# Interacting Many-Body Systems

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## Sheet 1

**Exercise 1:** Consider a two-particle system on the one-dimensional torus $[0, 2\pi]$. With $\phi(x) = (2\pi)^{-1/2}$ and $\eta(x) = (2\pi)^{-1/2}e^{ikx}$ two orthogonal and normalized wave-functions are given.

Consider now the two particle state given by a) the product b) the symmetrized product and c) the anti-symmetrized product of these two functions.

Calculate in each of these three cases the probability that both particles have distance smaller than $\pi$ as a function of $k$. What happens if $k$ gets large?

**Exercise 2:** Let $\phi, \eta$ be vectors in the Hilbert-space of square integrable functions mapping $\mathbb{R}^3$ to $\mathbb{C}$. Assume that $\|\phi\| = \|\eta\| = 1$ and $N$ is an even natural number.

Let $\Psi := \frac{1}{\sqrt{2}} \prod_{j=1}^{N} \phi(x_j) + \frac{1}{\sqrt{2}} \prod_{j=1}^{N} \eta(x_j)$, $\chi := \left( \prod_{j=1}^{N/2} \phi(x_j) \prod_{j=N/2+1}^{N} \eta(x_j) \right)_{sym}$ and
$\Theta := \prod_{j=1}^{N/2} \phi(x_j) \prod_{j=N/2+1}^{N} \eta(x_j)$.

Calculate the reduced one-particle density matrix as well as the reduced two-particle density matrix of $\Psi$, $\chi$ and $\Theta$.

**Exercise 3:** Let $\mu_\Psi$ be the reduced one-particle density matrix of some many body wave function $\Psi$. Show that

(a) $\mu_\Psi$ is non-negative, i.e. $\langle \phi, \mu_\Psi \phi \rangle \geq 0$ for any $\phi \in L^2(\mathbb{R}^3, \mathbb{C})$.

(b) Show that $Tr(\mu_\Psi) = 1$.

(c) Show that $\mu_\Psi$ is self-adjoint.

**Exercise 4:** Let $S(n) := \{\mu \in \mathcal{M}(n,n) : \mu \geq 0 \text{ and } Tr\,(\mu) = 1 \text{ and } \mu \text{ is self-adjoint}\}$ be the set of all density matrices of vectors in $\mathbb{C}^n$. $\mathcal{M}(n,n)$ stands for the set of all $n \times n$-matrices with complex entries. Show that $S(n)$ is convex and that the following properties are equivalent

(a) $\mu$ is a pure state, that is $\rho = |\phi\rangle\langle\phi|$ for some $\phi \in \mathbb{C}^n$

(b) $\mu$ is an orthogonal rank-one projection.

(c) $Tr(\mu^2) = 1$