Universität Tübingen, Fachbereich Mathematik
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Groups and Representations

Homework Assignment 4 (due on 12 November 2025)

Problem 15
Let G be a finite group and $\Gamma : G \to \mathrm{GL}(V)$ a finite dimensional representation. Prove that $|\det \Gamma(g)| = 1 \ \forall\, g \in G$.

Problem 16
Let G be a finite group, $|G| = n$. We enumerate the group elements, $G = \{g_j, j = 1 \dots n\}$, denote by m the number of conjugacy classes c (with n_c elements) and by p the number of non-equivalent irreducible representations Γ^i of G (with dimensions d_i).

Consider the matrix U with entries $u_{ja} = \sqrt{\frac{d_{i_a}}{n}}\, \Gamma^{i_a}(g_j)_{\mu_a \nu_a}$ with a triple $a = (i_a, \mu_a, \nu_a)$.

Employ the results of Sections 2.5 and 2.6 in order to solve the following problems.

a) Determine the dimensions of U and express the orthogonality relation for irreducible representations (Theorem 6) in terms of U.

b) Show:

(i) $\displaystyle\sum_{i \le p} d_i \operatorname{tr}\left(\Gamma^i(g_j)\Gamma^i(g_k)^\dagger\right) = n\delta_{jk}$,

(ii) $\displaystyle\sum_{g \in c} d_i\, \Gamma^i(g) = n_c\, \chi^i_c\, \mathbb{1}$ and

(iii) $\displaystyle\sum_{i \le p} n_c\, \chi^i_c\, \overline{\chi^i_{c'}} = n\delta_{cc'}$.

c) Conclude that $m = p$.

Problem 17 (Continuation of Problem 13)
We now determine all irreducible representations of D_4 (up to equivalence):

d) What are the dimensions of the irreducible representations?

e) Find all one dimensional irreducible representations.
Hint: First consider irreducible representations of quotient groups, cf. the remarks on (non-)faithful representations in Section 2.1.

f) Determine the character table and the remaining representation(s).

Problem 18
Let V be a finite-dimensional vector space and $P : V \to V$ a linear operator with $P^2 = P$.

a) Show that there exist subspaces U and W with $V = U \oplus W$, $P|_U = \mathbb{1}$ and $P|_W = 0$.

Let $\langle \cdot, \cdot \rangle$ be a scalar product on V, and let $P^\dagger = P$.

b) Show that $U = W^\perp$.

Problem 19

Let $g = \begin{pmatrix} u & -\overline{v} \\ v & \overline{u} \end{pmatrix}$, $u, v \in \mathbb{C}$ with $|u|^2 + |v|^2 = 1$.

a) Verify that $g \in \mathrm{SU}(2)$, and explain why every $g \in \mathrm{SU}(2)$ can be written in this way.

The basis vectors $|{\uparrow}\rangle$ and $|{\downarrow}\rangle$ of $\mathbb{C}^2$, as defined in the lecture[0], transform in the two-dimensional representation $\Gamma^2(g) = g \;\forall\, g \in \mathrm{SU}(2)$.

b) Write $\Gamma^2(g)|{\uparrow}\rangle$ and $\Gamma^2(g)|{\downarrow}\rangle$ as linear combinations of $|{\uparrow}\rangle$ and $|{\downarrow}\rangle$.

Consider now $\mathbb{C}^2 \otimes \mathbb{C}^2$ with the product basis $|{\uparrow\uparrow}\rangle = |{\uparrow}\rangle \otimes |{\uparrow}\rangle$ etc. (cf. lecture). Under $\mathrm{SU}(2)$ this basis transforms in $\Gamma^{2\otimes 2} = \Gamma^2 \otimes \Gamma^2$.

c) Expand $\Gamma^{2\otimes 2}|{\uparrow\uparrow}\rangle$ etc. in the product basis.

d) Show: $\mathrm{span}(|0,0\rangle)$ and $\mathrm{span}(|1,1\rangle, |1,0\rangle, |1,-1\rangle)$ (as defined in the lecture) are invariant under $\mathrm{SU}(2)$, and thus carry one- and three-dimensional representations of $\mathrm{SU}(2)$, respectively, i.e. $\Gamma^{2\otimes 2} = \Gamma^1 \oplus \Gamma^3$.

e) Explicitly determine the representation matrices $\Gamma^1(g)$ and $\Gamma^3(g)$.