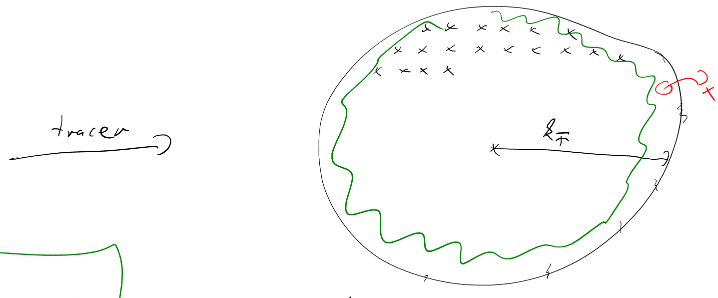
momentum space



$$\boxed{H = -\Delta_y + \sum_{j=1}^{N} -\Delta_{x_j} + \lambda \sum_{j=1}^{N} V(x_j - y)}$$

$$\bigwedge_{|k|<k_F} e_k \qquad e_k = e^{ikx} \text{ on the torus } [0,2\pi[^d$$

Theorem: For $\lambda = k_F^{-\frac{1}{2}}$ (in $d=3$) the leading order dynamics is given by free evolution on a time scale of order one.
In more details: If $\hat{V}$ is bounded and compactly supported, then it holds for any $t>0$ that $\|\psi_t - \psi_t^{free}\|_2 \xrightarrow[k_F\to\infty]{} 0$.

Remark: a) The # of particles $N \sim k_F^3$, density of the gas $\sim k_F^3$.
For a classical gas we would expect fluctuations in the force of order $k_F^{-\frac{1}{2}} \cdot k_F^{3/2} \approx k_F$, thus on a time-scale of order one a large fluctuation.

b) The Theorem holds in fact for arbitrarily large volume, considering the torus $[0,L[^3$, the estimate of $\|\psi_t - \psi_t^{free}\|_2$ will be uniform in L!

"Proof": We will not give a full rigorous proof, but control the most interesting terms.

$$H^{free} = -\Delta_y + \sum_{j=1}^{N} -\Delta_{x_j} + C$$

$$i\frac{d}{dt}\psi_t^{free} = H^{free}\psi_t^{free} \qquad \psi_0 = \psi_0^{free} = \eta(y)\cdot\bigwedge_{|k|<k_F} e_k$$

$$i\frac{d}{dt}\psi_t = H\psi_t$$

$U(t,s)$ is the propagator generated by H $\qquad \psi_t = U(t,0)\psi_0$

$U^{free}(t,s)$ " " " " H^{free} $\qquad \psi_t^{free} = U^{free}(t,0)\psi_0^{(free)}$

$$\boxed{U^{free}(t,s) - U(t,s)} = -i\int_s^t i\frac{d}{d\tau}\Big(U(t,\tau)\,U^{free}(\tau,s)\Big)\,d\tau = \boxed{-i\int_s^t \underbrace{U(t,\tau)}_{U-U^{free}+U^{free}}\left(H^{free}-H\right)U^{free}(\tau,s)\,d\tau}$$

$$\int_s^t i\frac{d}{d\tau}\Big(U(t,\tau)U^{free}(\tau,s)\Big)d\tau = \left[U(t,\tau)U^{free}(\tau,s)\right]_s^t$$

$$\Rightarrow U^{free}(t,s) - U(t,s) = -i\int_s^t U^{free}(t,\tau)\left(H^{free}-H\right)U^{free}(\tau,s)\,d\tau + i\int_s^t\left(-i\int_\tau^t U(t,\sigma)\left(H^{free}-H\right)U^{free}(\sigma,\tau)\right)\left(H^{free}-H\right)U^{free}(\tau,s)\,d\sigma\,d\tau + \dots$$

It follows that:

$$\psi_t^{free} - \psi_t = -i\int_0^t U^{free}(t,\tau)\left(H^{free}-H\right)U^{free}(\tau,0)\,\psi_0\,d\tau + \int_0^t\int_\tau^t U(t,\sigma)\left(H^{free}-H\right)U^{free}(\sigma,\tau)\left(H^{free}-H\right)U^{free}(\tau,0)\,\psi_0\,d\sigma\,d\tau + \dots$$

To estimate $\psi_t^{free} - \psi_t$ it is sufficient to control $P(\psi_t^{free} - \psi_t)$ where P projects on that part of Hilbert space where the Fermions are in the state $\bigwedge_{|k|<k_F} e_k$:

$$P\psi_t^{free} = \psi_t^{free} \Rightarrow \|\psi_t^{free} - \psi_t\|_2 = \|P\psi_t^{free} - P\psi_t\|_2 + \underbrace{\|(1-P)\psi_t^{free} - \psi_t\|_2}_{\|(1-P)\psi_t\|_2}$$

Since ψ and ψ^{free} are normalized to one it suffices to control

$$\|P\psi_t^{free} - P\psi_t\|_2$$

Note that $H^{free} - H = C_1' - \lambda \sum_{j=1}^{N} V(x_j - y)$.

