Recall

Theorem Let $V \in L^1 \cap L^\infty$, let $\beta < \frac{1}{6}$, assume that $\exists T > 0$ such that the solution of the Hartree is L^∞- bounded on $[0,T]$ $\left(\|\varphi_t\|_\infty < C \quad \forall t \in [0,T] \right)$, Then

$$\| \gamma_t^{\psi} - |\varphi_t\rangle\langle\varphi_t| \|_{tr} \xrightarrow[N\to\infty]{} 0$$

φ_t solution of the Hartree : $i\frac{d\varphi_t(x)}{dt} = \left(-\Delta + N^{3\beta} V(N^\beta \cdot) * |\varphi_t|^2(x)\right)\varphi_t(x)$

ψ_N^t solution of Many-body Schrödinger: $i\frac{d\psi_N}{dt} = \sum_{j=1}^{N}(-\Delta_j) + \sum_{j\neq k} V_N(x_j - x_k)$

$V_N(x) = N^{-1+3\beta} V(N^\beta x)$

$\to$ We will prove $\alpha = \langle \psi_N, \hat{\omega}\, n\psi \rangle \xrightarrow[N\to\infty]{} 0$ $\qquad \omega(k) = \begin{cases} \frac{k}{N^\lambda} & k \leq N^\lambda \\ 1 & \text{otherwise.} \end{cases}$

Lemmas from previous lectures: $\quad , \ \hat{n} = \widehat{\sqrt{\frac{k}{N}}}$

(i) $\|q_1 \hat{\omega} \psi\| \leq \|\hat{n}\hat{\omega}\psi\|$

(ii) $\|q_1 q_2 \hat{\omega}\psi\| \leq \|\hat{n}^2 \hat{\omega}\psi\|$

(iii) $\|A\|_{op}^2 \leq \|A^* A\|_{op}$

Proof $\frac{d\alpha}{dt}$ involves three terms:

$\mathrm{I} = 0$

$\mathrm{II} := N(N-1)\langle \psi, P_1 P_2 O_{12}\, q_1 q_2 (\hat{\omega}_2 - \hat{\omega})\psi\rangle$

$\mathrm{III} := N(N-1)\langle \psi, q_1 q_2 O_{12} P_1 q_2 (\hat{\omega}_1 - \hat{\omega})\psi\rangle$

$O_{12} = V_N(x_1 - x_2) - V_N * |\varphi_t|^2(x_1) - V_N * |\varphi_t|^2(x_2)$. $\quad ; \quad V_N(x) = N^{-1+3\beta} V(N^\beta x)$.

We start by III.

$|\mathrm{III}| = N(N-1)\langle \psi, P_1 q_2 O_{12}\, q_1 q_2 (\hat{\omega}_1 - \hat{\omega})\hat{\mathbb{1}}_{k\leq N^\lambda - 1}\psi\rangle$

$= N(N-1)\langle \psi, \hat{\mathbb{1}}_{k\leq N^\lambda} P_1 q_2 O_{12}\, q_1 q_2 (\hat{\omega}_1 - \hat{\omega})\psi\rangle$

$\underset{\text{Cauchy Schwarz}}{\leq} N^2 \underbrace{\|q_2 \hat{\mathbb{1}}_{k\leq N^\lambda}\psi\|_2}_{(a)} \underbrace{\|P_1 O_{12}\|_{op}}_{(b)} \underbrace{\|q_1 q_2 (\hat{\omega}_1 - \hat{\omega})\psi\|_2}_{(c)}$

For (a)

$\|q_2 \hat{\mathbb{1}}_{k\leq N^\lambda}\psi\|_2^2 \underset{(i)}{\leq} \|\hat{n}\, \mathbb{1}_{k\leq N^\lambda}\psi\|_2^2 = \langle \psi, \widehat{\frac{k}{N}}\, \mathbb{1}_{k\leq N^\lambda}\psi\rangle \leq N^{\lambda-1}\alpha_t$

since $\frac{k}{N}\mathbb{1}_{k\leq N^\lambda} \leq N^{\lambda-1}\omega(k)$

$\Rightarrow (a) \leq N^{\frac{\lambda-1}{2}}\sqrt{\alpha_t}$.

