Universität Tübingen, Fachbereich Mathematik
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Groups and Representations

Homework Assignment 5 (due on 19 November 2025)

Problem 20
We consider again the CO_2 molecule of Problem 11.

a) How many non-equivalent irreps does the symmetry group V_4 have, and what are their dimensions?

b) Determine the character table for V_4.

In Problem 11 we found a six-dimensional representation of V_4.

c) Which irreps are contained in this six-dimensional representation?

d) Decompose the six-dimensional carrier space into irreducible invariant subspaces by applying the generalised projection operators.

Problem 21
Three spin-$\frac{1}{2}$ particles[1] define a representation D of S_3 on $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 \cong \mathbb{C}^8$ by permutations of the particles, i.e. e.g. $D\left((12)\right)|{\uparrow\downarrow\uparrow}\rangle = |{\downarrow\uparrow\uparrow}\rangle$.

Which irreps of S_3 are contained in D and how often does each of them appear?

Problem 22 (continuation of Problem 19)
On $\mathbb{C}^2 \otimes \mathbb{C}^2$ also acts – as in Problem 21 – a representation D of $S_2 \cong \mathbb{Z}_2 = \{e, (12)\}$.

e) In which representations of S_2 do the vectors $|1,1\rangle$, $|1,0\rangle$, $|1,-1\rangle$ and $|0,0\rangle$ transform?

Problem 23
We consider a rotationally invariant Hamiltonian. Let E be an eigenvalue of H with eigenspace V_E spanned by the spherical harmonics $Y_{1m}(\vartheta,\varphi) = \cos\vartheta\, \mathrm{e}^{\mathrm{i}m\varphi}$ with a fixed radial part R, i.e. $V_E = \mathrm{span}(\{R(r)Y_{1m}(\vartheta,\varphi) : m = -1, 0, 1\})$.[2]

V_E carries a three-dimensional irreducible representation of $\mathrm{O}(3)$, defined by $\left(\Gamma(U)\psi\right)(x) = \psi(U^{-1}x)$. $\mathrm{O}(3)$ contains the subgroup $D_3 = \{e, C, \bar{C}, \sigma_1, \sigma_2, \sigma_3\} \cong S_3$, where C and $\bar{C}$ denote rotations about the z-axis (cf. Section 2.4.1).

Study the effect of perturbations that are only invariant under D_3 or $\mathbb{Z}_3 \cong \{e, C, \bar{C}\}$. Determine the relevant irreducible representations with their multiplicities and sketch the possible splitting of energy levels.

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[1] If "spin-$\frac{1}{2}$ particle" doesn't mean much to you, then just ignore the word. We introduced this manner of speaking in Section 2.8, and the only thing you need to know for this homework assignment are the definitions

$$|{\uparrow}\rangle = \begin{pmatrix}1\\0\end{pmatrix}, \qquad |{\downarrow}\rangle = \begin{pmatrix}0\\1\end{pmatrix}, \qquad |{\downarrow\uparrow}\rangle = |{\downarrow}\rangle \otimes |{\uparrow}\rangle \quad \text{etc.}$$

[2] We use spherical coordinates

$$\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}r\sin\vartheta\cos\varphi\\ r\sin\vartheta\sin\varphi\\ r\cos\vartheta\end{pmatrix}.$$