Lecture - 14, 10.02.2023

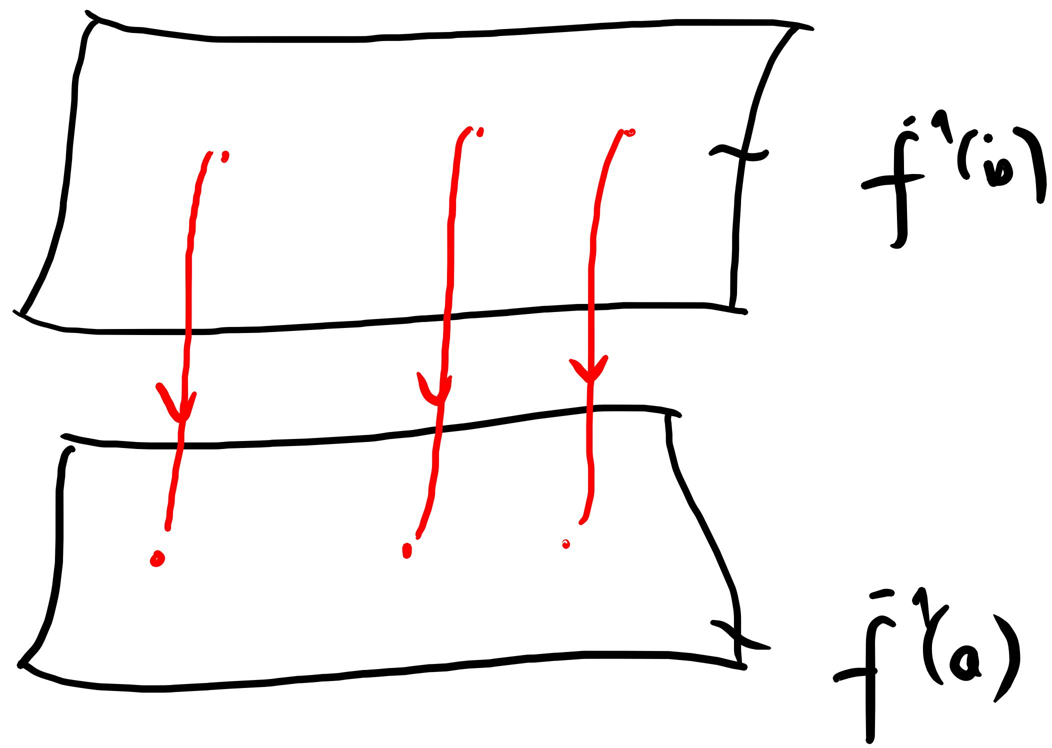
⌈ Recall: M cpct. sm. mfd., $f: M \to \mathbb{R}$ smooth, $a < b$, $f^{-1}([a,b])$ is without critical points

$\Rightarrow$ M^a is a (strong) deformation retract of M^b ⌋

(13.3) Remark (a) So the inclusion $i: M^a \hookrightarrow M^b$ is a homotopy equivalence.

(b) A strong deformation retraction $r: X \to A$ is a retraction ($r \circ i = id_A$) such that there exists a homotopy (h_t) from id_X to $i \circ r$ such that even $h_t|A = id_A$, $\forall t \in [0,1]$.

(c) The idea of proof is to follow the gradient line of the gradient of f w.r.t. an arbitrary Riemannian metric (in fact $-\operatorname{grad}(f)$).



Proof. We scale the gradient field by a smooth function $\rho : M \to \mathbb{R}$ as follows. Let

$$\rho(q) := \begin{cases} \|\operatorname{grad}(f)(q)\|^{-2} & \text{for } q \in f^{-1}([a,b]) \\ 0 & \text{outside a compact nhd} \\ & \text{of } f^{-1}([a,b]) \end{cases}$$

This can be done using a cut-off function w.r.t. to the compact subset $f^{-1}([a,b])$ and a cpct. nhd. K of $f^{-1}([a,b])$.

Then $X \in \mathfrak{X}(M)$,

$$X := \rho \cdot \operatorname{grad}(f)$$

and since M is compact the corresponding flow $(\varphi^t : M \to M)_{t \in \mathbb{R}}$

is global, i.e., all solution curves exist for all $t \in \mathbb{R}$. We now define $r^t : M^b \longrightarrow M^a$ by

$$r^t(q) := \begin{cases} q & \text{if } q \notin f^{-1}([a,b]) \\ \varphi^{t(a-f(q))}, & \text{if } q \in f^{-1}([a,b]) \end{cases}$$

and $r := r^1$.

