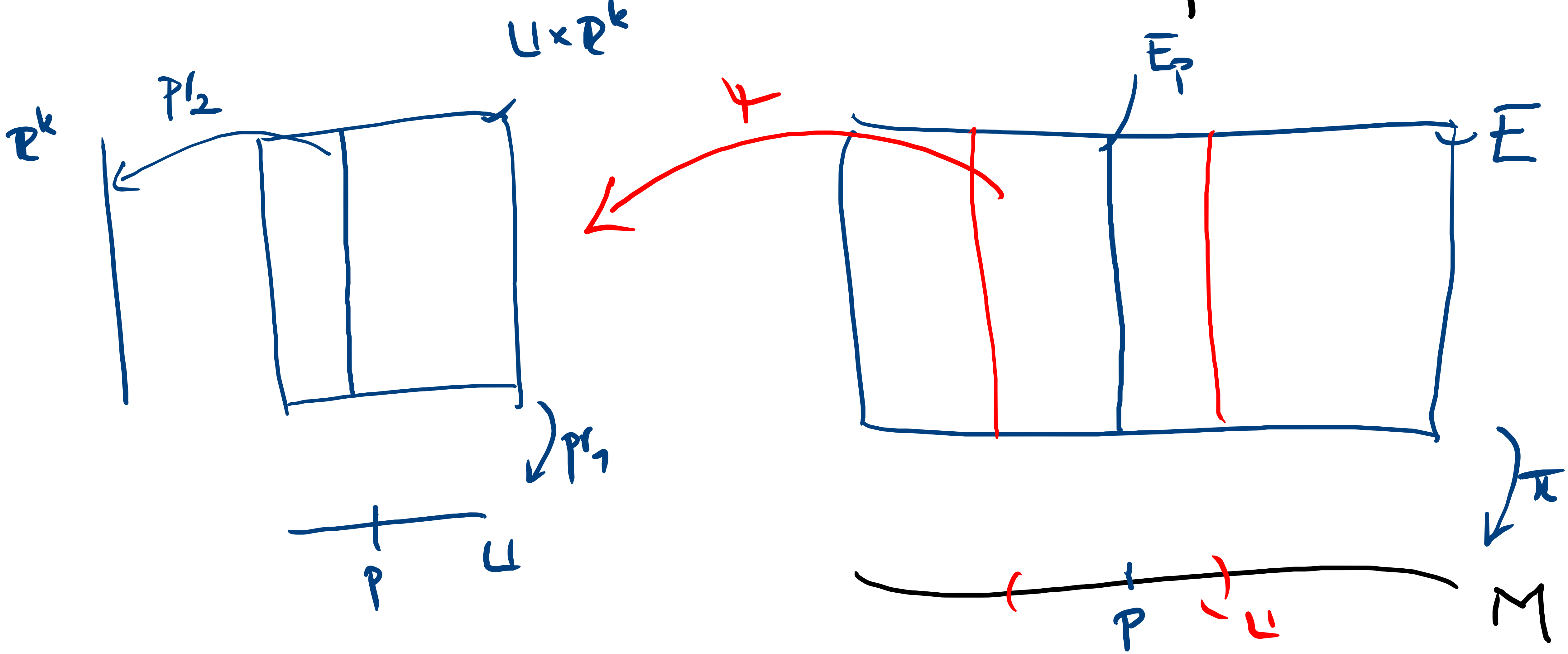
Lecture 12, 27.01.2023

⌈ Remember: A vector bundle over a mfd. M^n:



Operations from (multi-) linear algebra, which are functorial, can be done with vector bundles, e.g

$$V \longmapsto V^*$$

$$(V,W) \longmapsto V \oplus W$$

$$(V,W) \longmapsto V \otimes W$$

$\lrcorner$

(c) Example. Remember the universal property of the tensor product $V \times W \xrightarrow{\pi} V \otimes W$, $(v,w) \longmapsto v \otimes w$:

bilinear

$$\begin{tikzcd}
V \times W \arrow[r, "s"] \arrow[d, "\pi"'] \arrow[dr, phantom, "\circlearrowleft"] & L \\
V \otimes W \arrow[ur, dashed, "\exists \Phi"'] &
\end{tikzcd}$$

