First quantized language

$\mathcal{H} := L^2(\mathbb{R}^{3N})$

$\psi \in \mathcal{H}$ $\quad \psi(x_1 \dots x_N)$

Transition between one-particle states caused V, etc.

natural unitary map

Second quantised

Given Basis $\{e_k\}$ (ONB) of one-particle Hilbert space $\rightarrow$ Fock space

$\hat{\psi} \in \mathrm{span}(n_0, n_1 \dots n_{k \dots \infty})$ $\quad n_k = \#$ particles in e_k

$\sum n_k = N$

creation, annihilation operators. (plus further)

We found that in some cases there might be a special state that does not play an important role in our considerations, for example a φ_0 that contains practically all particles and has constant density.

$\hat{\psi} = \mathrm{span}(\cancel{n_0}, n_1 \dots)$ $\qquad \sum_{k=1}^{\infty} n_k \leq N$

Bogoliubov - theory

Consider a Bose gas, where most particles are in a given mode φ_t.
We will show that the time-evolution can be approximated, such that it has very beautiful properties one can read off in the second quantised language.
To get these nice properties we have to enlarge our space, we do so by dropping the condition $\sum_{k=1}^{\infty} n_k \leq N$.

The big insight of Bogoliubov is, that the leading order comes from transitions that are of second order in creation, annihilation - operators.

First quant.

II: $q_1 q_2 V(x_1-x_2) p_1 p_2$

c. c.

III: $p_1 q_2 V(x_1-x_2) q_1 q_2$

Second quantised

II: $a_k^* a_e^* V_{ke00} a_0 a_0 +$

$\langle e_k|_1 \langle e_{e_2} V(x_1-x_2)|e_0\rangle|e_0\rangle$

III: ...

second quant. setting $\Omega = \prod \varphi_t$

$\mathrm{II} = a_k^* a_e^* V_{ke00} N$

$\mathrm{III} \approx \dots$

$$H = \sum_{j=1}^{N} -\Delta_j + \frac{1}{N-1} \sum_{j \neq k} V(x_j - x_k) = \sum_{j=1}^{N} -\Delta_j + \frac{1}{N-1} \sum_{j \neq k} (p_j + q_j)(p_k + q_k) V(x_j - x_k)(p_j + q_j)(p_k + q_k)$$

$$H^{2nd} := \sum -\Delta_j + \frac{1}{N-1} \sum_{j \neq k} p_j p_k V(x_1 - x_k) p_j p_k \qquad 0$$

$$+ \frac{1}{N-1} \sum_{j \neq k} p_j p_k V(x_1 - x_k) p_j q_k + \dots \qquad \mathrm{I}$$

$$+ \frac{1}{N-1} \sum_{j \neq k} p_j q_k V(x_1 - x_k) q_j p_k + \dots \qquad \mathrm{II} + \text{extra}$$

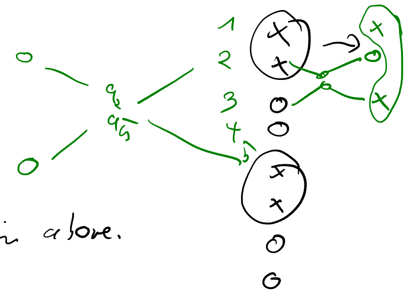
2 3

We omit the 3q and 4q term.

We will show that ψ^{2nd} given by $i\frac{d}{dt}\psi^{2nd}_0 = H^{2nd}\psi^{2nd}_t$ is close to ψ_t in norm!

$\lim_{N \to \infty} \|\psi_t^{2nd} - \psi_t\| = 0$ $\quad \forall t.$ $\qquad (\psi_0 = \psi_0^{2nd} = \prod \varphi_0)$

The H^{2nd}-dynamics has the interesting feature, that it only has pairs that we care later.



Theorem: $\lim_{N\to\infty} \| \psi_t^{2nd} - \psi_t \| = 0$

under the conditions of the Hartree-derivation above.

$(\psi_0 = \psi_0^{2nd} = \Pi \varphi_0 , \quad V \in L^\infty)$

Proof: We showed already that: $\| q_1 \psi \| \sim \frac{1}{\sqrt{N}}$ $\quad \| q_1 q_2 \psi \| \sim \frac{1}{N}$

$\| q_1 q_2 q_2 \psi \| \sim N^{-\frac{3}{2}}$

In the same way $\| q_1 \psi^{2nd} \| \sim \frac{1}{\sqrt{N}}$ etc.

$$\boxed{\| \psi^{2nd} - \psi \|_2^2 = 2 \,\mathrm{Re} \langle \psi_t^{2nd}, \psi_t^{nd} - \psi_t \rangle} = 2\,\mathrm{Re}\left(\|\psi^{2nd}\|_2^2 - \langle \psi^{2nd}, \psi \rangle \right)$$

$$\| \psi^{2nd} - \psi \|_2^2 = \underbrace{\| \psi^{nd} \|^2}_{1} + \underbrace{\| \psi \|^2}_{1} - 2\,\mathrm{Re} \langle \psi^{nd}, \psi \rangle$$

$$\langle \psi_t^{2nd}, \psi_t^{2nd} - \psi_t \rangle = - \int_0^t \frac{d}{ds} \langle \psi_t^{2nd}, U_{t-s}^{2nd} \psi_s \rangle \, ds = \langle \psi_t^{2nd}, \cancel{U_{t-t}} \psi_t \rangle - \langle \psi_t, \underbrace{U_{t-0} \psi_0}_{\psi_t^{nd}} \rangle$$

$$= -i \int_0^t \langle \psi_t^{nd}, U_{t-s}^{2nd} (H^{2nd} - H) \psi_s \rangle =$$

$\mathcal{N}_{odd} + \mathcal{N}_{even}$

$$= -i \int_0^t \langle \psi_s^{nd}, N \tfrac{(N-1)}{(N-2)} q_1 q_2 V(x_1 - x_2) q_1 p_2 \psi_s \rangle + \ldots \qquad \text{III}$$

$$+ \left(-i\right) \int_0^t \langle \psi_s^{2nd}, N \, q_1 q_2 V(x_1 - x_2) q_1 q_2 \psi_s \rangle \qquad \text{IV}$$

$$| \ldots | \le \int_0^t \| q_1 q_2 \psi_s^{2nd} \| N C \cdot \| q_1 \psi \| + \ldots \qquad \text{III}$$

$$+ \int_0^t \| q_1 q_2 \psi_s^{2nd} \| \cdot N \cdot C \quad \| q_1 q_2 \psi_s \|$$

$$\le \underbrace{C_t \quad N \cdot \frac{1}{N} \cdot \frac{1}{\sqrt{N}}}_{\text{III}} + C_t \, N \cdot \frac{1}{N} \frac{1}{N} \le \tilde{C}_t \frac{1}{\sqrt{N}}$$

$$\Rightarrow \quad \| \psi_t^{2nd} - \psi_t \|_2^2 \le C_t \frac{1}{\sqrt{N}}$$