Observe first that by rescaling the images of the solution curves stay the same as for grad(f), but the parametrization changes. In fact, since

(φ^t) is the flow w.r.t. $X = \rho \cdot \operatorname{grad}(f)$, i.e. $\frac{d\varphi^t}{dt} = X(\varphi^t)$

$$\frac{d}{dt}\, f \circ \varphi^t(q) = \left\langle \operatorname{grad}(f)(\varphi^t(q)),\ \frac{d\varphi^t}{dt}(q) \right\rangle =$$

($\langle -,- \rangle$ denoting the Riemannian metric here)

$$= \left\langle \operatorname{grad}(f),\ \frac{1}{\|\operatorname{grad}(f)\|^2} \cdot \operatorname{grad}(f) \right\rangle (\varphi^t(q)) \rangle = +1$$

for $q \in f^{-1}([a,b])$ and all $t \in [a-c, b-c]$ ($c := f(q)$)

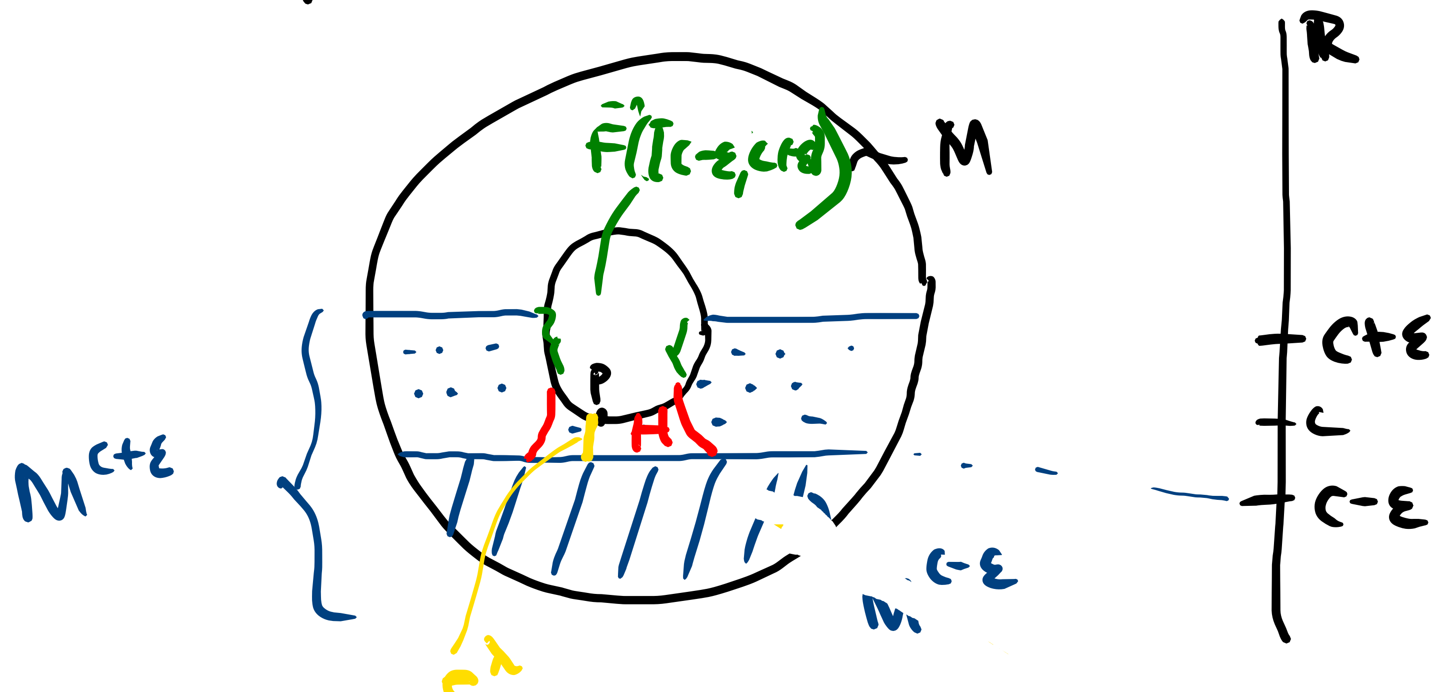
$$\Rightarrow f \circ \varphi^t(q) = f(q) + t,$$

