Interacting Many-Body Systems

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Sheet 4

Exercise 1: Let $\phi, \eta \in L^2(\mathbb{R}^3 \to \mathbb{C})$. Define the projectors $p := |\eta\rangle\langle\eta|$ and $q = 1 - p$. Let $a := \langle\phi, \eta\rangle$. Show that the following inequalities hold

(a) $\|\phi - \eta\|_2^2 = 2\,(1 - \mathrm{Re}(a))$ (Re stands for the real part.)

(b) For any complex phase $\lambda \in \mathbb{C}$, $|\lambda| = 1$ it holds that $\|\lambda\phi - \eta\|_2^2 \geq \frac{1}{2}\langle\phi, q\phi\rangle$

(c) There is a $\lambda \in \mathbb{C}$ with $|\lambda| = 1$ such that $\|\lambda\phi - \eta\|_2^2 \leq \langle\phi, q\phi\rangle$

Exercise 2: Let $V \in L^\infty$ and ϕ_t, η_t be solutions of the Hartree equation

$$i\frac{d}{dt}\phi_t = \left(-\Delta + V \star |\phi_t|^2\right)\phi_t$$
$$i\frac{d}{dt}\eta_t = \left(-\Delta + V \star |\eta_t|^2\right)\eta_t$$

Show that there is a $C < \infty$ such that $\|\phi_t - \eta_t\|_2^2 \leq e^{Ct}\,\|\phi_0 - \eta_0\|_2^2$. Hint: Show that $\|\phi_t - \eta_t\|_2^2$ fulfills a Grönwall inequality. Use the results from exercise 1.

Exercise 3: Let W be a three particle interaction, i.e. $W : \mathbb{R}^9 \to \mathbb{R}$. Consider a system of N-Bosons under influence of the 3-particle interaction. The system is described by a solution Ψ_t of the Schrödinger equation with Hamiltonian

$$H = \sum_{j=1}^{N} -\Delta_j + N^{-2} \sum_{j \neq k \neq l \neq j} W(x_j, x_k, x_l)\,.$$

Assume that $\Psi_0(x_1, \ldots, x_N) = \prod_{j=1}^N \phi_0(x_j)$ for some one-particle wave function ϕ_0.

Give the equation for ϕ_t for which you expect that it effectively describes this system (no rigorous proof needed).

Exercise 4: Consider a mixture of two different Bose gases, with N respectively M particles that interact.

The system is described by a solution Ψ_t of the Schrödinger equation with Hamiltonian

$$H = \sum_{j=1}^{N} -\Delta_{x_j} + \sum_{j=1}^{M} -\Delta_{y_j} + \frac{1}{N} \sum_{j \neq k; j,k=1}^{N} V(x_j - x_k) + \frac{1}{N+M} \sum_{j=1}^{N} \sum_{k=1}^{M} W(x_j - y_k) + \frac{1}{M} \sum_{j \neq k; j,k=1}^{M} U(y_j - y_k)$$

where $V, W, U : \mathbb{R}^3 \to R$ with $V, W, U \in L^\infty$.

Assume that $\Psi_0(x_1, \ldots, x_N) = \prod_{j=1}^{N} \phi_0(x_j) \prod_{j=1}^{M} \eta_0(x_j)$ for some one-particle wave functions ϕ_0, η_0.

Give the coupled equations for ϕ_t, η_t for which you expect that they effectively describe this system (no rigorous proof needed).