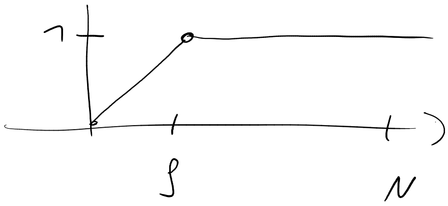
$$\boxed{\mathring{\alpha} \le C\left(\frac{\sqrt{N}}{\varrho} + \alpha\right)}$$



$$N = \varrho \cdot \Lambda$$

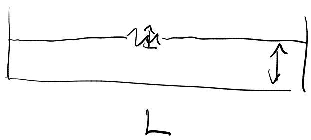
$$\mathring{\alpha} \le C\left(\sqrt{\frac{\Lambda}{\varrho}} + \alpha\right)$$

↑ very small iff $\Lambda \ll \varrho$

Speed of sound in the Bose gas

The goal is to show that there are sound waves travelling through the gas, i.e. waves with velocity bounded from below. We will get an understanding starting from the N-body Schrödinger.

We already did N-body to Hartree. Thus we show that solutions of the Hartree-equation show this behaviour.



Torus, Volume $\Lambda = L^d \gg 1$

$d = 1$

$$\varphi_0 = \frac{1}{\sqrt{\Lambda}}(1+\varepsilon) \qquad \|\varphi_0\|_2 = 1$$

$$\Downarrow$$

$$\varphi_t, \qquad i\partial_t \varphi_t = \left(-\Delta + \frac{N}{\varrho} V * |\varphi_t|^2\right)\varphi_t \qquad \Big|\cdot\sqrt{\Lambda} \qquad \varphi_t =: (1+\varepsilon_t)\frac{1}{\sqrt{\Lambda}}$$

$$\Downarrow$$

$$i\frac{d}{dt}\varepsilon = -\Delta\varepsilon + \cancel{\frac{1}{\Lambda}} V * (1+\varepsilon)(1+\varepsilon^*)\,(1+\varepsilon)$$

In order to see the sound wave we assume that ε is not only localized, but has also small amplitude. We linearize the equation and get a good approx, since the quadratic and cubic terms (in ε) are small compared to the linear ones:

$$\boxed{i\frac{d}{dt}\varepsilon = -\Delta\varepsilon + \cancel{\frac{1}{\Lambda}\hat{V}(0)\,\varepsilon} + \cancel{\frac{1}{\Lambda}} V * \varepsilon + \cancel{\frac{1}{\Lambda}} V * \varepsilon^*}$$

Remark: The constant $\frac{\hat{V}(0)}{\Lambda}$ gives only a global phase for φ_t.

In other words: we assume w.l.o.g that $\hat{V}(0) = 0$ from the very beginning.
(the general case only has a different global phase for φ_t)

In Fourier space the equation looks beautiful:

$$i\frac{d}{dt}\hat{\varepsilon} = k^2\hat{\varepsilon} + \hat{V}\cdot\hat{\varepsilon} + \hat{V}\cdot\widehat{\varepsilon^*}$$

A few words on the effective potential: Coupling parameter: $\frac{1}{\varrho}$
total particle #: $N = \varrho\cdot\Lambda$

$$\widehat{\varepsilon^*}(k) = \widehat{\varepsilon(-k)}^*$$

$$\hat{\varepsilon}(k) = \int e^{ikx}\,\varepsilon(x)\,d^3x$$

$$\left(\hat{\varepsilon}(k)\right)^* = \int e^{-ikx}\,\varepsilon^*(x)\,d^3x = \widehat{\varepsilon^*}(-k)$$

$$i\frac{d}{dt}\hat{\varepsilon}(\ell) = \ell^2\hat{\varepsilon} + \hat{V}\hat{\varepsilon} + \hat{V}\left(\hat{\varepsilon}(-\ell)\right)^* \qquad \Big| \; *$$

$$-i\frac{d}{dt}\hat{\varepsilon}^*(-\ell) = \ell^2\hat{\varepsilon}^*(-\ell) + \hat{V}\hat{\varepsilon}^*(-\ell) + \hat{V}\hat{\varepsilon}(\ell)$$

$$i\frac{d}{dt}\begin{pmatrix}\hat{\varepsilon}(\ell)\\ \hat{\varepsilon}(-\ell)^*\end{pmatrix} = \begin{pmatrix}\ell^2+\hat{V} & \hat{V}\\ -\hat{V} & -\ell^2-\hat{V}\end{pmatrix}\begin{pmatrix}\hat{\varepsilon}(\ell)\\ \hat{\varepsilon}^*(-\ell)\end{pmatrix}$$

we can treat all ℓ's separately

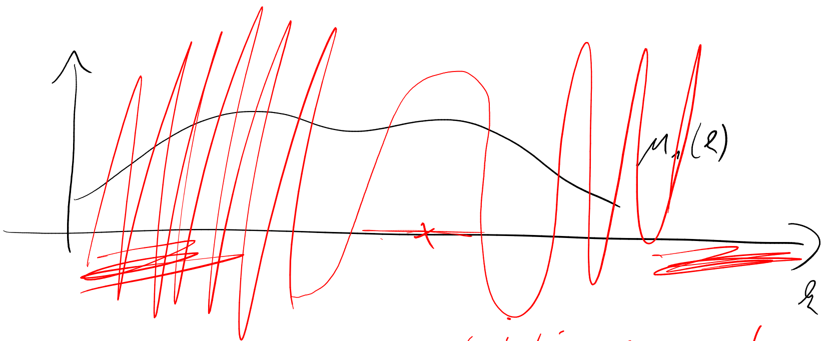
Remark: The matrix is not self-adjoint, so $\hat{\eta}(\ell) := \begin{pmatrix}\hat{\varepsilon}(\ell)\\ \hat{\varepsilon}^*(-\ell)\end{pmatrix}$ changes its L^2-norm.