For example, consider for a given vector space V the bilinear map

$$V^* \times V^* \longrightarrow \mathrm{Bil}(V, \mathbb{R}), \ (\lambda, \mu) \longmapsto \big((v, w) \longmapsto \lambda(v)\mu(w) \big)$$

induces a linear map $\Phi : V^* \otimes V^* \longrightarrow \mathrm{Bil}(V, \mathbb{R})$.

$$\lambda \otimes \mu \longmapsto \big((v, w) \mapsto \lambda(v)\mu(w) \big),$$

which turn to be an isomorphism.

So, a sm section g in $T^*M \otimes T^*M$ is a family of bilinear products ($g_p : TM_p \times TM_p \to \mathbb{R}$) depending smoothly on p.

(11.20) Definition. A Riemannian metric on a smooth mfd. M is a global section in the bundle $T^*M \otimes T^*M \to M$, $g \in \Gamma(M; T^*M \otimes T^*M)$, which is pointwise symmetric and positive definite.

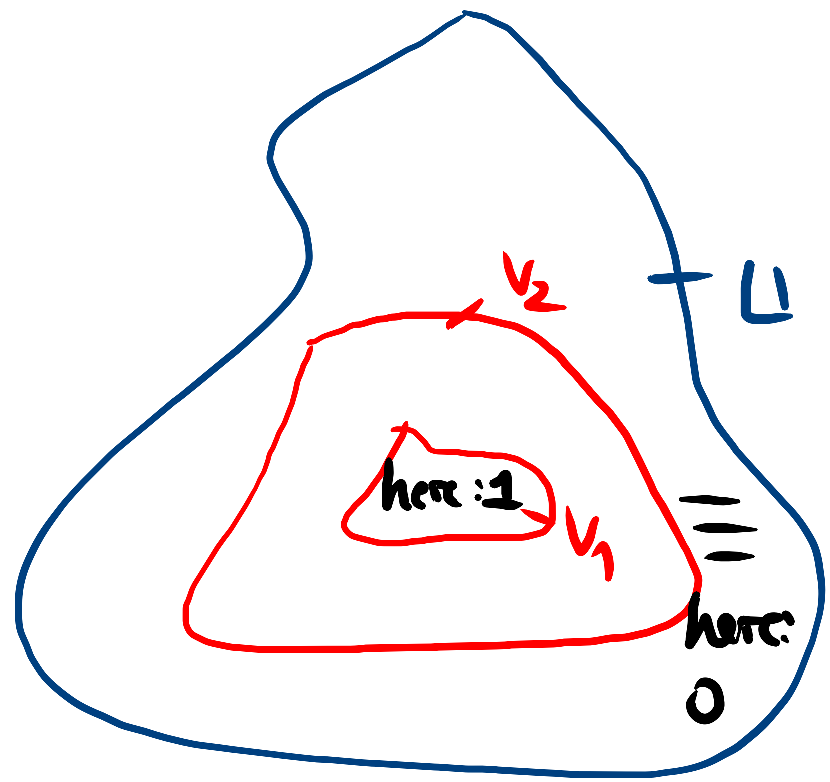
(11.21) Remark. (a) If $(v_1, \dots, v_n)$ is a basis of a vector space V and $(\lambda^1, \dots, \lambda^n)$ is the dual basis of V^*, then $(\lambda^i \otimes \lambda^j)$ is a basis of $V^* \otimes V^*$.

(b) Therefore, if g is a Riemannian metric on M and $x : U \to V \subseteq \mathbb{R}^n$ is a chart of M, then $g|U$ has a unique representation as

$$g|U = g_{ij}(x)\, dx^i \otimes dx^j$$

with $g_{ij} \in \mathcal{E}(V)$ and $(g_{ij}(x))$ being symmetric and positive definite, for all $x \in V$.

