Second quantization

We will introduce the formalism called "second quantization", to start with for a very simple systems: We consider a two-dimensional one-body Hilbert space.

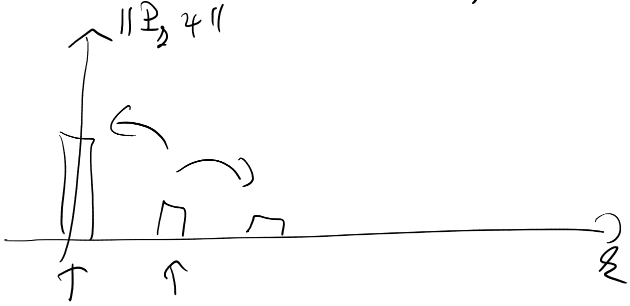
$$\mathcal{H} = \text{span}\{\varphi, \eta\}$$

We will consider N-body systems described by $\psi_t \in \mathcal{H}^{\otimes N}$

$$i\partial_t \psi_t = H\psi = \sum_{j=1}^{N} h_j \psi_t \qquad \text{(symmetric situation)}$$

ψ_0 is symmetric, $h_j = 1 \otimes 1 \otimes \dots \otimes h \otimes \dots$

We call φ the "good" and η the "bad" state (just to connect to our understanding above)



$$\psi_t^k := P_k \psi_t$$

To solve the Schrödinger equation we are interested in the transitions caused by H:

$$\langle \psi^l, \underbrace{H}_{\text{transition}} \psi^k \rangle = \langle \psi^l, \sum_{j=1}^{N} h_j \psi^k \rangle$$

$$= N\langle \psi^l, h_1 \psi^k \rangle = N\langle \psi^l (p_1 + q_1) h_1 (p_1 + q_1) \psi^k \rangle =$$

$$= N\langle \psi^l p_1 h_1 q_1 \psi^k \rangle = N\langle \psi^l, \hat{n}^\eta (\hat{n}^\eta)^{-1} p_1 h_1 q_1 \hat{n}^{-1} \hat{n} \psi^k \rangle =$$

$$\hat{n}^\eta = \sqrt{\frac{N-k}{N}}^{\,\varphi}$$

$$= N\langle (\hat{n}^\eta)^{-1} p_1 \psi^l, \hat{n}^\eta |\varphi\rangle\langle\varphi|_1 h_1 |\eta\rangle\langle\eta| \hat{n}^{-1} \hat{n} \psi^k \rangle$$

$$= N\langle \cancel{(\hat{n}^\eta)^{-1}} p_1 \psi^l, |\varphi\rangle\langle\varphi| h_1 |\eta\rangle\langle\eta|_1 \hat{n}^\eta_1 \cancel{\hat{n}^{-1}} \hat{n} \psi^k \rangle$$

$$(*) = \langle \varphi, h_1 \eta \rangle$$

$$\langle \psi^l, |\varphi\rangle\langle\eta| \cdot \sqrt{\widehat{k+1}^{\,\eta}} \sqrt{\hat{k}^\eta} \psi^k \rangle$$

combinatorical factors.

annihilates a particle in state η

creates a particle in state φ

We define $a(\varphi) := \langle\varphi| \sqrt{\hat{k}^\varphi}$ same with η

$a^*(\varphi) := |\varphi\rangle \sqrt{\widehat{k+1}^{\,\varphi}}$

$$(*) = \langle \varphi, h_1 \eta \rangle \cdot \langle \psi^l, a^\varphi a^*(\eta) \psi^k \rangle$$

Time evolution in this new language: $H \to \langle \varphi, h_1 \eta \rangle \cdot a(\varphi) a^*(\eta) + \dots$

Properties: a^* is the adjoint of a.

$$\left(|\varphi\rangle \sqrt{\ell+1}\right)^{adj} = \sqrt{\ell+1}\,\langle\varphi| = \langle\varphi|\sqrt{\ell+1} = a^*(\varphi)$$

$$[a^*_\varphi, a_\varphi] = \Big[\underbrace{a^*(\varphi)\,a(\varphi) - a(\varphi)\,a^*(\varphi)}_{\underbrace{\sqrt{\ell}\,\sqrt{\ell} - \sqrt{\ell+1}\,\sqrt{\ell+1}}_{-1}}\Big]\,|\psi^\ell\rangle$$

$$[a^*_\varphi, a_\eta] = 0$$

Summarizing we can write ψ_t as a vector on Fock space.

(linear combination of Elements in the image of P_ℓ)

$$H = a^*(\varphi)\,a(\varphi)\langle\varphi, h\varphi\rangle + a^*(\eta)\,a(\varphi)\langle\varphi, h\eta\rangle + \dots$$

This can be generalized to Hilbert spaces of higher dimension:

Let e_ℓ, $\ell \in \mathbb{N}$ a Basis of $\mathcal{H}$. (ONB)

For each e_ℓ we write $a_\ell := a(e_\ell)$ $\quad a^*_\ell := a^*(e_\ell)$

and get $[a^*_\ell, a_\ell] = -1$ $\quad [a^*_\ell, a_j] = 0$.

C-number replacement

Consider a Bose gas with most particles in a state φ

Assume $\{\overset{= e_0}{\varphi}, e_1, \dots e_\ell, \dots\}$ is a ONB.

This means that $a(\varphi), a^*(\varphi)$ create/annihilate a particle of state φ and multiply with $\sim \sqrt{N}$.

We can approximate the system by writing

$$H = N\langle\varphi, h\varphi\rangle + \sum_{\ell=1}^{\infty} a^*_\ell \sqrt{N}\langle\varphi, h e_\ell\rangle + c.c.$$
$$+ \sum_{\ell,e=1}^{\infty} a^*_\ell a_e \langle e_e, h e_\ell\rangle$$

The ψ now lives on a Fock space, which ignores particles in φ

$\Pi\varphi \to \Omega$ etc. $\qquad \psi_t = \sum P_\ell \psi_t$

$\tilde{\psi}_t^\ell$