It is convenient to use the language of second quantization (w.l.o.g. $\hat V(0) = 0$)

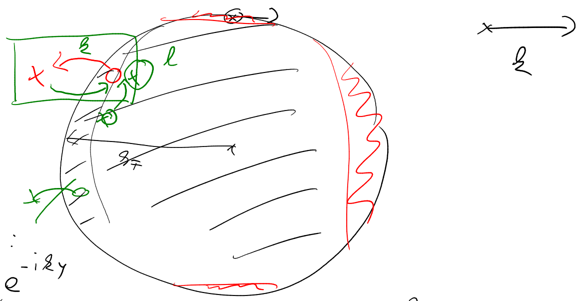
$$H^{free} - H = C - \lambda \sum_{\ell} \sum_{k \neq 0} \hat V(k)\, a^*(k+\ell)\, a(\ell) \underbrace{\sum_j b^*(j-k)\, b(j)}_{e^{iky}}$$

where a, a^* are the creation and annihilation operators of fermions b^*, b for tracer

It follows that $P(H^{free} - H)\, U^{free} \Psi_0 = P \cdot C\, U^{free} \Psi_0$!

Let us look at the second order of Duhamel: The goal is to find a constant C_1' such that the leading order contribution from V drops.

$$P \int U^{free}(t,\tau)\, C_1\, U^{free}(\tau, \sigma)\, C_1\, U^{free}(\sigma, 0)\, \Psi_0\, d\sigma\, d\tau + O + \boxed{P \int U^{free}(t,\tau)\, \underline{V}\, U^{free}(\tau,\sigma)\, \underline{\underline{V}}\, U^{free}(\sigma,0)\, \Psi_0\, d\sigma\, d\tau}$$



The only contribution in the green box comes from:

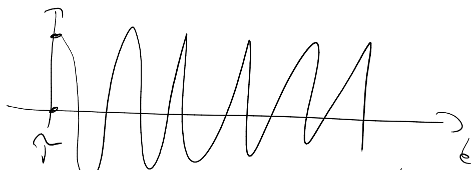
$$\lambda^2\, P \int U^{free}(t,\tau) \sum_{\substack{k, |\ell| < k_F \\ |k+\ell| \geq k_F}} a(k+\ell)\, \overset{e^{iky}}{a^*(\ell)}\, U^{free}(\tau,\sigma)\, a^*(k+\ell)\, \overset{e^{-iky}}{a(\ell)}\, U^{free}(\sigma,0)\, \Psi_0\, d\sigma\, d\tau\ \ |\hat V(k)|^2 = (*)$$

$t \leq \tau \leq \sigma \leq 0$

Assume that the tracer has momentum j the

$e^{-i(E^{GS} + j^2)(t-\tau)}$ $\quad e^{-i\left(E^{GS} - \ell^2 + (k+\ell)^2 + (j-k)^2\right)\boxed{(\tau-\sigma)}}$ $\quad e^{-i(E^{GS} + j^2)\sigma}$

The σ-integral



For most momenta $-\ell^2 + (k+\ell)^2 \sim k_F$

$$(*) = \lambda^2 |\hat V(k)|^2 \int e^{\cdots} \sum e^{-i\, E^{GS} - \ell^2 + (k+\ell)^2 + (j-k)^2 (\tau - s) - i(E^{GS} + j^2)\sigma} \cdots\ d\sigma\, d\tau$$

The σ-dependence in the exponent is $e^{+i(\underbrace{(k+\ell)^2 - \ell^2}_{\sim k_F} + \underbrace{(j-k)^2 - j^2}_{\sim 1})\sigma} = e^{i((k+\ell)^2 - \ell^2 + 1)\sigma}\, e^{i((j-k)^2 - j^2 - 1)\sigma}$

$$\frac{\frac{d}{d\sigma} e^{i((k+\ell)^2 - \ell^2 + 1)\sigma}}{(k+\ell)^2 - \ell^2 + 1}$$

non-negative

Integrating by parts we get boundary terms and another integral. The boundary-term will be of leading order!

Since for the other integral we can again integrate by parts and get an additional factor $\sim \frac{1}{k_F}$

$\sigma = t$ vanishes since this implies $\tau = t$, thus the τ-integral is over an interval of length 0 $\int_t^t \cdots d\tau = 0$

So the main contribution is:

$$\sum_{k, \ell} \lambda^2 |\hat V(k)|^2 \int_0^t U^{free}(t,\tau)\, \underset{\uparrow}{\ } U^{free}(\tau, 0)\, \Psi_0\, d\tau \qquad \frac{(j-k)^2 - j^2 - 1}{(k+\ell)^2 - \ell^2 + 1}$$

we can choose C_1 such that the first order Duhamel drops with this expression.

Order of magnitude: $\frac{\lambda^2}{k_F} \cdot k_F^2$ area of the Fermi-surface

$\lambda \sim k_F^{-\frac{1}{2}}$ the leading order is a) constant b) of order one.