Interacting Many-Body Systems

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Sheet 8

Exercise 1:

(a) Let $\{e_k\}$, $k \in \mathbb{Z}$ be an ONB of the one-particle Hilbert space.

Show that $\sum_{k\in\mathbb{Z}} \langle \Psi, a^\star(e_k) a(e_k) \Psi\rangle$ gives the expected particle-number of Ψ.

(b) Let ϕ be a normalized one-body wave function, Ω the vacuum-state w.r.t ϕ, i.e. $a^\star(\phi)\Omega = 0$.

Find real and positive numbers C_k such that the $\Psi_k := C_k a(\phi)^k \Omega$ are normalized. Show, that $\langle \Psi_k, \Psi_l \rangle = 0$ whenever $k \neq l$.

(c) Find an eigenfunction Ψ for $a^\star(\phi)$ with eigenvalue $\lambda > 0$. It is best to write this eigenfunction using the elements from b), i.e. $\Psi = \sum_{k=0}^{\infty} \lambda_k \Psi_k$

What is the probability distribution for the number of particles in state ϕ for that eingenfunction (i.e. the distribution of the $|\lambda_k|^2$)?

Show that there is no eigenfunction of $a^\star(\phi)$.

Exercise 2: Find a way to write the expression $\alpha_t^1 = \langle \Psi_t, q_1^\phi \Psi_t \rangle$ and $\alpha_t^2 = \langle \Psi_t, q_1^\phi q_2^\phi \Psi_t \rangle$ using creation- and annihilation operators of the state ϕ.

Exercise 3:

(a) Let ϕ, η be normalized and orthogonal.

Show that $a(\mu\phi + \tau\eta) = \mu a(\phi) + \tau a(\eta)$ and $a^\star(\mu\phi + \tau\eta) = \mu a^\star(\phi) + \tau a^\star(\eta)$ for any $|\mu|^2 + |\tau|^2 = 1$.

Given this it is most reasonable to generalize the definition of the creation operators to non-normalized states via $a(\phi) := \|\phi\|_2 a(\frac{\phi}{\|\phi\|_2})$ and $a^\star(\phi) := \|\phi\|_2 a^\star(\frac{\phi}{\|\phi\|_2})$.

(b) Assume that ϕ_t solves $i\frac{d}{dt}\phi_t = h\phi_t$. Find an expression for $\frac{d}{dt}a(\phi_t)$ and $\frac{d}{dt}a^\star(\phi_t)$.