(11.22) Remark (a) If $U \subseteq M$ and $V_1 \subseteq V_2 \subseteq U$ are relatively compact (i.e., $\overline{V_2} \subseteq U$, $\overline{V_2}$ compact, $\overline{V_1} \subseteq V_2$, $\overline{V_1}$ compact), it is always possible to construct a smooth cut-off-function $\varphi : U \to \mathbb{R}$, i.e.,



- $\varphi(p) \in [0,1]$, $\forall p \in U$
- $\varphi|V_1 = 1$
- $\varphi|U \setminus V_2 = 0$

Therefore

$$\dim_{\mathbb{R}} \Gamma(M;E) = \infty$$

if $\mathrm{rk}(E) \geq 1$, since, by local triviality, you have many local sections (as vector valued smooth functions on open subsets of $\mathbb{R}^n$), then you can cut-off them by multiplying with φ and then continue it trivially with zero over the whole mfd.

(6) If $\mathfrak{U} = (U_i)_{i \in I}$ is an open cover of a mfd. M, then a family $(\chi_i)_{i\in I}$ of smooth functions on M is called a *partition of unity subordinate to $\mathfrak{U}$*, if:

- $\mathrm{supp}(\chi_i) = \overline{\{p \in M : \chi_i(p) \neq 0\}} \subseteq U_i$

- $\forall p \in M$ $\exists$ only finitely many $i \in I$ with $\chi_i(p) \neq 0$ and

$$\sum_{i \in I} \chi_i(p) = 1$$

Partitions of unity exist for all covers $\mathcal{U} = (U_i)$ of M.

(c) From this one can see that there exist always many Riemannian metrics on M. If you pick an atlas $\mathcal{U} = (U_\alpha \to V_\alpha)_\alpha$ of M, you can define first Riemannian metrics $g^{(\alpha)}$ on U_α by taking $g^\alpha(x) = (\delta_{ij})$.

for example. Then using a partition of unity subordinate to $\mathcal{U}$, say (χ_α), define

$$g := \sum_{\alpha \in I} \chi_\alpha \, g^{(\alpha)}$$

giving a Riemannian metric on M (since positive definiteness is a convex condition).

(11.23) Remark. (a) A non-degenerate bilinear form $g : V \times V \to \mathbb{R}$, e.g. a scalar product, on a real vector space V induces a distinguished isomorphism $\Phi : V \to V^*$, given by

$$v \longmapsto (w \longmapsto g(v,w))$$

Therefore, a Riemannian metric g on a mfd. M, makes it possible always to switch between TM and T^*M. If locally g is given by $(g_{ij}(x))$ and $(g^{ij}(x))$ denotes the inverse matrices, i.e.,

$$g^{ij} \cdot g_{jk} = \delta^i_k \quad (i,k = 1,\dots,n),$$

then the transformation, say, from a 1-form $\alpha = \eta_j dx^j$

to a vector field $X = \xi^i \partial/\partial x^i$ via g is done by using (g^{ij}) (or (g_{ij})) to pull the index from below above. So if

$$\alpha(Y) = g(X, Y),$$

then

$$\xi^i = g^{ij} \eta_j .$$

(b) In this way, on a Riemannian mfd. (M, g) we have for any smooth $f \in \mathcal{E}(M)$ a *gradient of f* given by

$$g(\mathrm{grad}(f), Y) = df(Y)$$

and locally by

$$\mathrm{grad}(f)^i = g^{ij}\cdot\frac{\partial f}{\partial x^j} \qquad (i=1,\dots,n).$$

§12. Morse functions and the Morse Lemma

(12.1) Definition. Let M be a smooth function on a smooth mfd. M^n, $f: M \to \mathbb{R}$. A point $p \in M$ is called a critical point of f, if $df_p = 0$.

(12.2) **Remark.** (a) Recall, if you choose a chart x around p, then p is critical, if and only if, for f in local coordinates x holds:

$$\frac{\partial f}{\partial x^j}(x_0) = 0 \qquad (j = 1, \dots, n)$$

$(x_0 = x(p))$.