$$(b) = \|P_1 O_{12}\|_{op} = \|P_1 \left(V_N(x_1-x_2) - V_N * |\varphi_t|^2(x_1) - V_N * |\varphi_t|^2(x_2)\right)\|_{op}$$

$$\leq 2\|P_1\|_{op} \|V_N * |\varphi_t|^2\|_\infty + \|P_1 V_N(x_1-x_2)\|_{op}$$

$$\underset{(iii)}{\leq} 2\|\varphi_t\|_\infty^2 \|V_N\|_1 + \sqrt{\underbrace{\|P_1 V_N^2(x_1-x_2) P_1\|_{op}}}$$

$\leq \|V_N\|_2^2 \ \|\varphi_t\|_\infty^2$

$$\leq 2\|\varphi_t\|_\infty^2 \|V_N\|_{L^1} + \|\varphi_t\|_\infty \|V_N\|_2 .$$

$$\|V_N\|_1 = \int N^{-1+3\beta} |V(N^\beta x)| \, d^3x \underset{\substack{y = N^\beta x \\ d^3y = N^{3\beta} d^3x}}{=} N^{-1} \int |V(y)| \, d^3y = N^{-1}\|V\|_1 .$$

$$\|V_N\|_2^2 = \int N^{-2+6\beta} |V(N^\beta x)|^2 d^3x = N^{-2+3\beta} \|V\|_2^2 \implies \|V_N\|_2 = N^{-1+\frac{3}{2}\beta} \|V\|_2 .$$

$$\|P_1 O_{12}\|_{op} \leq C N^{-1+\frac{3}{2}\beta}.$$

For term (c)

$$\|q_1 q_2 (\hat{w}_1 - \hat{w})\psi\|_2 \underset{(ii)}{\leq} \|\hat{n}^2 (\hat{w}_1 - \hat{w})\psi\|_2 = \underbrace{\sqrt{\langle \psi, \widehat{\left(\tfrac{K}{N}\right)^2} (\hat{w}_1 - \hat{w})^2 \psi\rangle}}_{\leq C N^{-1}\sqrt{\alpha_t}}$$

Since $\left(\frac{K}{N}\right)^2 (w_1 - w)^2 \leq N^{-2} w(K)$ (where we have used $K < N^\lambda$ on support of $(w_1 - w)$ and $|w_1 - w| < N^{-\lambda}$)

This implies combining (a), (b) & (c),

$$|\mathrm{III}| \leq N^2 C . N^{\frac{\lambda-1}{2}} \sqrt{\alpha_t} \ C N^{-1+\frac{3}{2}\beta} N^{-1}\sqrt{\alpha_t}$$

$$\leq C \alpha_t N^{\frac{-1+\lambda+3\beta}{2}} .$$

Now, we estimate term II.

$$\mathrm{II} := N(N-1) \langle \psi, P_1 P_2 O_{12} q_1 q_2 (\hat{w}_2 - \hat{w}) \psi \rangle$$

$$P_1 P_2 \ V_N * |\varphi_t|^2 (x_1) \ q_1 q_2 = 0$$

$$P_1 P_2 \ V_N * |\varphi_t|^2 (x_2) \ q_1 q_2 = 0$$

$$w_2(K) = \begin{cases} \frac{K+2}{N^\lambda} & K \leq N^\lambda - 2 \\ 1 & \text{if } K \geq N^\lambda - 1 \end{cases}$$

$$w(K) = \begin{cases} \frac{K}{N^\lambda} & \text{if } K \leq N^\lambda \\ 1 & \text{if } K \geq N^\lambda + 1 \end{cases}$$

$$\mathrm{II} = N(N-1) \langle \psi, P_1 P_2 \ V_N(x_1 - x_2) \ q_1 q_2 \ (\hat{w}_2 - \hat{w}) \hat{\mathbb{1}}_{K \leq N^\lambda} \psi \rangle$$

$$= N(N-1) \langle \psi, \hat{\mathbb{1}}_{K \leq N^\lambda + 2} \ P_1 P_2 \ V_N(x_1 - x_2) \ q_1 q_2 \ (\hat{w}_2 - \hat{w}) \psi \rangle$$

