Universität Tübingen, Fachbereich Mathematik
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Groups and Representations

Homework Assignment 3 (due on 8 November 2023)

Problem 9

Let $\phi : \mathrm{SL}(2,\mathbb{C}) \to \mathrm{O}(3,1)$ be the homomorphism to the Lorentz group, as introduced in the lectures. Let $\alpha, \beta \in [0, 2\pi]$, $r > 0$ and

$$U = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}, \qquad V = \begin{pmatrix} \mathrm{e}^{-i\beta} & 0 \\ 0 & \mathrm{e}^{i\beta} \end{pmatrix}, \qquad B = \begin{pmatrix} r & 0 \\ 0 & \frac{1}{r} \end{pmatrix}.$$

Show:

a) $\phi(U)$ is a rotation about the x_2-axis by an angle 2α.

b) $\phi(V)$ is a rotation about the x_3-axis by an angle 2β.

c) $\phi(B)$ is a boost in x_3-direction, i.e.

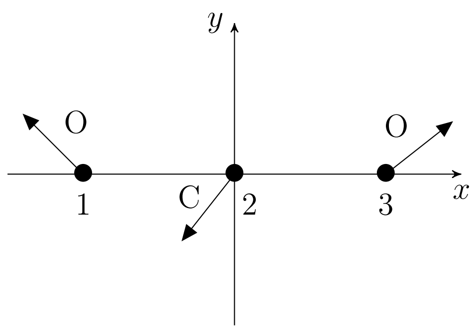
$$\phi(B) = \begin{pmatrix} \cosh t & 0 & 0 & \sinh t \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh t & 0 & 0 & \cosh t \end{pmatrix}$$

for some $t \in \mathbb{R}$.

Problem 10

CO_2 is a linear molecule; in its ground state the carbon atom sits in the middle between the two oxygen atoms. The symmetry group of this system is isomorphic to the Klein four group $V_4 \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ and has the following elements: the identity (e), reflections (σ_x and σ_y) across the x- and y-axis, respectively, and a rotation (R) by 180° about the origin.

A coplanar vibration entails displacements of the 3 atoms in a fixed plane. It can be characterised by a vector $(x_1, y_1, x_2, y_2, x_3, y_3) \in \mathbb{R}^6$.



Determine the action of the symmetry group on the canonical basis of $\mathbb{R}^6$. Write down the resulting six dimensional representation of V_4.

Problem 11 (Continuation of Problem 9)
Let $\Lambda \in \mathrm{O}(3,1)$ be time orientation preserving, i.e. $d(e_0, \Lambda e_0) > 0$. Show that there exist $U, V \in \mathrm{O}(3)$ and a boost B in x_3-direction, such that

$$\Lambda = UBV\,.$$

HINT: First consider Λe_0 and find U and B such that $B^{-1}U^{-1}\Lambda e_0 = e_0$.

Problem 12
Let D_4 be the symmetry group of a square. We denote by R the rotation by $\frac{\pi}{2}$ and by σ the reflection across the diagonal through the lower left and upper right corner. We write all group elements as $R^k \sigma^\ell$ for some k and ℓ. (Why is this possible and which values do k and ℓ take?)

a) Find all conjugacy classes.
   HINT: Determine $\sigma R \sigma$ first, this simplifies calculations a lot.
b) Determine all normal subgroups and the isomorphism types of the corresponding quotient groups (i.e. name known groups to which they are isomorphic).
c) Is D_4 isomorphic to a direct product of non-trivial subgrous?

Problem 13
Let $(\mathbb{R}, +)$ be the additive group of real numbers, and

$$\Gamma(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\,, \qquad x \in \mathbb{R}\,,$$

a representation on $\mathbb{C}^2$. Find all invariant subspaces. Is Γ completely reducible?