we have that if $q \in f^{-1}(c)$, then $\varphi^t(q) \in f^{-1}(c+t)$. Therefore $r^1(f^{-1}(b)) = f^{-1}(a)$ and (r^t) is a homotopy from $r^0 = id$ to r^1. Since $r^t | M^a = id_{M^a}$, r is in fact a strong deformation retraction from M^b to M^a.

$\square$

(13.4) Theorem. Let $f: M^n \to \mathbb{R}$ be a Morse function and $p \in M$ be a critical point of index $\lambda \in \{0, \dots, n\}$. Let $\varepsilon > 0$ and let p the only critical point of f in $f^{-1}([c-\varepsilon, c+\varepsilon])$, where $c = f(p)$. Then, if $\varepsilon > 0$ is small enough, $M^{c+\varepsilon}$ is homotopy equivalent to $M^{c-\varepsilon}$ with a λ-cell attached.

(13.5) Idea of proof. We will construct a new function $F: M \to \mathbb{R}$ which agrees with f outside of a small neighborhood $U \subseteq M$ of p and which in U slightly smaller than f, but has the same critical points as f. Therefore $F^{-1}((-\infty, c-\varepsilon])$ will be $M^{c-\varepsilon}$ together with a small region, a so-called handle H near p. For the height function of the torus the picture is as follows:



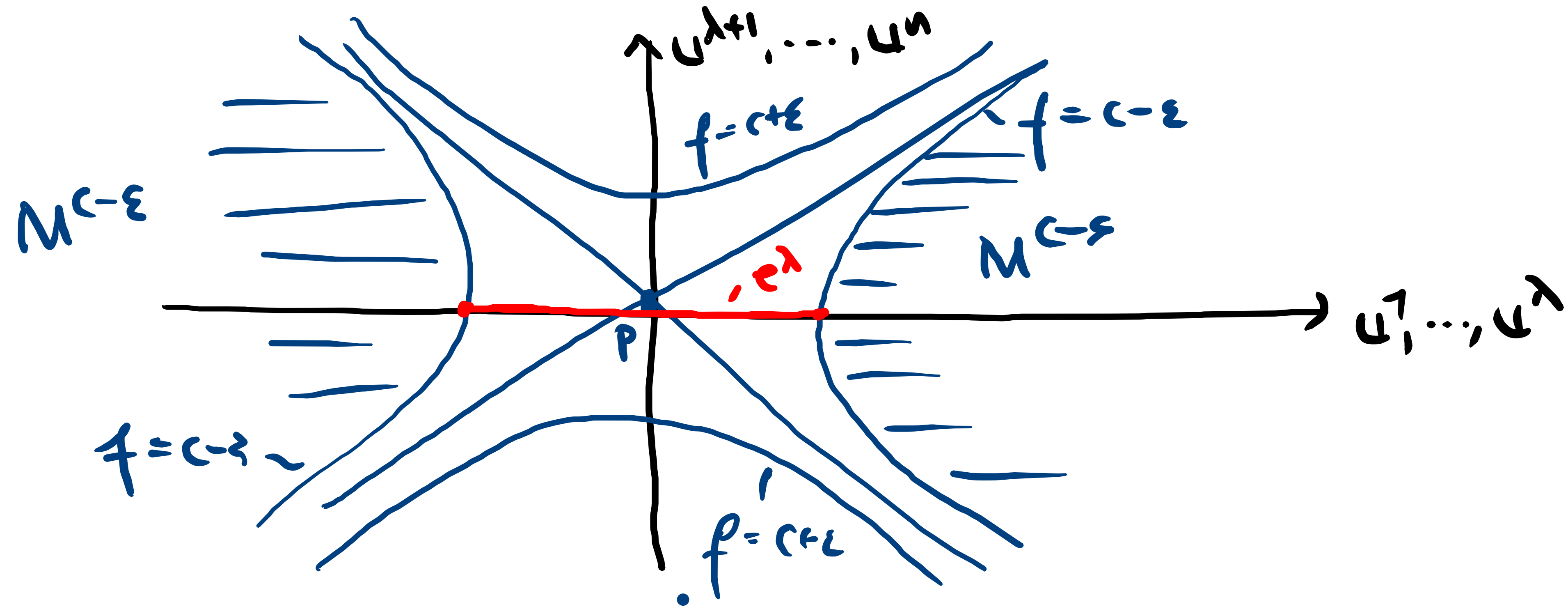
Since F has no critical points in $F^{-1}([c-\varepsilon, c+\varepsilon])$ by applying (13.2) we get that $M^{c-\varepsilon} \cup H = F^{-1}((-\infty, c+\varepsilon])$ will be a deformation retract of $M^{c+\varepsilon}$. Finally, by a direct argument by pushing along horizontal lines, we will see that for a certain λ-cell $e^\lambda \subseteq H$ $M^{c-\varepsilon} \cup e^\lambda$ will be a deformation retract of $H \cup M^{c-\varepsilon}$.
Putting these things together shows the theorem.

