

Groups and Representations

Homework Assignment 6 (due on 26 November 2025)

Problem 24
We consider the abelian group $C_3 = \{e, a, a^{-1}\} \cong \mathbb{Z}_3$.

a) How many (non-equivalent) irreps does C_3 have, what are their dimensions and how often do they appear in the regular rep?

b) Show that
$$e_1 = \frac{1}{3}(e + a + a^{-1})$$
is a primitive idempotent, generating the trivial rep.

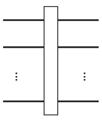
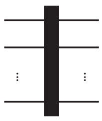
c) Use the ansatz
$$e_2 = xe + ya + za^{-1}$$
in order to find all primitive idempotents.

d) For each primitive idempotent find out whether it generates a new (non-equivalent) irrep or an irrep equivalent to one generated by a previous idempotent.

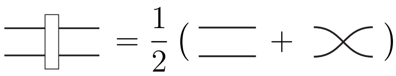
e) Specify all minimal left ideals and construct the corresponding irreps of C_3. Collect your results in a table.

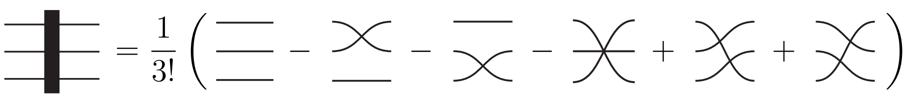
Problem 25
Let V be a (finite dimensional) vector space and $A : V \to V$ a linear map. Show that if A is nilpotent (i.e. if for some $n \in \mathbb{N}$ we have $A^n v = 0\ \forall\, v \in V$) then $\operatorname{tr} A = 0$.

Problem 26 [3]
We can also write elements of the $\mathcal{A}(S_n)$ in birdtrack notation. In particular, we denote symmetrisers and anti-symmetrisers by open and solid bars, respectively, i.e.

$\frac{1}{n!}s = \frac{1}{n!}\sum_{p\in S_n} p =$  and $\frac{1}{n!}a = \frac{1}{n!}\sum_{p\in S_n} \operatorname{sgn}(p)p =$  .

Note that we include a factor of $\frac{1}{n!}$ in the definition of bars over n lines. For instance,

 and

 . (∗)

Notice that in birdtrack notation the sign of a permutation, $(-1)^K$, is determined by the number K of line crossings; if more than two lines cross in a point, one should slightly perturb the diagram before counting, e.g.  $\rightsquigarrow$  (K=3).

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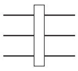
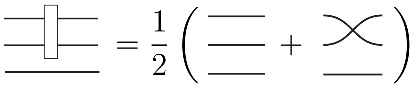
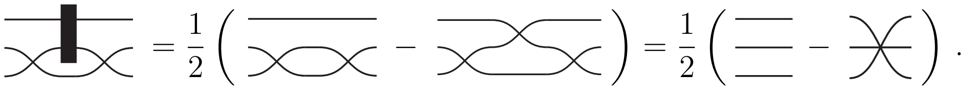
[3] will be discussed in the lecture

a) Expand  and  as in (∗).

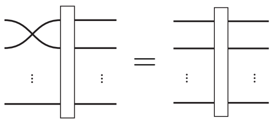
We also use the corresponding notation for partial (anti-)symmetrisation over a subset of lines, e.g.



or



It follows directly from the definition of S and A that when intertwining any two lines S remains invariant and A changes by a factor of (-1), i.e.



