Bogolioubov approximation

$\rightsquigarrow \Psi_0 = \prod \varphi_0$ for some φ_0.

$$\rightsquigarrow H^{2nd} = \sum_{j=1}^{N} \left(-\Delta_j + V * |\varphi_t|(x_j)\right) + \frac{1}{N} \sum_{j \neq k} q_j q_k V(x_j - x_k) P_j P_k$$

+ all other terms that contain 2 p's and 2 q's.

$\rightsquigarrow \| \Psi_t - \Psi_t^{2nd} \|_2 \leq C_t N^{-\frac{1}{2}}$.

$$\rightsquigarrow \sum_{j \neq k} q_j q_k V(x_j - x_k) P_j P_k \xrightarrow{\text{second quantization}} \sum_{a,b} a_a^* a_b^* V_{ab00}\, a_0 a_0$$

$= \langle e_a(x_1) e_b(x_2), V(x_1 - x_2) e_0(x_1) e_0(x_2) \rangle$

$\rightsquigarrow \{ e_0 = \varphi, e_1, \dots \}$ is ONB of the one-body Hilbert space.

Goal $\| \Psi_t - \varphi_t^{Bog} \|_2$ small? i.e. φ_t^{Bog} approximates Ψ_t.

where Bog stands for the Bogoliubov approximation, i.e. the dynamics given by Hamiltonian where you replace $a_0 \sim \sqrt{N}$

$$\| \Psi_t - \varphi_t^{Bog} \|_2 \leq \underbrace{\| \Psi_t - \Psi_t^{2nd} \|_2}_{\leq C_t N^{-\frac{1}{2}}} + \underbrace{\| \varphi_t^{2nd} - \varphi_t^{Bog} \|_2}_{\leq ?}$$

$\rightsquigarrow$ We assume that $V \in L^\infty$, $\| q_1 \chi \| \leq C_t N^{-\frac{1}{2}}$ $\quad \chi \in \{ \Psi_t, \Psi_t^{2nd} \}$

$\| q_1 q_2 \chi \| \leq C_t N^{-1}$

and
$$(*) \begin{cases} \sum_{a,b \neq 0} \langle \varphi_s^{Bog}, a_a^* a_b^* a_a a_b \varphi_s^{Bog} \rangle \leq C_s \\ \sum_{a \neq 0} \langle \varphi_s^{Bog}, a_a^* a_a \varphi_s^{Bog} \rangle \leq C_s \end{cases}$$

We have proven $(*)$ for Ψ_t, the proof for φ_t^{Bog} and φ_t^{2nd} is similar.

Nbr of particles not in zero_mode.

$$\| \varphi_t^{2nd} - \varphi_t^{Bog} \|_2^2 = 2 \operatorname{Re} \left(\langle \varphi_t^{Bog}, \varphi_t^{Bog} - \varphi_t^{2nd} \rangle \right)$$

$$\langle \varphi_t^{Bog}, \varphi_t^{Bog} - \varphi_t^{2nd} \rangle = -\underbrace{\int_0^t \frac{d}{ds} \langle \varphi_t^{Bog}, U_{t,s}^{Bog} U_{s,0}^{2nd} \varphi_0 \rangle \, ds}_{}$$

$$'' \quad \left[\langle \varphi_t^{Bog}, U_{t,s}^{Bog} U_{s,0}^{2nd} \varphi_0 \rangle \right]_0^t$$

$$= \left[\langle \varphi_t^{Bog}, \varphi_t^{2nd} \rangle - \langle \varphi_t^{Bog}, \varphi_t^{Bog} \rangle \right]$$

$$\|\varphi_t^{2nd} - \varphi_t^{Bog}\|_2^2 \leq \int_0^t |\langle \varphi_s^{Bog}, (H^{Bog} - H^{2nd}) \varphi_s^{2nd} \rangle| \, ds$$

$$\lesssim \sum_{k=0}^{N} \sum_{j \neq \ell} \int_0^t |\langle \varphi_s^{Bog}, a_j^* a_\ell^* V_{j\ell 00} (a_0 a_0 - N) P_k \varphi_s^{2nd} \rangle| \, ds$$

$$+ \sum_{k=0}^{N} \sum_{j \neq \ell} \int_0^t |\langle \varphi_s^{Bog}, (a_0 a_0 - N) V_{j\ell 00} a_j^* a_\ell^* P_k \varphi_s^{2nd} \rangle| \, ds + \dots$$

We will control one of the terms, and the others can be controlled similarly:

$$\sum_{k=0}^{N} \sum_{j \neq \ell} \int_0^t |\langle \varphi_s^{Bog}, a_j^* a_\ell^* V_{j\ell 00} (a_0 a_0 - N) P_k \varphi_s^{2nd} \rangle| \, ds$$

$$\lesssim \int_0^t \underbrace{\Big\| \sum_{j \neq \ell} a_j a_\ell V_{j\ell 00} \varphi^{Bog} \Big\|_2}_{(1)} \underbrace{\Big\| \sum_{k=0}^{N} (a_0 a_0 - N) P_k \varphi_s^{2nd} \Big\|_2}_{(2)} \, ds$$

$$(1) \lesssim \sum_{j \neq \ell} |V_{j\ell 00}| \, \|a_j a_\ell \varphi_s^{Bog}\|_2$$

$$\lesssim \sqrt{\sum_{j \neq \ell} |V_{j\ell 00}|^2 \sum_{j \neq \ell} \|a_j a_\ell \varphi_s^{Bog}\|_2^2}$$