$$= N \sum_{j=2}^{N} \langle \psi, \hat{\mathbb{1}}_{K \leq N^\lambda + 2} P_1 P_j \ V_N(x_1 - x_j) \ q_1 q_j \ (\hat{w}_2 - \hat{w}) \psi \rangle$$

$$\underset{\text{Cauchy Schwarz}}{\leq} N \left\| \sum_{j=2}^{N} q_j V_N(x_1-x_j) P_1 P_j \hat{\mathbb{1}}_{K \leq N^\lambda+2} \Psi \right\|_2 \underbrace{\| q_1 (\hat{\omega}_2 - \hat{\omega}) \Psi \|_2}$$

$$\cdot \ \| q_1 (\hat{\omega}_2 - \hat{\omega}) \Psi \|_2 \underset{(i)}{\leq} \| \hat{n} (\hat{\omega}_2 - \hat{\omega}) \Psi \|_2 = \sqrt{\left\langle \Psi, \frac{\hat{K}}{N} (\hat{\omega}_2 - \hat{\omega})^2 \Psi \right\rangle}$$

$$\frac{K}{N} (\omega_2 - \omega)^2 \leq 4 N^{-1-\lambda} \omega(\kappa)$$

$$\| q_1 (\hat{\omega_2} - \hat{\omega}) \Psi \|_2 \leq C N^{\frac{-1-\lambda}{2}} \sqrt{\alpha_t}$$

Let us estimate the first term:

$$\left\| \sum_{j=2}^{N} q_j V_N(x_1-x_j) P_1 P_j \hat{\mathbb{1}}_{K \leq N^\lambda+2} \Psi \right\|_2^2$$

$$= \sum_{j,\ell=2}^{N} \left\langle \Psi, \hat{\mathbb{1}}_{K \leq N^\lambda+2} P_1 P_\ell \underbrace{V_N(x_1-x_\ell)} q_\ell q_j \underbrace{V_N(x_1-x_j) P_1 P_j} \hat{\mathbb{1}}_{K \leq N^\lambda+2} \Psi \right\rangle$$

$$= \sum_{j=2}^{N} \left\langle \Psi, \hat{\mathbb{1}}_{K \leq N^\lambda+2} P_1 P_j V_N(x_1-x_j) q_j V_N(x_1-x_j) P_1 P_j \hat{\mathbb{1}}_{K \leq N^\lambda+2} \Psi \right\rangle \quad \ldots \quad \text{(diag)}$$

$$+ \sum_{j \neq \kappa} \left\langle \Psi, \hat{\mathbb{1}}_{K \leq N^\lambda+2} P_1 P_\ell V_N(x_1-x_\ell) q_\ell q_j V_N(x_1-x_j) P_1 P_j \hat{\mathbb{1}}_{K \leq N^\lambda+2} \Psi \right\rangle \quad \ldots \quad \text{(off-diag)}$$

$$\text{(diag)} = N \left\langle \Psi, \hat{\mathbb{1}}_{K \leq N^\lambda+2} P_1 P_2 V_N(x_1-x_2) q_2 V_N(x_1-x_2) P_1 P_2 \hat{\mathbb{1}}_{K \leq N^\lambda+2} \Psi \right\rangle$$

$$\leq N \| P_1 V_N(x_1-x_2) \|_{op}^2 \leq N \| P_1 V_N^2 (x_1-x_2) P_1 \|_{op} = N \| V_N^2 * |\varphi_t|^2 (x_2) P_1 \|_{op}$$

$$\leq C N \| V_N \|_2^2 \leq C N . N^{-2+3\beta} = C N^{-1+3\beta}.$$

$$\text{(off-diag)} = \sum_{j \neq \ell = 2}^{N} \left\langle \Psi, \hat{\mathbb{1}}_{K \leq N^\lambda+2} P_1 P_\ell V_N(x_1-x_\ell) q_\ell q_j V_N(x_1-x_j) P_1 P_j \hat{\mathbb{1}}_{K \leq N^\lambda+1} \Psi \right\rangle$$

$$= \sum_{j \neq \ell = 2}^{N} \left\langle q_j \hat{\mathbb{1}}_{K \leq N^\lambda+2} \Psi, P_1 P_\ell \sqrt{V_N(x_1-x_j)} \sqrt{V_N(x_1-x_\ell)} \sqrt{V_N(x_1-x_j)} \sqrt{V_N(x_1-x_\ell)} P_1 P_j q_\ell \hat{\mathbb{1}}_{K \leq N^\lambda+2} \Psi \right\rangle$$

