

Groups and Representations

Homework Assignment 11 (due on 14 January 2026)

Problem 43

Let Γ^j be the irreps of $SU(2)$ (or of $SO(3)$) as constructed in the lecture.

a) Prove the following formula for the characters

$$\chi^j(\alpha) = \frac{\sin((2j+1)\alpha/2)}{\sin(\alpha/2)}.$$

b) Consider the product representation $\Gamma^j \otimes \Gamma^k$ with $j \geq k$. Show that every irrep Γ^ℓ with $\ell = j - k, \dots, j + k$ appears exactly once in the decomposition of $\Gamma^j \otimes \Gamma^k$, and that all other irreps are absent.

Problem 44

Let Γ be an irrep of $SU(2)$ on \mathbb{C}^n . Show that there exists a $T \in GL(n, \mathbb{C})$ with $T^2 \in \{\pm \mathbf{1}\}$ s.t.

$$\overline{\Gamma(g)} = T\Gamma(g)T^{-1} \quad \forall g \in SU(2).$$

In which cases do we have $T^2 = \mathbf{1}$?

HINT: First find T for the defining representation. Observe that $T \in SU(2)$ and then investigate $\Gamma(T)$.

Problem 45

Let G be a Lie group with Lie algebra \mathfrak{g} , and let ad be the adjoint representation of \mathfrak{g} , i.e. $\text{ad}_X(Y) = [X, Y]$. The map

$$\begin{aligned} K : \mathfrak{g} \times \mathfrak{g} &\rightarrow \mathbb{R} \\ (X, Y) &\mapsto K(X, Y) = \text{tr}(\text{ad}_X \circ \text{ad}_Y) \end{aligned}$$

is called Killing-Form. Show:

- a) K is bilinear and symmetric.
- b) $K(\text{Ad}_g(X), \text{Ad}_g(Y)) = K(X, Y) \quad \forall X, Y \in \mathfrak{g} \text{ and } \forall g \in G$.

Remark: For semi-simple Lie groups (which we haven't defined, but the classical groups $SU(n)$ and $SO(n)$ are examples) K is positive definite, i.e. it defines a scalar product.

Let G now be such that K is positive definite. We choose an orthonormal basis $\{X_j\}$ with respect to K , i.e. $K(X_j, X_k) = \delta_{jk}$, and define $C_2 \in E(\mathfrak{g})$ by

$$C_2 = \sum_j X_j X_j.$$

Show:

- c) C_2 is independent of the choice of basis.
- d) C_2 is a Casimir operator (the so-called *quadratic Casimir operator*), i.e.

$$\text{Ad}_g(C_2) = C_2 \quad \forall g \in G.$$