Universität Tübingen, Fachbereich Mathematik
Dr. Stefan Keppeler & Saradha Senthil-Velu Wintersemester 23/24

Groups and Representations

Homework Assignment 9 (due on 20 December 2023)

Problem 33

For a fixed partition λ of $n \in \mathbb{N}$ we define an ordering of standard tableaux as follows. Consider the sequence $(r_\lambda^p)_j$, $j = 1, \ldots, n$, of numbers in the boxes of Θ_λ^p starting with the first row read from left to right, then the second row from left to right etc. We say that $\Theta_\lambda^p > \Theta_\lambda^q$ if the first non-vanishing term in the sequence $(r_\lambda^p)_j - (r_\lambda^q)_j$, $j = 1, \ldots, n$, is positive. Then, e.g., the standard tableaux for $\lambda = (3, 2)$ are ordered as

$$\begin{array}{|c|c|c|}\hline 1&2&3\\\hline 4&5\\\cline{1-2}\end{array} < \begin{array}{|c|c|c|}\hline 1&2&4\\\hline 3&5\\\cline{1-2}\end{array} < \begin{array}{|c|c|c|}\hline 1&2&5\\\hline 3&4\\\cline{1-2}\end{array} < \begin{array}{|c|c|c|}\hline 1&3&4\\\hline 2&5\\\cline{1-2}\end{array} < \begin{array}{|c|c|c|}\hline 1&3&5\\\hline 2&4\\\cline{1-2}\end{array} .$$

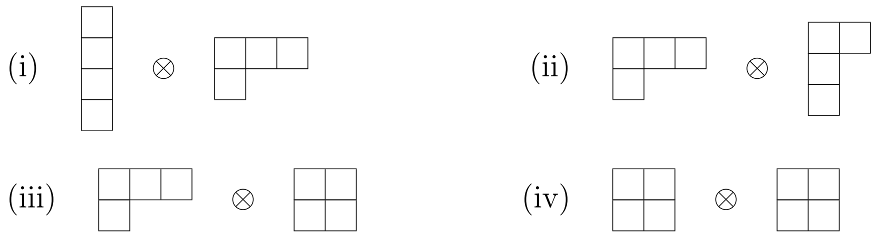
a) Prove that $e_\lambda^p e_\lambda^q = 0$ if $\Theta_\lambda^p > \Theta_\lambda^q$.

b) Show that (a) implies that the left ideals generated by the standard tablaux for a fixed partition are linearly independent.

Problem 34

a) Determine the character table of S_4 using the methods of Section 5.5.

b) Consider the following product representations of S_4, determine which irreps they contain and how many times. Also write their decomposition into irreps in terms of Young diagrams.



Problem 35

In the quark model baryons are made out of three quarks. The latter are characterised i.a. by the quantum numbers[4] I (isospin) and Y (hyper charge).

We have $(I, Y) = \left(\frac{1}{2}, \frac{1}{3}\right)$ for the up-quark, $|u\rangle$, $(I, Y) = \left(-\frac{1}{2}, \frac{1}{3}\right)$ for the down-quark, $|d\rangle$, and $(I, Y) = \left(0, -\frac{2}{3}\right)$ for the strange-quark, $|s\rangle$. For products like $|udd\rangle = |u\rangle \otimes |d\rangle \otimes |d\rangle$ the values of I and Y are given by the sums of the values for the individual quarks.

For combinations of 3 quarks (each up, down or strange) we thus have a 27-dimensional space V, which carries a representation of S_3 (by permutation of the factors).

a) Which irreps are contained in this representation and what are their multiplicities?

b) Let $U \subset V$ be an irreducible invariant subspace. What can we say about the values of I and Y on U?

c) In a (I, Y)-diagram mark all points corresponding to vectors transforming in the irrep defined by $\yng(3)$.

d) Repeat part (c) for the irrep with Young diagram $\yng(2,1)$. You find some potentially useful `Octave`/MATLAB-Code on the course webpage.

Problem 36

The so-called Lie algebra of SU(2) is the (real) vector space

$$\mathfrak{su}(2) = \{X \in \mathbb{C}^{2\times 2} : \mathrm{tr}(X) = 0, X^\dagger = X\}\,.$$

A basis is given by the Pauli matrices (see Problem 31). Show:

a) SU(2) acts on $\mathfrak{su}(2)$ by conjugation: $X \mapsto UXU^\dagger$.

b) $\langle X, Y\rangle = \frac{1}{2}\,\mathrm{tr}(XY)$ defines a scalar product on $\mathfrak{su}(2)$.
HINT: Begin by calculating $\mathrm{tr}(\sigma_i \sigma_j)$.

c) Every $U \in \mathrm{SU}(2) \cong S^3$ (cf. Problem 20) can be written as $\mathrm{e}^{-\frac{1}{2}\mathrm{i}\alpha\vec{\sigma}\cdot\vec{n}}$ with $\vec{n} \in S^2 \hookrightarrow \mathbb{R}^3$ (cf. Problem 31). Over which values does α run?

[4] Think of "quantum number" as "eigenvalue of some linear operator".