Interacting Many-Body Systems

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Sheet 3

Exercise 1: Consider the weakly interacting Bose gas in one dimension on a torus $[0, L]$. The Hamiltonian is given by

$$H = -\sum_{j=1}^{N} \Delta_{x_j} + \frac{1}{N} \sum_{j \neq k} V(x_j - x_k) \, .$$

Assume that $\widehat{V} \geq 0$ and that $\sum_{k=0}^{\infty} \widehat{V}(k) < \infty$. In class we showed that for $L = 2\pi$ the ground state of the system shows 100% condensation, in this exercise we assume that $L = N^{\mu}$ for some $\mu > 0$.

For which choices of μ can you still proof 100% condensation following the proof we did in class? How does μ depend on the spatial dimension of the gas?

Exercise 2: Let $V : \mathbb{R}^3 \to \mathbb{R}$ be a compactly supported potential. Consider a system of N Bosons subject to three-particle interactions given by $W(x, y, z) = V(|x - y| + |x - z| + |y - z|)$. The respective Hamiltonian reads

$$H = -\sum_{j=1}^{N} \Delta_{y_j} + \sum_{j \neq k \neq l \neq j} W(x_j, x_k, x_l) \, .$$

Assume that there is a balance between kinetic and potential terms and that (after rescaling space and time variables) the rescaled Laplacian has no N-dependent factor (similar to the Gross-Pitaevskii scaling discussed in class).

Give the respective scaling factors for the new space- and time-variables.

Exercise 3: Consider the following differential inequality: $\dot{f}(t) < G(f, t)$ for some $G : \mathbb{R} \times \mathbb{R}_0^+ \to \mathbb{R}$. Assume there exists a solution $h : I \to \mathbb{R}$ of the differential equation $\dot{h}(t) = G(h, t)$ with $h(0) = f(0)$ on some interval $I = [0, a)$ with $a \in \mathbb{R}^+ \cup \{\infty\}$.

Show that $f(t) \leq h(t)$ for all $t \in I$.

Show also, that the condition $\dot{f}(t) \leq G(f, t)$ is not sufficient, i.e. find a G, h and f with $\dot{f}(t) \leq G(f, t)$, $\dot{h}(t) = G(h, t)$ and $f(0) = h(0)$ such that $f(t) \leq h(t)$ does *not* hold for all $t \geq 0$.

Exercise 4: Find a proof of Grönwall's Lemma in the case that f is continuous and differentiable from the left and $\dot{f} \leq Cf + \varepsilon$ for some $C, \varepsilon > 0$. Here $\dot{f}$ stands form the left-derivative of f.