(13.6) The details. (a) Let $u = (u^1, \dots, u^n)$ be Morse coordinates in $U \subseteq M$ around $p \in M$ according to the Morse Lemma, so

$$f(u) = c - (u^1)^2 - \dots - (u^\lambda)^2 + (u^{\lambda+1})^2 + \dots + (u^n)^2 . \qquad (u(p) = 0)$$

We choose $\varepsilon > 0$ so small that f has no other critical points in $f^{-1}([c-\varepsilon, c+\varepsilon])$. Further, the image of $u : U \to V \subseteq \mathbb{R}^n$ should contain the ball $B_{\sqrt{2\varepsilon}}(0)$. Finally let $e^\lambda \subseteq U$ be given by the condition

$$\xi(u) := (u^1)^2 + \dots + (u^\lambda)^2 \leq \varepsilon \,, \quad u^{\lambda+1} = \dots = u^n = 0$$

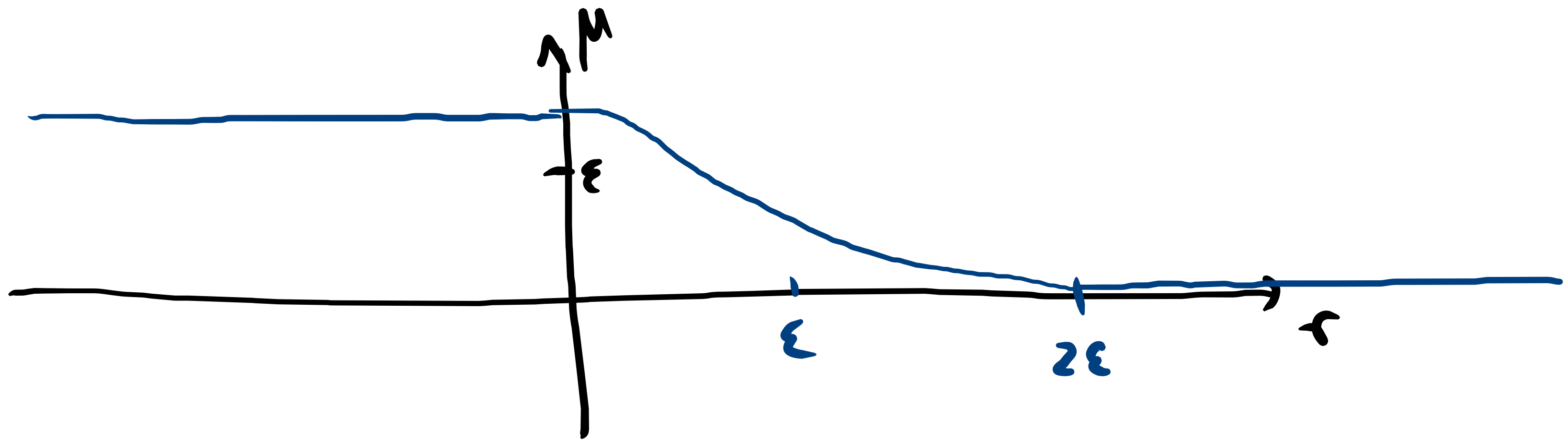


Note that the boundary $\dot{e}^\lambda$ of e^λ is precisely $e^\lambda \cap M^{c-\varepsilon}$, so e^λ is a λ-cell attached to $M^{c-\varepsilon}$.

