Lecture 11, 20.01.2023

[Recall: M smooth, $U \subseteq M$ open, $f \in \mathcal{E}(U)$. For every $p \in M$ we defined the differential of f in p $df_p : TM_p \to \mathbb{R}$ (i.e., $df_p \in TM_p^*$),

$$df_p(\xi) := \xi(f_p)$$

and the differential $d : \mathcal{E}(U) \longrightarrow \mathcal{E}^{(1)}(U)$ by

$$df(p) := df_p \quad]$$

(11.9) Remark. (a) If $x : U \to V \subseteq \mathbb{R}^n$ is a chart, we denote by $f(x)$ also the coordinate rep.'n of $f : U \to \mathbb{R}$, then df is in fact a smooth form on U with

$$df = \frac{\partial f}{\partial x^i} \quad dx^i$$

($df_p \in TM_p^*$ is obvious), since, by definition,

$$df_p\left(\frac{\partial}{\partial x^i}\Big|_p\right) = \frac{\partial}{\partial x^i}\Big|_p (f_p) = \frac{\partial(f \circ x^{-1})}{\partial x^i}(x_0) = \frac{\partial f}{\partial x^i}(x_0),$$

i.e.,

$$df_p = \frac{\partial f}{\partial x^i}(x_0) \cdot dx^i_p$$

and therefore

$$df = \frac{\partial f}{\partial x^i} dx^i .$$

(b) For $\mathcal{U} \subseteq \mathbb{R}^n$ and $f : \mathcal{U} \to \mathbb{R}$ smooth one knows the gradient vector field

$$\operatorname{grad}(f) = \left(\frac{\partial f}{\partial x^1}, \ldots, \frac{\partial f}{\partial x^n} \right) : \mathcal{U} \to \mathbb{R}^n$$

which is considered as a vector field, of course. However, on a smooth mfd., If you change the chart, $x = \tau(y)$, the components of the gradient of $g := f \circ \tau$ transform like the coeficients of a differential form

$$\frac{\partial g}{\partial y^j} = \frac{\partial f}{\partial x^i} \circ \tau \cdot \frac{\partial x^i}{\partial y^j}$$

(by the chain rule), i.e., like the coefficients of a dif. form (and not of a vector field). So, on a smooth mfd. (without additional structure) one has no gradient of a function f but only the differential $df \in \mathcal{E}^{(1)}(U)$ of f.

(11.10) Definition. Let M and N be smooth mfds. and $\Phi : M \to N$ be smooth, $p \in M$. We define the differential of Φ in p as the linear map

$$D\Phi_p : TM_p \longrightarrow TN_{\Phi(p)}$$

by

$$D\Phi_p(\xi)(g_q) := \xi\big((g \circ \Phi)_p\big)$$

for $\xi \in TM_p$, $g_q \in \mathcal{E}_q(N)$, $q := \Phi(p)$ (and $g \in \mathcal{E}(V)$ a repr.)

(11.11) Remark. (a) It is easy to see that $D\Phi_p$ is indeed a linear map (and is well defined).

(b) If $x : \mathcal{U} \to V \subseteq \mathbb{R}^n$ is a chart around p and $y : \tilde{\mathcal{U}} \to \tilde{V} \subseteq \mathbb{R}^r$ is a chart around $q = \Phi(p)$, and if $\Phi : V \to \tilde{V}$ denotes also the coordinate rep. of Φ w.r.t. x and y, then

$$D\Phi_p\left(\frac{\partial}{\partial x^i}\Big|_p\right) = \frac{\partial \Phi^j}{\partial x^i}(x_0) \cdot \frac{\partial}{\partial y^j}\Big|_q ,$$

i.e., the Jacobian of $y \circ \Phi \circ x^{-1}$ represents $D\Phi_p$ w.r.t. to these coordinates.