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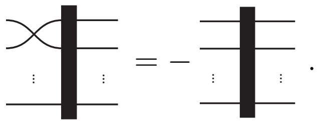
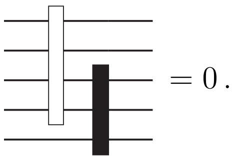
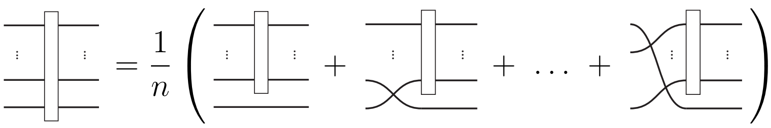
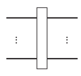


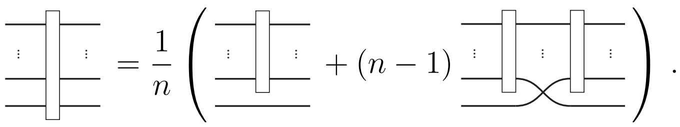
b) Explain why this implies that whenever two (or more) lines connect a symmetriser to an anti-symmetrizer the whole expression vanishes, e.g.



Symmetrisers and anti-symmetrisers can be built recursively. To this end notice that on the r.h.s. of



we have sorted the terms according to where the last line is mapped – to the nth, to the $(n-1)$th, . . . , to the first line line. Multiplying with  from the left and disentangling lines we obtain the compact relation



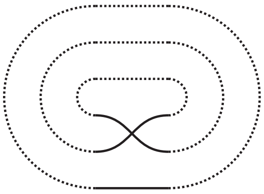
c) Derive the corresponding recursion relation for anti-symmetrisers.

Problem 27
For $\sigma \in S_n$ and $j = 1, \dots, n$ let $k_j(\sigma)$ be the number of (disjoint) cycles of length j in σ, e.g. $k_1(e) = n$ and $k_j(e) = 0 \; \forall \, j > 1$. Show:

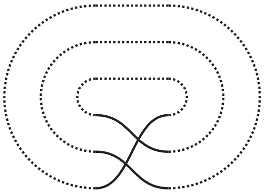
a) The conjugacy class of σ is determined by its cycle structure, i.e.

$$[\sigma] := \{\tau\sigma\tau^{-1} : \tau \in S_n\} = \{\tau \in S_n : k_j(\tau) = k_j(\sigma), j = 1, \dots, n\} .$$

It's almost trivial using the birdtrack notation (see Section 1.4)!
Hint: In order to make the cycle structure visible consider the birdtrack diagram of σ and connect the first line on the left to the first line on the right etc.; e.g. for $(12), (132) \in S_3$ consider



and



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b) The number of elements of a class is given by

$$|[\sigma]| = \frac{n!}{\prod\limits_{j \le n} k_j! j^{k_j}} .$$

c) Fun exercise (optional): Watch the video *An Impossible Bet* by `minutephysics`,

https://youtu.be/eivGlBKlK6M ,

and come up with a good strategy. Don't watch the solution! Think about cycles instead.

Problem 28
For $A \in \mathbb{C}^{n\times n}$ the matrix exponential is defined as

$$\mathrm{e}^A = \exp(A) = \sum_{\nu=0}^{\infty} \frac{A^\nu}{\nu!} .$$

Prove:

a) The series converges absolutely and uniformly.
Hint: On $\mathbb{C}^{n\times n}$ use the operator norm

$$\|A\| = \sup_{v \in \mathbb{C}^n \setminus \{0\}} \frac{|Av|}{|v|} ,$$

for which we have $\|AB\| \le \|A\| \, \|B\|$.

b) For $T \in \mathrm{GL}(n)$ we have $\mathrm{e}^{TAT^{-1}} = T\mathrm{e}^A T^{-1}$.

c) e^{tA} is the unique solution of the initial value problem $\dot{X}(t) = AX(t)$, $X(0) = \mathbb{1}$.

d) For $t, s \in \mathbb{C}$ we have $\mathrm{e}^{(t+s)A} = \mathrm{e}^{tA}\mathrm{e}^{sA}$.

e) $\left(\mathrm{e}^A\right)^\dagger = \mathrm{e}^{(A^\dagger)}$.

f) $\det \mathrm{e}^A = \mathrm{e}^{\operatorname{tr} A}$.