Claim: $M^{c-\varepsilon} \cup e^\lambda$ is a (strong) deformation retract of $M^{c+\varepsilon}$.

(b) Let $\mu : \mathbb{R} \to \mathbb{R}$ now be a smooth fct'n satisfying

- $\mu(0) > \varepsilon$
- $\mu(r) = 0$ for $r \geq 2\varepsilon$
- $-1 < \mu'(r) \leq 0, \ \forall r \in \mathbb{R}$



Then we introduce $F \in \mathcal{E}(M)$ by setting $F = f$ outside of U and

$$F(u) = f(u) - \mu \cdot \big(\xi(u) + 2\eta(u)\big)$$

with

$$\eta(u) := (u^{k+1})^2 + \dots + (u^n)^2 .$$

<u>Assertion 1.</u> $F^{-1}((-\infty, c+\varepsilon]) = M^{c+\varepsilon}$

Pf. Outside of the ellipsoid $\{u \in U : \xi(u) + 2\eta(u) \leq 2\varepsilon\}$ the functions F and f agree. Inside the ellipsoid we have

$$F \leq f = c - \xi + \eta \leq c + \frac{1}{2}\xi + \eta \leq c + \varepsilon .$$

$\blacksquare$

<u>Assertion 2.</u> F and f have the same critical points.

Pf. We have only to check this inside the ellipsoid which lies in $\mathcal{U}$. Observe

$$\frac{\partial F}{\partial \xi} = -1 - \mu'(\xi + 2\eta) < 0$$

$$\frac{\partial F}{\partial \eta} = +1 - 2\mu'(\xi + 2\eta) \geq 1$$

Now the covectors $d\xi$ and $d\eta$ vanish simultaneously only at the origin and are linearly independent where they don't vanish both. Since

$$dF = \frac{\partial F}{\partial \xi} \cdot d\xi + \frac{\partial F}{\partial \eta} \cdot d\eta,$$

this shows that $F|U$ has only a critical pt. in p.

(c) Consider now the region $F^{-1}([c-\varepsilon, c+\varepsilon])$. Since

$$F(p) = c - \mu(0) < c - \varepsilon,$$

$F^{-1}([c-\varepsilon, c+\varepsilon])$ contains no critical point. Therefore, by (13.2), $F^{-1}((-\infty, c-\varepsilon])$ is a (strong) deformation retract of $F^{-1}((-\infty, c+\varepsilon])$ $= f^{-1}((-\infty, c+\varepsilon]) = M^{c+\varepsilon}$.

We set

$$H := \overline{F^{-1}((-\infty, c-\varepsilon]) \setminus M^{c-\varepsilon}}.$$

