Lecture - 13, 03.02.2023

Recall (i) V fin.-dim. real vector space, $s : V \times V \to \mathbb{R}$ bilinear, symmetric, $V_- \subseteq V$ max. subspace, so that $s|V_- \times V_-$ is negative definite.

$$\lambda := \dim V_-$$

is called <u>the index of s</u>.

(ii) $s : V \times V \to \mathbb{R}$ bilinear, sym. is non-degenerate, if

$$s(v,w)=0,\ \forall w\in V \implies v=0. \quad \lrcorner$$

(12.7) <u>Definition</u>. (a) A smooth function $f: M \to \mathbb{R}$ on a smooth mfd. M is called a <u>Morse function on M</u>, if for every critical point $p \in M$ of f the Hessian $\mathrm{Hess}(f)(p)$ is non-degenerate.

(b) Let $f: M \to \mathbb{R}$ be a Morse fctn. on M and $p \in M$ a critical point of f. Then we call the index of the symmetric bilinear form $\mathrm{Hess}(f)(p): TM_p \times TM_p \to \mathbb{R}$ <u>the index of p</u>:

(12.8) Remark. There exists an appropriate topology on the space $\mathcal{E}(M)$ of smooth functions of M. W.r.t. this topology the subspace of Morse functions is open and dense (i.e, the closure of it is the whole space). Thus „almost all" smooth functions on M are Morse in this sense. You can always disturb a non-Morse function slightly to make it Morse. In particular, Morse functions always.

(12.9) Motivation (a) A normal form of a non-degenerate bilinear form $s: V \times V \to \mathbb{R}$ is given by a basis $(v_1, \dots, v_n)$ s.th. the corresponding matrix is simply

$$(s_{ij}) = \mathrm{diag}(\underbrace{-1, \dots, -1}_{n_- = \lambda}, \underbrace{+1, \dots, +1}_{n_+})$$

(6) Therefore, if one starts with an arbitrary chart around a critical point $p \in M$ of a Morse function $f: M \to \mathbb{R}$ of index λ, one can always the chart by a linear transformation so that f in the coordinates x looks like the following (say $x_0 = 0 \in \mathbb{R}^n$):

$$f(x) = f(0) + \left(-(x^1)^2 - \dots - (x^\lambda)^2 + (x^{\lambda+1})^2 + \dots + (x^n)^2\right) + \text{h.o.t.}$$

It is a surprising fact, and indeed fundamental for the further analysis of Morse functions, that you can achieve, that the higher order terms „h.o.t." vanish, if you change the coordinates once more in a tricky way (using the implicit function theorem, of course). That's the content of the famous

(12.10) Morse Lemma. Let $p \in M^n$ be a critical point of a Morse function $f: M \to \mathbb{R}$ of index $\lambda \in \{0, \dots, n\}$. Then there exists a chart x around p so that (with $x_0 = x(p) = 0$)

$$f(x) = f(0) - \sum_{j=1}^{\lambda} (x^j)^2 + \sum_{j=\lambda+1}^{n} (x^j)^2 .$$

(12.11) Remark. The proof is based on a „parameter dependent" diagonalization construction for bilinear forms as follows:

(12.12) Sublemma. Let $A = \mathrm{diag}(a_1, \dots, a_n)$ with $a_j \in \{-1, +1\}$, for all $j = 1, \dots, n$. Then there exists an open nhd. $U \subseteq \mathrm{Sym}_n \mathbb{R}$ $(= \mathbb{R}^{\frac{1}{2} n \cdot (n+1)}$; all symmetric $n \times n$-matrices) and a smooth map $Q : U \to GL_n \mathbb{R} \subseteq \mathrm{Mat}_n \mathbb{R} = \mathbb{R}^{n^2}$, so that

- $Q(A) = \mathbb{1}$
- $Q(B)^T \cdot B \cdot Q(B) = A$.

Pf. Let $B \in \mathrm{Sym}_n \mathbb{R}$ so near to A s.th. $b_{11} \neq 0$ and has the same sign as a_1. Then consider first the linear coordinate change T in $\mathbb{R}^n$ given by $x = Ty$

$$x^1 = \left(y^1 - \frac{b_{12}}{b_{11}} y^2 - \cdots - \frac{b_{1n}}{b_{11}} y^n \right) \Big/ \sqrt{|b_{11}|}$$

$$x^k = y^k \qquad \text{for } k = 2, \ldots, n$$

Then one verifies that

$$T^t \cdot B \cdot T = \left(\begin{array}{c|ccc} a_1 & 0 & \cdots & 0 \\ \hline 0 & & & \\ \vdots & & B_1 & \\ 0 & & & \end{array}\right)$$

with some symmetric $(n-1)\times(n-1)$-matrix B_1. If B is close enough to A, then B_1 will be as close to $A_1 := \mathrm{diag}(a_2, \dots, a_n)$ as we want.