- $$\sum_{j \neq \ell} |V_{j\ell 00}|^2 = \sum_{j \neq \ell} \langle e_0(x_1) e_0(x_2) V(x_1 - x_2) e_j(x_1) e_\ell(x_2) \rangle \langle e_j(x_1) e_\ell(x_2) V(x_1 - x_2) e_0(x_1) e_0(x_2) \rangle$$

$$\leq C.$$

- $$\sum_{j \neq \ell} \|a_j a_\ell \varphi_s^{Bog}\|_2^2 = \sum_{j \neq \ell} \langle \varphi_s^{Bog}, a_j^* a_\ell^* a_j a_\ell \varphi_s^{Bog} \rangle \leq C_s$$

$$\Rightarrow (1) \leq C_s$$

$$(2) = \left\| \sum_{k=0}^{N} (a_0^* a_0 - N) P_k \psi_s^{2nd} \right\|_2 \;=\!=\; \left\| \sum_{k=0}^{N} \left(\sqrt{(N-k-1)(N-k)} - N\right) P_k \psi_s^{2nd} \right\|_2$$

$$(a_0 a_0 - N) P_k = \left(\sqrt{(N-k)(N-k-1)} - N \right) P_k .$$

$$\lesssim \left\| \widehat{k+1} \, P_k \psi_s^{2nd} \right\|_2$$

$$\lesssim C_s .$$

Summarizing, we get

$$\| \psi_t^{2nd} - \psi_t^{Bog} \|_2 \leq C_t N^{-1}.$$

$$\Longrightarrow \| \psi_t - \psi_t^{Bog} \| \lesssim C_t N^{-\frac{1}{2}} + C_t N^{-1}.$$

2§ Ground stat of the Bose gas on a torus:

⇝ Compute the Ground stat of Bose-gas on a one-dimensional torus of length 2π using second quantization language.

⇝ ONB is given by $\left\{ e_k = \frac{1}{\sqrt{2\pi}} e^{ikx}, k \in \mathbb{Z} \right\}$

⇝ In second quantization, the N-body Hamiltonian reads

$$H = \sum_{k \neq 0} k^2 a_k^* a_k + \frac{1}{N} \sum_{a,b,c,d} a_c^* a_d^* a_b a_c \underbrace{V_{abcd}}_{\langle e_c e_d ,\, V(x_j - x_k) ,\, e_a e_b \rangle}$$

⇝ $\langle e_c e_d , V(x_j - x_k) e_a e_b \rangle$ is computed in one of the previous lectures.

$\langle e_c e_d , V(x_j - x_k) e_a e_b \rangle = 0$ whenever $(a+b \neq c+d) \to$ momentum conservation.

⇝ Assume without loss of generality that $\hat{V}(0) = 0$. and call k the momentum transfer.

$$H = \sum_{k \neq 0} k^2 a_k^* a_k + \frac{1}{N} \sum_k \hat{V}(k) \sum_{a,b} a_{a+k}^* a_{b-k}^* a_a a_b$$

$$= \sum_{k \neq 0} k^2 a_k^* a_k + \frac{1}{N} \sum_k \hat{V}(k) \sum_{a,b} a_{a+k}^* a_a a_{b-k}^* a_b - N^{-1} \sum_{k \neq 0} \hat{V}(k) \sum_a a_a^* a_a$$

$\langle \Omega, H\Omega \rangle = 0$ since at least one annihilation operator acting on Ω.

So, all the summands are zero. This implies that the ground state energy is below zero.

For any Ψ:

$$\langle \Psi, H\Psi \rangle = \sum_{k\neq 0} k^2 \langle \Psi, a_k^* a_k \Psi \rangle + \frac{1}{N} \sum_{k\neq 0} \hat{V}(k) \left\| \sum_{a\in\mathbb{Z}} a_{a+k}^* a_a \Psi \right\|^2 - \frac{1}{N} \sum_{k\neq 0} \hat{V}(k) \langle \Psi, \sum_e a_e^* a_e \Psi \rangle$$

We have $\sum_{k\neq 0} \hat{V}(k) = V(0)$, we get for all Ψ

$$\langle \Psi, H\Psi \rangle = \sum_{k\neq 0} k^2 \langle \Psi, a_k^* a_k \Psi \rangle + \frac{1}{N} \sum_{k\neq 0} \hat{V}(k) \left\| \sum_{a\in\mathbb{Z}} a_{a+k}^* a_a \Psi \right\|^2 - \frac{1}{N} V(0) \langle \Psi, \sum_e a_e^* a_e \Psi \rangle$$

$$\geq \sum_{k\neq 0} k^2 \langle \Psi, a_k^* a_k \Psi \rangle - \frac{1}{N} V(0) \underbrace{\langle \Psi, \sum_e a_e^* a_e \Psi \rangle}$$

Since the Ground State is below zero, we get

$$0 > \langle \Psi^G, H\Psi^G \rangle \geq \sum_{k\neq 0} \langle \Psi^G, a_k^* a_k \Psi^G \rangle - \frac{1}{N} V(0) \underbrace{\langle \Psi^G, \sum_e a_e^* a_e \Psi^G \rangle}_{=N}$$

where we have used that $\sum_e a_e^* a_e$ gives the total nb. of particles.

$$\Rightarrow \langle \Psi^G, \sum_{k\neq 0} a_k^* a_k \Psi^G \rangle \leq V(0).$$

$\Rightarrow$ This expression means that the expected number of particles not in the 0-mode is bounded by a constant.