Then we have

$$F^{-1}((-\infty, c-\varepsilon]) = M^{c-\varepsilon} \cup H.$$

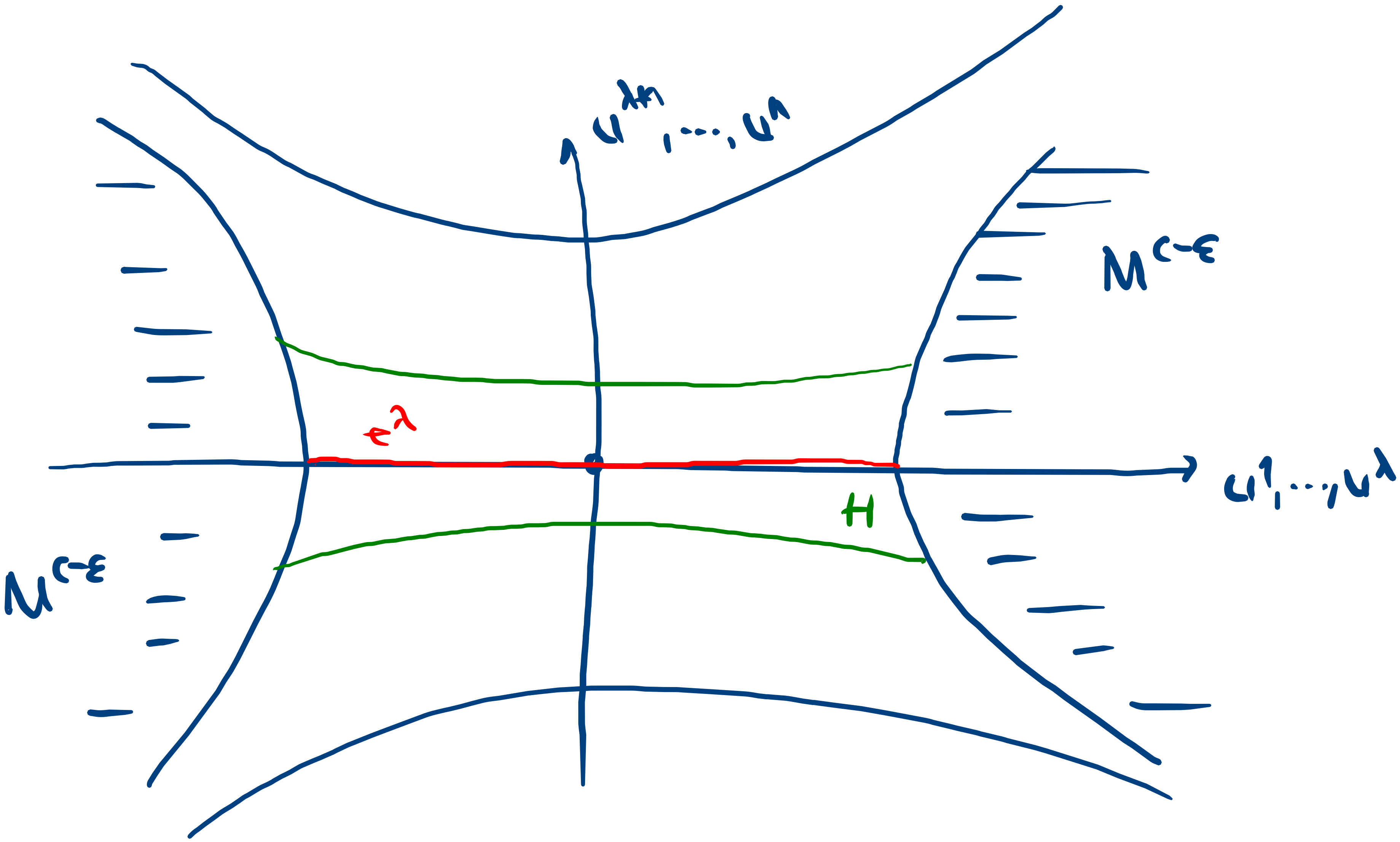
Next we consider the λ-cell $e^\lambda \subseteq H$ given by

$$e^\lambda = \{ u \in U : \xi(u) \leq \varepsilon, \ \eta(u) = 0 \}$$

Then, since $\partial F / \partial \xi < 0$, we have that

$$F(q) \le F(p) < c-\varepsilon.$$

So $q \in H$.



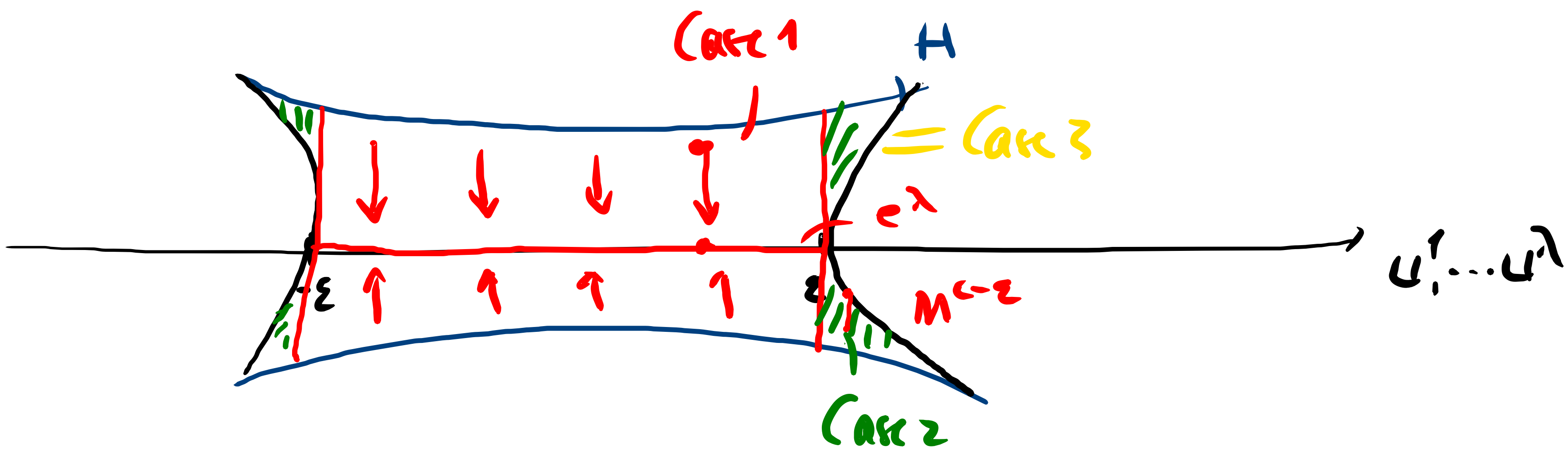
<u>Assertion 3.</u> $M^{c-\varepsilon} \cup e^{\lambda}$ is a (strong) deformation retract of $M^{c-\varepsilon} \cup H$.