(c) If $\Phi : M \to N$ is smooth, $p \in M$ and $\xi = \dot{\alpha}(0) \in TM_p$ for a smooth curve $\alpha : (-\varepsilon, \varepsilon) \to M$ through p, $\alpha(0) = p$, then

$$D\Phi_p(\xi) = (\Phi\circ\alpha)^{\bullet}(0)$$

Pf. Let $s = g_q \in \mathcal{E}_q(N)$ arbitrary, $g \in \mathcal{E}(V)$, V nhd. of $q \in N$. With $\beta := \Phi\circ\alpha$ we find that

$$D\Phi_p(\dot\alpha(0))(g_q) = \dot\alpha(0)\big((g\circ\Phi)_p\big) = \frac{d}{dt}\Big|_{t=0}(g\circ\Phi\circ\alpha)(t)$$

$$= \frac{d}{dt}\Big|_0 (g\circ\beta)(t) = \dot\beta(0)(g_q).$$

(11.12) Additional algebraic structure on $\mathfrak{X}(U)$

Remind: A Lie bracket on a real vector space V is a

- $\mathbb{R}$-bilinear map $[-,-] : V \times V \longrightarrow V$ satisfying

(i) $[v, w] = -[w, v]$, $\forall v, w \in V$;

(ii) $[v,[w,u]] + [w,[u,v]] + [u,[v,w]] = 0$, $\forall v, w, u \in V$.

(Jacobian identity).

$(V, [-,-])$ is a *Lie algebra*.

(1.13) *Definition.* Let X, Y be smooth vector fields on an open subset $U \subseteq M$ of a smooth mfd. M, $X, Y \in \mathfrak{X}(U)$. Then we define the Lie bracket $[X, Y] \in \mathfrak{X}(U)$ as

$$[X,Y]_p(f_p) := X_p\,(Yf)_p - Y_p((Xf)_p)$$

(11.14) Remark. (a) Here a vector field $X \in \mathfrak{X}(U)$ acts on a smooth fctn. $f \in \mathcal{E}(U)$ as

$$Xf(p) := X_p(f_p)$$

(b) The definition is well defined, i.e., is doesn't depend on the choice of the representative f of the germ $s \in \mathcal{E}_p(M)$, $s = f_p$.

(c) $[-,-] : \mathfrak{X}(U) \times \mathfrak{X}(U) \to \mathfrak{X}(U)$ is indeed a Lie bracket,

$(\mathfrak{X}(U), [-,-])$ is a Lie algebra.

(d) In local coordinates, if $X = \xi^i \frac{\partial}{\partial x^i}$ and $Y = \eta^j \frac{\partial}{\partial x^j}$, one finds that

$$[X,Y] = \left(\xi^i \frac{\partial \eta^k}{\partial x^i} - \eta^j \frac{\partial \xi^k}{\partial x^j} \right) \cdot \frac{\partial}{\partial x^k}.$$

