If $x : U \to V$ is a chart on M, and

$X : U \to TM$, $\omega : U \to TM^*$ are a vector field resp differential form on U, then according to (10.15) there exist uniquely determined functions $\xi^1, \dots, \xi^n : V \to \mathbb{R}$ resp. $\eta_1, \dots, \eta_n : V \to \mathbb{R}$ so that

$$X = \xi^i \frac{\partial}{\partial x^i} \quad , \qquad \omega = \eta_j \, dx^j$$

(to be exact we had to write $X = \xi^i \circ \varphi \cdot \frac{\partial}{\partial x^i}$, $\omega = \eta_j \circ \varphi \, dx^j$, if $\varphi : U \to V$ is the chart).

(b) Observe next the <u>transformation behaviour</u> of vector fields resp. differential forms if you change the chart. So let $x : \mathcal{U} \to V_1$, $y : \mathcal{U} \longrightarrow V_2$ be two charts (on the same open set $\mathcal{U} \subseteq M$) and let

$$X = \xi^i \frac{\partial}{\partial x^i} = \eta^j \frac{\partial}{\partial y^j},$$

for ξ^i, η^j the resp. fctn's on V_1 resp. V_2. Then according to Prop. (10.18) we find that

$$\xi^i = \frac{\partial x^i}{\partial y^j} \eta^j \qquad (i = 1, \dots, n)$$

(since $\frac{\partial}{\partial y^j} = \frac{\partial x^i}{\partial y^j} \cdot \frac{\partial}{\partial x^i}$ and comparison of the coefficients yield the result).

(c) Similarly the repr. of a diff. form w.r.t. coordinates

$$\omega = \xi_i \, dx^i = \eta_j \, dy^j$$

has the transformation rule

$$\xi_i = \frac{\partial y^j}{\partial x^i} \cdot \eta_j \, .$$

(d) In particular, since the Jacobian of the transition $\left(\frac{\partial y^i}{\partial x^j}\right)$ and its inverse are smooth, the functions $\xi^1, \dots, \xi^n$ are smooth if and only if $\eta^1, \dots, \eta^n$ are smooth (and similarly for forms). Thus the smoothness of the coefficient functions do not depend on the chart.

(1.5) Definition. Let M be smooth and $U \subseteq M$ be open.

(a) A vector field $X : U \longrightarrow TM$ is called smooth, if for any chart $x : \tilde{U} \to \hat{V} \subseteq \mathbb{R}^n$ (with $U \cap \tilde{U} \neq \emptyset$) we have: If

$$X|_{U\cap\tilde{U}} = \xi^i \frac{\partial}{\partial x^i}$$

with $\xi^i : x(U\cap\tilde{U}) \to \mathbb{R}$ $(i = 1, \dots, n)$, then these functions have to be smooth.

(b) In the same manner we call a diff. form $\omega : U \to TM^*$ smooth, if all coordinate rep'ns $\omega = \eta_j dx^j$ on charts have smooth fctn's η_j.

(11.6) Remark. (a) We denote by

$$\mathfrak{X}(U) := \{ X : U \to TM \; : \; X \text{ is a smooth vector field} \}$$

and

$$\mathcal{E}^{(1)}(U) := \{ \omega : U \to TM^* : X \text{ is a smooth dif. form} \}.$$

Observe that $\mathfrak{X}(U)$ as well as $\mathcal{E}^{(1)}(U)$ have the structure of an $\mathcal{E}(U)$-module via pointwise construction:

$$(X_1 + X_2)(p) := X_1(p) + X_2(p)$$

$$(f \cdot X)(p) := f(p) \cdot X(p)$$

for $X_1, X_2 \in \mathfrak{X}(U)$, $f \in \mathcal{E}(U)$, and similarly for $\mathcal{E}^{(1)}(U)$.

(b) Observe that if you have a chart $x: U \to V \subseteq \mathbb{R}^n$, then as $\mathcal{E}(U)$-modules

$$\mathfrak{X}(U) \cong \mathcal{E}(U)^n, \quad \mathcal{E}^{(1)}(U) \cong \mathcal{E}(U)^n,$$

i.e., these modules are *free of rank n*. If U is not the domain of a chart (e.g. $U = M$), then in general $\mathfrak{X}(U)$ resp. $\mathcal{E}^{(1)}(U)$ will not be free

(11.7) Example. A famous theorem in Algebraic Topology says that every continuous vector field on S^2 must have a zero. („Hairy Ball theorem"). If $\mathfrak{X}(S^2)$ would be $\mathcal{E}(S^2)^2$, then there would be a $\mathcal{E}(S^2)$-basis (X_1, X_2) of $\mathfrak{X}(S^2)$, i.e., $(X_1(p), X_2(p))$ must be a basis of TS^2_p, for all $p \in S^2$. So it cannot be that $X_1(p) = 0$ for some $p \in S^2$.

(11.8) Definition. Let M be smooth, $U \subset M$ be open and $f \in \mathcal{E}(U)$. For every $p \in U$ we define the differential of f in p by $df_p : TM_p \to \mathbb{R}$, (i.e. $df_p \in TM_p^*$), by

$$df_p(\xi) := \xi(f_1)$$

and the differential $d : \mathcal{E}(U) \longrightarrow \mathcal{E}^{(1)}(U)$ by

$$df(p) = df_p .$$