Pf. The deformation retract $r = r^{\Phi}$ together with its homotopy (r^t) from $r^0 = \overset{r}{\cancel{\mathrm{id}}}$ to $\hat{r}\ \cancel{= \mathrm{id}}$ is done now „by hands" (along vertical lines and not using the gradient flow of f or F). We distinguish three cases:

<u>Case 1.</u> $\{ u \in H : \xi(u) \leq \varepsilon \}$

Here we set

$$r^t(u^1,\dots,u^n) := (u^1,\dots,u^\lambda, t\cdot u^{\lambda+1},\dots,tu^n).$$



Case 2. Let $\{u\in H:\ \varepsilon \le \xi(u) \le \eta(u)+\varepsilon\}$ . .

Here we take the convex combination along vertical lines

- - . -

$$r^t(u) = (u^1, \dots, u^\lambda, s_t u^{\lambda+1}, \dots, s_t u^n)$$

with

$$s_t = t + (1-t)\sqrt{\frac{\xi-\varepsilon}{\eta}} \qquad (\Rightarrow r(u) \in M^{c-\varepsilon}).$$

Case 3. Within the region $\eta + \varepsilon \leq \xi$ (i.e., inside of $M^{c-\varepsilon}$) we let $r^t = \mathrm{id}$, of course.

One checks now easily that the r^t's in the different cases agree at their joint boundaries and therefore give indeed a deformation retraction of $M^{c-\varepsilon} \cup H$ onto $M^{c-\varepsilon} \cup e^\lambda$.

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Combining (c) with assertion 3 this finishes the proof of Theorem (13.4).

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(13.7) Remark. (a) A CW-structure on a topological space X is given by a filtration of closed subspaces $(X^k)_{k\in\mathbb{N}}$,

$$X^0 \subseteq X^1 \subseteq \dots X^k \subseteq \dots$$

such that X^{k+1} is obtained from X^k by attaching (locally finite) $(k+1)$-cells to X^k. It must be that .

$$X = \bigcup X^k$$

and X must have the weak topology w.r.t. (X^k), i.e., $A \subseteq X$ is closed if $A \cap X^k$ is closed, $\forall k \in \mathbb{N}_0$.

(b) There exists a homology theory adapted to CW-structure, the so-called cellular homology of X, where the k. chain group is generated by the k-cells of X.

(c) CW-complexes $(X, (X^k))$ behave well in homology and homotopy theory

(d) It is now not hard to prove (see e.g. Milnor on p. 20) that a compact mfd. has the homotopy type of a finite

CW-complex. More precisely, if $f: M \to \mathbb{R}$ is a Morse function with ν_λ critical points of index λ (we say f is of type $(\nu_0, \dots, \nu_n)$) then there exist a CW-structure on a homotopy equivalent space X with ν_λ cells of dimension λ. This opens the door to compute many topological invariants of M, e.g.,

$$b_k = \operatorname{rk} H_k(M;\mathbb{Z})$$

or the Euler characteristique

$$\chi(M) = \sum_{k=0}^{n} (-1)^k b_k .$$