Fermi - gases

One goal of this chapter is to prove the validity of the "mean-field" approximation.
There are two main differences compared to Bosons:

1) We will have a set of N different "orbitals" that describe the motion of the gas.
2) Considering a gas of fixed volume, the kinetic energy will grow much faster compared to Bosons. Kinetic energy per particle $N^{\frac{2}{d}} \left(= \left(N^{\frac{1}{d}}\right)^2 = (k_{max})^2\right)$
   $\rightarrow$ If we want a balance between kinetic and potential terms, interaction must be stronger.

One important argument will be, that on the other hand antisymmetrization reduces fluctuations in the force.

Outlook:
1) Discuss the fluctuations for N Fermions
2) Consequence of 1) on the motion of an impurity
3) Come back to prove mean-field theory.

Fluctuations in the dense Fermi-gas

Let $\{e_k : k \in I\}$ be an ONB of the one-particle Hilbert space (I countable)

$$\psi = \bigwedge_{j=1}^{N} e_j(x_j), \qquad \text{let } V \text{ be some multiplication operator.} \quad V \in L^\infty$$

$$\mathbb{E}(\Sigma V) := \sum_{j=1}^{N} \langle \psi, V(x_j)\, \psi \rangle \qquad \mathbb{E}((\Sigma V)^2) := \sum_{j,k=1}^{N} \langle \psi\, V(x_j)\, V(x_k)\, \psi \rangle$$

$$\mathrm{Var}(\Sigma V) = \mathbb{E}((\Sigma V)^2) - \mathbb{E}(\Sigma V)^2$$

We use $\sum_{j\in I} |e_j\rangle\langle e_j|_k = id \quad \forall k$

$$\mathbb{E}(\Sigma V) = N \langle \psi, V(x_1)\, \psi \rangle = N \sum_{j,k\in I} \underbrace{\langle \psi, P_j^1\, V(x_1) P_k^1\, \psi\rangle}_{=0 \text{ if } j\neq k} \qquad P_j^k = |e_j\rangle\langle e_j|_k$$

$$= N \sum_{j\in I} \langle \psi, P_j^1\, V(x_1)\, P_j^1\, \psi\rangle = N \sum_{j=1}^{N} \langle \psi, P_j^1\, V(x_1)\, P_j^1\, \psi \rangle =$$

$$= N \sum_{j=1}^{N} \langle \psi | e_j\rangle \langle e_j | V(x_1)\, e_j\rangle \langle e_j | \psi\rangle = N \sum_{j=1}^{N} \langle e_j\, V(x_1)\, e_j\rangle . \qquad \underbrace{\langle \psi | e_j\rangle\langle e_j | \psi\rangle}_{1/N}$$

$$= \sum_{j=1}^{N} \langle e_j\, V(x)\, e_j \rangle$$

$$\mathbb{E}((\Sigma V)^2) = \underbrace{N(N-1)\, \langle \psi, V(x_1)\, V(x_2)\, \psi\rangle}_{a} + \underbrace{N \langle \psi, V(x_1)^2\, \psi\rangle}_{b}$$

$$b) = \sum_{j=1}^{N} \langle e_j\, V^2(x)\, e_j\rangle = \sum_{j=1}^{N} \sum_{k\in I} \langle e_j\, V(x_1) | e_k\rangle\langle e_k | V(x_1)\, e_j\rangle$$

$$a)\ N(N-1) \sum_{j,k,l,m\in I} \underbrace{\langle \psi, P_j^1 P_k^2\, V(x_1)\, V(x_2)\, P_l^1 P_m^2\, \psi\rangle}_{=0 \text{ if } \{j,k\}\neq\{l,m\}} = N(N-1) \sum_{j,k=1}^{N} \langle \psi P_j^1 P_k^2\, V(x_1)\, V(x_2)\, P_j^1 P_k^2\, \psi\rangle$$
$$+ \langle \psi\, P_j^1 P_k^2\, V(x_1)\, V(x_2)\, P_k^1 P_j^2\, \psi\rangle$$

$$= N(N-1) \sum_{j,k=1}^{N} \langle e_j\, V(x)\, e_j\rangle \langle e_k\, V(x)\, e_k\rangle \overbrace{\langle \psi, |e_j^1\rangle |e_k^2\rangle \langle e_k^1| \langle e_k^2|\, \psi\rangle}^{\frac{1}{N(N-1)}}$$
$$+ \langle e_j\, V(x)\, e_k\rangle \langle e_k\, V(x)\, e_j\rangle \langle \psi, |e_j^1\rangle |e_k^2\rangle \langle e_j^1| \langle e_k^2\, \psi\rangle = -\frac{1}{N(N-1)}$$

$$= \sum_{j,q=1}^{N} \langle e_j V(x) e_j \rangle \langle e_q V(x) e_q \rangle - \langle e_j V(x) e_q \rangle \langle e_q V(x) e_j \rangle$$

$$= \mathbb{E}(qV_j)^2$$

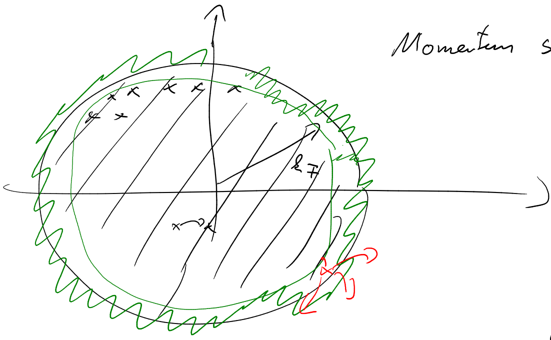
$$\mathrm{Var}(qV) = \sum_{j=1}^{N} \sum_{q \notin \{1,..,N\}} \langle e_j V(x) e_q \rangle \langle e_q | V(x) | e_j \rangle$$

In the classical setting we expect a variance of order N.

Let us assume that a) $\hat{V}$ has compact support

b) the occupied states are the ones with smallest momentum

Momentum space



Since $\hat{V}$ has compact support:

$\langle e_j V e_q \rangle = 0$ whenever $|j-q| > q_{max}$

Only values of j close to the surface contribute.

Given j there are finitely many possible "partners" q

$$q_F = N^{\hat{d}}$$

$$\mathrm{Var}(qV) = q_F^{d-1} \cdot C'$$

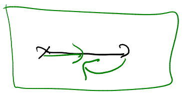
At high temperature, respectively when the particles are scattered in momentum space, the "surface" might be as large as the volume and the variance will be of the same order as in the classical situation.

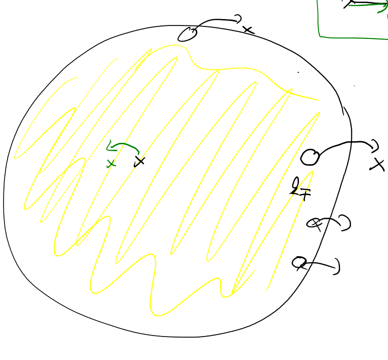
The tracer and the Fermi gas

To better understand the physical impact of the last chapter on the dynamics we consider a "tracer" particle interacting with an ideal Fermi gas.

The effect we found above (reduction of fluctuations of the potential) already gives that the time the tracer moves frictionless in the gas is much larger than in the classical or Bosonic case.

In addition to that the fluctuations come from particles of high momenta, thus they only have a short live-time. So the impulse ($F\Delta t$) is further suppressed.





typically $\Delta E \sim q_F$

transition $\int_0^t A e^{-i\Delta E s} ds$