Interacting Many-Body Systems

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Sheet 7

Exercise 1: Let ϵ_t be a solution of the effective equation we found in class to describe the time evolution of a small perturbation with respect to a Bose gas of constant density. Assume that $\widehat{V}$ is everywhere negative, smooth, radially symmetric and $\lim_{k\to\infty}\widehat{V}(k)=0$.

Argue that ϵ_t is in general unstable. Which momenta lead to instability, which don't?

To solve this exercise it is important to note, that the eigenvalues of the Hamiltonian describing the motion of the small perturbations is given by

$$\rho\lambda\sqrt{k^4+k^2\widehat{V}(k)}$$

where ρ is the density of the gas and λ the coupling constant. We chose $\lambda=\rho^{-1}$ so the pre-factor was one in class.

If $\widehat{V}$ is continuous, negative and decaying, it follows that there are intervals $[0,k_0]$ and $[k_1,\infty[$ such that $\sqrt{k^4+k^2\widehat{V}(k)}$ is imaginary on $[0,k_0]$ and real on $[k_1,\infty[$.

For any initial state which is a linear combination of eigenfunctions with real eigenvalue $\pm\sqrt{k^4+k^2\widehat{V}(k)}$ we get a particle that moves as discussed in class.

For that part with negative eigenvalues, the eigenfunctions propagate according to $e^{-\sqrt{|k^4+k^2\widehat{V}(k)|}t}\phi_1(k)$ respectively $e^{\sqrt{|k^4+k^2\widehat{V}(k)|}t}\phi_2(k)$ where $\phi_1(k)$ and $\phi_2(k)$ are the respective eigenfunctions with eigenvalue $\pm i\sqrt{|k^4+k^2\widehat{V}(k)|}$.

While linear combinations of the $\phi_1(k)$ show stability in time, there is a blow-up for linear combinations of $\phi_2(k)$: The amplitude of those waves grows exponentially in time. With growing density, the effective interaction grows. So we (roughly) expect that the density at the perturbation will behave like

$$\sqrt{\rho_t}=1+e^{\sqrt{|k^4+k^2\widehat{V}(k)|}\int_0^t\rho_s ds}\varepsilon_0$$

This equation show finite-time blowup: Define $f_t=\sqrt{\rho_t}-1$.

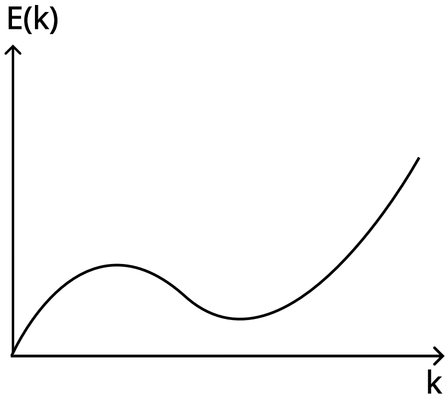
It follows that

$$\frac{d}{dt}f_t=\rho_t f_t=f_t(f_t+1)^2\leq f_t^3$$

So f_t is bounded from below by the solutions of $\frac{d}{dt}h = h^3$.

One can show that this has a finite-time blow-up using the seperation of variables.

Exercise 2: (Superfluid Helium-4)



On the left you see a sketch of the dispersion relation for small perturbations in superfluid Helium-4. In the graph, k stands for the momentum of the wave, $E(k)$ for the respective eigenvalue of the Hamiltonian of the effective description. Note, that in contrast to Helium-3, Helium-4 atoms are Bosons.

Give a rough picture of how you expect the interaction between Helium-4-atoms to look like.

Argue that there is a second speed of sound in ultra-cold Helium-4, i.e. a wave that travels with a certain speed and decays slower than usual.

Superfluif Helium has many interesting properties that were found experimentally many decades ago. I recommend to have a look at these properties. This exercise here is about the second speed of sound in the gas.

An the graph one can see that the dispersion relation (ignoring constants we calculated $\sqrt{k^4 + k^2\widehat{V}(k)})$ has a local maximum and a local minimum. Inbetween there must be a point, where the second derivative is zero, in other words there is a k_0 such that $\frac{d^2}{dk^2}\sqrt{k^4 + k^2\widehat{V}(k)} = 0$.

In class we learned that the phase factor we get from the time evolution is $t(\sqrt{k^4 + k^2\widehat{V}(k)} - \frac{x}{t}k$. We can choose k such that the first derivative of the expression is zero. Thus there is a point, where first and second derivative are zero. It follows that the wave-function shows slower than ballistic decay in that direction (think of the stationary phase argument we had in class).

This means that there is a second speed of sound in the gas. The speed is given by the first derivative of the dispersion curve at that point, where the second derivative is zero.

Exercise 3: (Mixture of two condensates of identical particles)

In the following exercise, Ψ_t is a solution of the Schrödinger equation w.r.t. the Hamiltonian

$$H = \sum_{j=1}^{N} -\Delta_j + \frac{1}{N-1}\sum_{j\neg k} V(x_j - x_k)$$

Let ϕ_0, η_0 be two orthogonal and $L^2(\mathbb{R}^3)$-normalized one-particle wave functions.

In class we have seen, that there is no simple effective description of a mixture of two condensates w.r.t. ϕ_0 respectively η_0. For certain linear combinations of mixtures, however, one can in fact find an effective description.

So consider

$$\Psi_0 = \sum_{k=0}^{N} C_k \left(\prod_{i=1}^{k} \phi_0(x_i) \prod_{j=k+1}^{N} \eta_o(x_j) \right)_{sym}$$

for $C_k \in \mathbb{C}$ with $\sum_{k=0}^{N} |C_k|^2 = 1$.

Find a choice of C_k and an evolution equation for ϕ_t and η_t such that Ψ_t is effectively described by

$$\sum_{k=0}^{N} C_k \left(\prod_{i=1}^{k} \phi_t(x_i) \prod_{j=k+1}^{N} \eta_t(x_j) \right)_{sym}$$

Hint: Think of a product of $2^{-1/2}(\phi_0 + \eta_0)$.

The trick is to consider an initial wave which is a product

$$\Psi_0 = \prod_{j=1}^{N} (a\phi(x_j) + b\eta(x_j))$$

with $a^2 + b^2 = 1$. One example is $a = b = N^{-1/2}$.

Then

$$\Psi_0 := 2^{-N/2} \sum_{k=1}^{N} \sqrt{\binom{N}{k}} \left(\prod_{j=1}^{k} \phi(x_j) \prod_{j=k+1}^{N} \eta(x_j) \right)_{sym}$$

(note that the symmetrization $()_{sym}$ includes a normalization factor $\left(\binom{N}{k}\right)^{-1/2}$). Thus $C_k = \sqrt{\binom{N}{k}}$.

As time evolves, the wave solves the Hartree equation, thus

$$i\frac{d}{dt}(\phi_t + \eta_t) = (-\Delta + V \star (\phi_t + \eta_t)(\phi_t + \eta_t)^\star)\,(\phi_t + \eta_t)$$

This can be written as a coupled equation

$$\begin{aligned} i\frac{d}{dt}\phi_t &= (-\Delta + V \star (\phi_t + \eta_t)(\phi_t + \eta_t)^\star)\,\phi_t \\ i\frac{d}{dt}\eta_t &= (-\Delta + V \star (\phi_t + \eta_t)(\phi_t + \eta_t)^\star)\,\eta_t \end{aligned}$$

it follows that

$$\sum_{k=0}^{N} C_k \left(\prod_{i=1}^{k} \phi_t(x_i) \prod_{j=k+1}^{N} \eta_t(x_j) \right)_{sym}$$