Interacting Many-Body Systems

Prof. Dr. P. Pickl

Sheet 6

Exercise 1: Let $\Psi \in L^2(\mathbb{R}^{3N})$, $\phi \in L^2(\mathbb{R}^3)$ with $\|\Psi\|_2 = \|\phi\|_2 = 1$.

Assume that $\langle \Psi, \widehat{\frac{k}{N}} \Psi \rangle \leq CN^{-1}$, $\langle \Psi, \widehat{\mathbb{1}}_{\text{odd}} \Psi \rangle \leq CN^{-1}$ and $\langle \Psi, \widehat{\frac{k}{N}} \widehat{\mathbb{1}}_{\text{odd}} \Psi \rangle \leq CN^{-2}$ (all weight operators are defined with respect to ϕ).

Show that $|\langle \Psi, A_1 \Psi \rangle - \langle \phi, A\phi \rangle|$ is of order N^{-1} for any operator A with bounded operator norm ($A_1 = A \otimes 1 \ldots \otimes 1$ denotes the operator A acting on the first particle).

What does this imply for $\left\| \mu^{\Psi} - |\phi\rangle\langle\phi| \right\|_{Tr}$?

Exercise 2: Let Ψ_t be a solution of the Schrödinger equation with Hamiltonian

$$H = \sum_{j=1}^{N} -\Delta_j + N^{-2} \sum_{j \neq k \neq l \neq j} W(x_j, x_k, x_l)$$

with potential $W(x, y, z)$ that is symmetric under the exchange of any of the variables x, y, z.

Assume that there is a C such that $W(x, y, z) < C$ for all $x, y, z \in \mathbb{R}^3$.

The goal s to convince yourself that $\alpha_t := \langle \Psi_t, q_1^{\phi_t} \Psi_t \rangle$ is small if ϕ_t is the solution of the effective equation you guessed in exercise 3 on sheet 4.

In contrast to the case of two body interactions, one has to control more terms than I, II and III. Give all respective terms one has to control.

Estimate the term with the operator $p_1 p_2 p_3 W(x_1, x_2, x_3) q_1 q_2 q_3$.

Exercise 3: The phase-function for the sound-wave in the cold Bose gas was given by

$$f(k) = \sqrt{k^4 + k^2 \widehat{V}(k)} - vk \ .$$

For this exercise we consider the case $k \in \mathbb{R}_0^+$ (negative k's can be treated equivalently)

Assume that $\widehat{V}(0) > 0$ and that $\widehat{V}(k)$ is two times continuously differentiable with respect to k. Show that in the case $v = \sqrt{\widehat{V}(0)}$ one gets: $f(0) = 0, \lim_{k \to 0} f'(k) = 0$ and $\lim_{k \to 0} f''(k) = 0$. Here $'$ stands for derivation with respect to k.

Show that if $\widehat{V}' \geq -\frac{3}{2}k$ there is no stationary point (i.e. $f'(k)$ can not be zero) for any $|v| < \sqrt{\widehat{V}(0)}$.