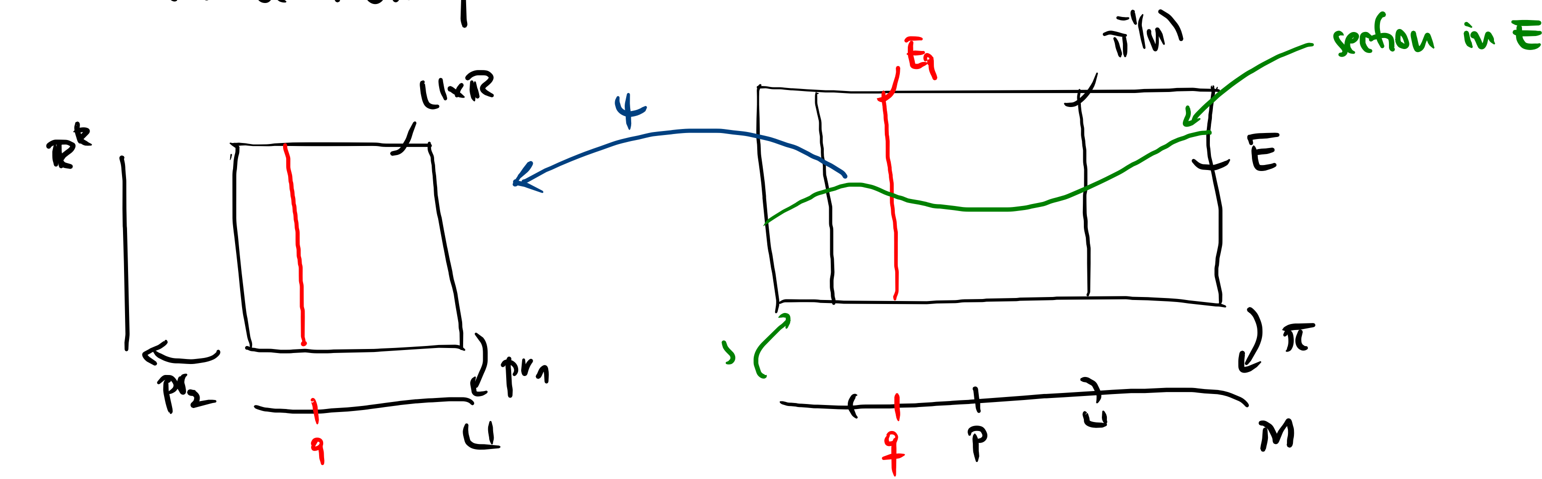
(1.15) Motivation. $TM \to M$ and $T^*M \to M$ are examples of so called vector bundles over M, i.e. maps $\pi : E \to M$, where each fibre $E_p := \pi^{-1}(p)$ has the structure of a real vector space of a fixed dimension and where the fibration is *locally trivial*, more precisely:

(11.16) *Definition.* Let E and M be smooth mfds. and $\pi : E \to M$ smooth. Let furthermore each fibre $E_p = \pi^{-1}(p) \subseteq E$ $(p \in M)$ be equipped with the structure of a real vector space of dimension $k \in \mathbb{N}_0$. Then we call $\pi : E \to M$ a *real vector bundle of rank k over M*, if for every $p \in M$ there exists an open nhd. $U \subseteq M$ of p and a bundle chart $\psi : \pi^{-1}(U) \longrightarrow U \times \mathbb{R}^k$ with $pr_1 \circ \psi = \pi$, i.e., a diffeomorphism so that for each $q \in U$ (denoting by ψ_q the restriction

$\psi|(E_q)$:

$$pr_2 \circ \psi_q : E_q \longrightarrow \mathbb{R}^k$$

is a linear isomorphism.



(11.17) **Remark**. (a) Observe that in general a vector bundle needs not to be *globally trivial*, i.e. $E \cong \underline{\mathbb{R}}^k := M \times \mathbb{R}^k$.
(b) We can equip TM and T^*M with the structure of a smooth mfd. such that $\pi : TM \to M$ resp. $\pi : T^*M \to M$ become real vector bundles of rank n over M.

(11.18) **Definition**. Let $\pi : E \to M$ be a vector bundle over M. A smooth map $s : M \to E$ with $\pi \circ s = id_M$ is called *a (smooth) section in E*. We denote

$$\Gamma(M;E) := \{ s : M \to E \text{ sections} \}$$

the space of smooth sections in E. It has the structure of an $\mathcal{E}(M)$-module.

(11.19) Remark. (a) Obviously

$$\mathfrak{X}(M) = \Gamma(M; TM), \quad \mathcal{E}^{(1)}(M) = \Gamma(M; T^*M)$$

(b) Operations in (multi-) linear algebra, which are independent of a chosen basis, e.g.

- the dual vector space V^* of a given vector space V

- $(V, W) \longrightarrow \mathrm{Hom}(V, W)$
- $(V, W) \longrightarrow V \otimes W$
- $(V, W) \longrightarrow V \oplus W$

can be done with vector bundles, a relative version of a vector space, as well.