By induction we find therefore $Q_1(B_1) \in Gl_{n-1}\mathbb{R}$, depending smoothly on B_1 with $Q_1(A_1) = \mathbb{1}$ and $Q(B_1)^T \cdot B_1 \cdot Q(B_1) = A_1$. Finally define $Q(B) := T(B) \cdot S(B)$ with

$$S := \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & Q_1(B_1) & \\ 0 & & & \end{pmatrix}.$$

Then

$$Q(B)^T \cdot B \cdot Q(B) = A$$

and

$$Q(A) = \mathbb{1}.$$

$\blacksquare$

<u>Proof of the Morse Lemma.</u> We may assume that M is a convex open nhd. of $0 \in \mathbb{R}^n$, $p = 0$, $f(0) = 0$, and the matrix

$$A = \left(\frac{\partial^2 f}{\partial x^i \partial x^j}\right)(0)$$

is diagonal with entries ± 1. Of course, $Df(0) = 0$. Then, using Lemma (10.13) we find smooth functions $h_j \in \mathcal{E}(n)$ so that

$$f(x) = x^j \cdot h_j(x),$$

and applying the Lemma once more to each h_j we find

$b_{ij} \in \mathcal{E}(M)$ so that

$$f(x) = x^i \cdot x^j \cdot b_{ij}(x),$$

where we may assume that $b_{ij} = b_{ji}$ (by changing from (b_{ij}) to $\frac{1}{2}(b_{ij} + b_{ji})$, if necessary). Of course: $B(0) = A$, when we define $B_x = (b_{ij}(x))$. By the sublemma we find a matrix $Q = \underbrace{Q_x}_{:= Q_{B(x)}} \in GL_n \mathbb{R}$ so that $Q(0) = 1$ and

$$Q_x^T \cdot B_x \cdot Q_x = A.$$

for all x near 0. Now we define $\varphi : \mathcal{U} \to \mathbb{R}^n$, for a sufficiently small nbhd. $\mathcal{U} \subseteq \mathbb{R}^n$ of 0 by

$$\varphi(x) := Q(x)^{-1} \cdot x$$

Then $\varphi(0) = 0$, and by the product rule

$$D\varphi(0) = Q(0)^{-1} = \mathbb{1}.$$

Therefore, by restricting $\mathcal{U}$ if necessary, by the inverse function theorem $(\mathcal{U}, \varphi)$ is a chart around p. But for

$y = \varphi(x) = Q(x)^{-1} \cdot x$, so $Q_x y = x$, we finally conclude

$$f(y) = f(x(y)) = x^T \cdot B_x \cdot x = y^T \cdot \underbrace{Q_x^T \cdot B_x \cdot Q_x}_{= A} y = y^T \cdot A \cdot y$$

$$= \sum_{j=1}^{n} a_{jj} (y^j)^2 .$$

$\boxed{\text{VII}}$

(12.13) Corollary. Let $f: M \to \mathbb{R}$ be a Morse fctn. on a smooth mfd. Then the critical set

$$\mathrm{Crit}(f) := \{ p \in M : df_p = 0 \}$$

is closed and discrete.

Pf. Since $df: M \to T^*M$ is smooth, in particular continuous, the preimage of the image $Z \subseteq T^*M$ of the zero-section $\sigma: M \to T^*M$, $\sigma(p) = 0_p$, which is closed, $Z = \mathrm{im}(\sigma)$, i.e.,

$$\mathrm{Crit}(f) = df^{-1}(Z)$$

must be closed as well.

If $p \in M$, choose a Morse chart $x: \mathcal{U} \to V \subseteq \mathbb{R}^n$ according to the Morse Lemma around p and let

$V = B_r(0) \subseteq \mathbb{R}^n$ for $r > 0$ small enough. Since

$$\operatorname{grad}\left(-(x^1)^2 - \dots - (x^\lambda)^2 + (x^{\lambda+1})^2 + \dots + (x^n)^2\right)$$
$$= (-2x^1, \dots, -2x^\lambda, +2x^{\lambda+1}, \dots, 2x^n),$$

we see that p is the only crit. point of f in U. This shows that $\mathrm{Crit}(f) \subseteq M$ is discrete.

$\square$

§ 13. The main theorems of Morse Theory

(13.1) Motivation. (a) Let M^n be a compact smooth mfd. We want to understand the homotopy type of M using a Morse function $f: M \to \mathbb{R}$ and investigating the sublevel sets

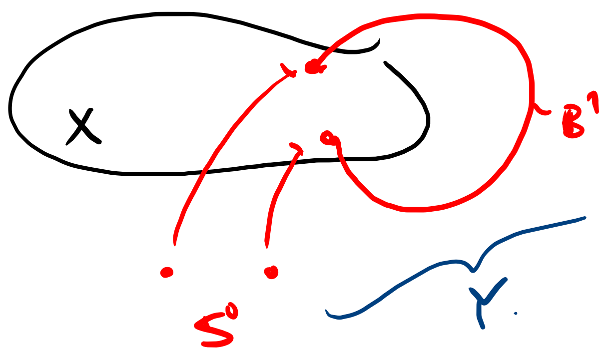
$$M^c := f^{-1}((-\infty, c]).$$

(If $c > \max(f)$, then $M^c = M$, of course.)

(b) Remember that we say that a topological space Y results from a topological space X by attaching a k-cell e^k, if we have an attaching map $f: S^{k-1} \longrightarrow X$ and

$$Y \cong X \cup_f e^k := (X \sqcup B^k)/\sim_f$$

where the equivalence relation $\sim_f$ is generated by identifying $x \in S^{k-1} \subseteq B^k$ with $f(x) \in X$.



(c) Since the induced topology of a discrete subset D of a topological space X is discrete (i.e., all subsets of D are open) a discrete and closed subset of a compact space must be finite. Thus $\mathrm{Crit}(f) \subseteq M$ is finite for a Morse function.

(d) The first observation is now that the homotopy type from M^a to M^b for $a<b$ does not change at all if $f^{-1}([a,b])$ contains no critical point.

(13.2) Proposition. Let M be a cpct. mfd. and $f: M \to \mathbb{R}$ smooth. Assume that for $a<b$ the compact subset $f^{-1}([a,b])$ contains no critical points. Then M^a is a (even strong) deformation retract of M^b.