$$\begin{vmatrix}\underbrace{\ell^2+\hat{V}}_{a}-\underbrace{\lambda}_{b} & \hat{V}\\ -\hat{V} & -\ell^2-\hat{V}-\lambda\end{vmatrix} = +\lambda^2 - (\ell^2+\hat{V})^2 + \hat{V}^2 = 0$$

$$\Rightarrow \lambda^2 = \ell^4 + 2\ell^2\hat{V} + \cancel{\hat{V}^2} - \cancel{\hat{V}^2}$$

$$\lambda_\ell = \pm|\ell|\sqrt{\ell^2+2\hat{V}}$$

$$\Rightarrow \begin{pmatrix}\varepsilon(x)\\ \varepsilon^*(x)\end{pmatrix} = \int e^{i\lambda_\ell t - i\ell x}\cdot\mu_1(\ell)\cdot e_1\, d^3x + \int e^{-i\lambda_\ell t - i\ell x}\mu_2(\ell)\, e_2\, d^3x$$



main contribution comes from places where $\frac{d}{d\ell}f'(\ell)=0$ for $f = i\lambda_\ell t - i\ell x$

$$\frac{d}{d\ell}f'(\ell) = i\left(t\left(\sqrt{\ell^2+\hat{V}} + \frac{\ell^2}{\sqrt{\ell^2+\hat{V}}}\right) - x\right) = 0 \quad\Rightarrow\quad \frac{x}{t} = \underbrace{\frac{\hat{V}+2\ell^2}{\sqrt{\ell^2+\hat{V}}}}_{\in\,[\sqrt{\hat{V}(0)},\,+\infty[}$$

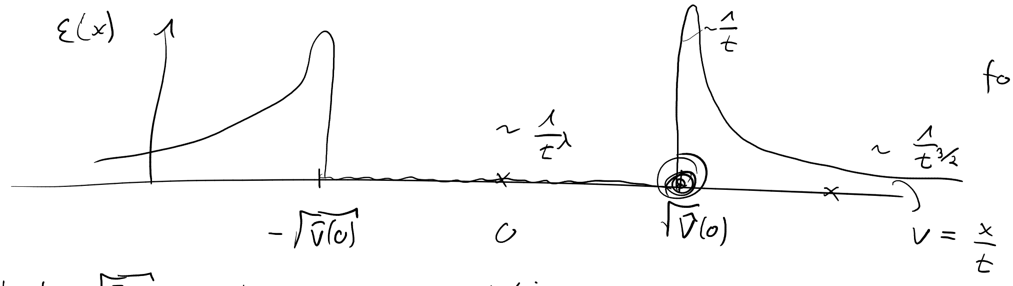
This stationary phase argument tells us, that for many potentials $\hat{V}$ there is a minimal speed for the travelling wave.

More rigorous: $\varepsilon(x) \sim \int e^{i(\lambda t - \ell x)}\mu_1(\ell)\, d^3\ell = \int e^{i\lambda t - \ell v t}\mu_1(\ell)\, d^3\ell \qquad v := \frac{x}{t}$

$$= \int e^{itf(\ell)}\mu_1(\ell)\, d\ell$$

$$= \int \underbrace{\left[itf'\, e^{itf(\ell)}\right]}_{\text{der. of } e^{itf(\ell)}}\frac{1}{itf'}\,\mu_1(\ell)\, d\ell \underset{\text{int. by parts}}{=} -\frac{1}{it}\int e^{itf(\ell)}\cdot\left(\frac{\mu_1(\ell)}{f'}\right)' d\ell$$

At positions x (resp. v) where $f = \lambda - \ell v$ has no stationary point $(f' \neq 0)$ we can do this many times $\Rightarrow \varepsilon(x) \sim \frac{1}{t^\lambda}\ \forall\lambda$



for $d=3$

For velocities $|v| < \sqrt{\hat{V}(0)}$ there are no stationary points $\sim \frac{1}{t^\lambda}$ beh.

" $|v| > \sqrt{\hat{V}(0)}$ there are stat. points. So depending on the dimension we can integrate by parts several times $\sim \frac{1}{t^{d/2}}$ ("ballistic behavior")

At $|v| = \sqrt{\hat{V}(0)}$ one can show that at the stationary point f' has a quadratic 0:

(Simple case $\hat{V} = \text{const}$) $f = \overbrace{\underbrace{|k|\sqrt{k^2+\hat{V}}}_{|k|\,(\cancel{\sqrt{V}} + \text{quadr. in } k)}}^{} - \cancel{kv} \sim k^3$ at $k=0$:

$$\sqrt{V+k^2} \sim \sqrt{V} + \frac{k^2}{2\sqrt{V}}$$