$$= \sum_{j \neq \ell = 2}^{N} \left\langle q_j \hat{\mathbb{1}}_{K \leq N^\lambda+2} \Psi, P_1 \sqrt{V_N(x_1-x_j)} P_\ell \sqrt{V_N(x_1-x_\ell)} \sqrt{V_N(x_1-x_j)} P_j \sqrt{V_N(x_1-x_\ell)} P_1 q_\ell \hat{\mathbb{1}}_{K \leq N^\lambda+2} \Psi \right\rangle$$

$$\underset{\text{Cauchy Schwarz}}{\leq} N^2 \| q_1 \hat{\mathbb{1}}_{K \leq N^\lambda+2} \Psi \|_2^2 \ \| P_1 \sqrt{V_N(x_1-x_j)} \|_{op}^4$$

$$\cdot \ \| P_1 \sqrt{V_N(x_1-x_j)} \|_{op}^4 \leq \| P_1 V_N(x_1-x_j) P_1 \|_{op}^2 = \| V_N * |\varphi_t|^2 (x_j) P_1 \|_{op}^2 \leq \| V_N * |\varphi_t|^2 \|_\infty^2 \| P_1 \|_{op}^2$$

$$\leq C \| V_N \|_1^2 \leq C N^{-2}.$$

$$\cdot \ \| q_1 \hat{\mathbb{1}}_{K \leq N^\lambda+2} \Psi \|^2 \underset{(i)}{\leq} \| \hat{n} \hat{\mathbb{1}}_{K \leq N^\lambda+2} \Psi \|_2^2 = \left\langle \Psi, \frac{\hat{K}}{N} \hat{\mathbb{1}}_{K \leq N^\lambda+2} \Psi \right\rangle$$

we have here $\frac{K}{N} \mathbb{1}_{K \leq N^\lambda + 2} \leq C N^{\lambda-1} \omega(k)$

$$\Rightarrow \|q_1 \hat{\mathbb{1}}_{K \leq N^\lambda + 2} \psi\|_2^2 \leq C N^{\lambda-1} \langle \psi, \hat{\omega} \psi \rangle = C N^{\lambda-1} \alpha_t .$$

$$\Rightarrow \|\sum_{j=2}^{N} q_j V_N(x_1 - x_j) p_1 p_j \mathbb{1}_{K \leq N^\lambda + 2} \psi\|_2^2 \leq C N^{-1+3\beta} + C N^{-1+\lambda} \alpha_t$$

$$\Rightarrow |\mathrm{III}| \leq C . N \, N^{\frac{-1-\lambda}{2}} \sqrt{\alpha} \left(C N^{-1+3\beta} + C N^{-1+\lambda} \alpha_t \right)^{\frac{1}{2}} \leq C . \alpha + C N^{\frac{3\beta-\lambda}{2}} \sqrt{\alpha_t} \leq C\alpha_t + C N^{3\beta-\lambda}$$

$$\Rightarrow |\mathrm{II}| + |\mathrm{III}| \leq C \alpha_t N^{\frac{-1+\lambda+3\beta}{2}} + C\alpha_t + C N^{3\beta-\lambda}$$

We choose $\lambda = 1 - 3\beta$, we get:

$$|\mathrm{II}| + |\mathrm{III}| \leq C\alpha + C N^{-1+6\beta}$$

$$\Rightarrow \frac{d\alpha}{dt} \leq C\alpha + C N^{-1+6\beta}$$

By using Grönwall; we get:

$$\alpha(t) \leq C_t \left(\alpha(0) + N^{-1+6\beta} \right)$$

for $\beta < \frac{1}{6}$, we can close Grönwall.

If we start from $\alpha(0) \xrightarrow[N \to \infty]{} 0$ then $\alpha(t) \xrightarrow[N \to \infty]{} 0$ $\Rightarrow \| \gamma_\psi^t - |\varphi_t\rangle\langle\varphi_t| \|_{tr} \xrightarrow[N \to \infty]{} 0$