(b) Also, if you in addition choose a Riemannian metric g on M, then p is critical for f, if and only if,

$$\mathrm{grad}_g(f)(p) = 0.$$

(c) Without further structure, e.g. a Riemannian metric, it is not possible to define a Hessian of f in $p \in M$, as a (symmetric) bilinear form on TM_p. However, if p is critical for f, this is possible.

(12.3) Definition. Let $f: M \to \mathbb{R}$ be a smooth function and $p \in M$ a critical point of f. Then we define the Hessian of f in p as follows:

$$\mathrm{Hess}(f)(p): TM_p \times TM_p \longrightarrow \mathbb{R},$$
$$\mathrm{Hess}(f)(p)(\xi, \eta) := \xi(Yf)_p.$$

Here $Y \in \mathfrak{X}(U)$ is a vector field in a neighborhood of p with $Y_p = \eta$.

(12.4) <u>Remark</u>. (a) Let us see, why this definition is independent of the local extension Y of η. Assume that X is a local continuation of ξ, $X_p = \xi$. (Such continuations, even to the whole of M, are always possible.) Then we can also consider $\eta((Xf)_p)$ by changing the rules of ξ and η. But

$$\xi((Yf)_p) - \eta((Xf)_p) = [X,Y]_p(f_p) = df_p([X,Y]_p) = 0.$$

since is critical, so $df_p(\zeta) = 0$, for all $\zeta \in TM_p$. But $\eta((Xf)_p)$ is independent of Y (and $\xi((Xf)_p)$ is independent of X), thus $\mathrm{Hess}(f)(p)$ is well-defined.

(b) Observe that by this argument we immediately see, that $\mathrm{Hess}(f)(p) : TM_p \times TM_p \to \mathbb{R}$ is indeed symmetric.

(c) Let us look at $\mathrm{Hess}(f)(p)$ in a local coordinate x around $p \in M$. In order to determine the matrix of $\mathrm{Hess}(f)(p)$ w.r.t. the basis $(\partial/\partial x^1|_p, \dots, \partial/\partial x^n|_p)$ we have to compute $\mathrm{Hess}(f)(p)(\partial/\partial x^i|_p, \partial/\partial x^j|_p)$. But for $\partial/\partial x^j|_p$, we can choose, of course, the continuation $\partial/\partial x^j$ on the chart domain and compute then that

$$h_{ij} = \mathrm{Hess}(f)(p)\left(\frac{\partial}{\partial x^i}\Big|_p, \frac{\partial}{\partial x^j}\Big|_p\right) = \frac{\partial}{\partial x^i}\Big|_p \left(\frac{\partial f}{\partial x^j}\right) = \frac{\partial^2 f}{\partial x^i \partial x^j}(x_0)$$

$(x_0 = x(p))$.

(12.5) Reminder. A symmetric bilinear form $s: V \times V \to \mathbb{R}$ on a (finite dimensional) real vector space V is called non degenerate, if for all $v \in V$ hold: If $s(v,w) = 0$, for all $w \in V$, then $v = 0$.

(12.6) Remark. (a) In case of a non-degenerate bilinear form $s: V \times V \to \mathbb{R}$ the induced linear map $V \to V^*$ is

The spaces V_- and V_+ are not uniquely determined but their dimensions are. (w.r.t. a basis $n_- := \dim(V_-)$ is the number of negative eigenvalues and $n_+ = \dim(V_+)$ is the number of positive eigenvalues of (s_{ij}).) The number

$$\lambda := n_- = \dim(V_-)$$

is called *the index of s*.

therefore an isomorphism.

(b) If $(v_1, \dots, v_n)$ is a basis of V, and $(s_{ij}) \in \mathrm{Sym}_n \mathbb{R}$ is the rep.'n of s w.r.t. this basis, then s is non-degenerate iff (s_{ij}) is invertible, $\det(s_{ij}) \neq 0$.

(c) If $s : V \times V \to \mathbb{R}$ is non-degenerate, then there exist maximal subspaces $V_-, V_+ \subseteq V$ so that

$$V = V_- \oplus V_+$$

and

$s|V_- \times V_-$ is negative definite

$s|V_+ \times V_+$ " positive "