The magic of Bogoliubov-dynamics

$$\Psi_t^{\text{Schrö}} \longrightarrow \Psi_t^{\text{2nd}} \longrightarrow \Psi_t^{\text{Bog}}$$

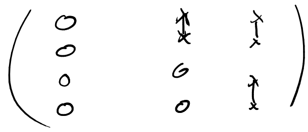
N-body — N-body — N-independent, arbitrary # of particles

first quant

2^{nd} q.

Definition: A quasi-free state is a wave function on Fock-space for which all correlations can be expressed via (products + sums) of pair correlations and a one-particle state.

Example: 4 particle-corr.



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We will learn today that the Bog. time-evolution maps quasi-free $\to$ quasi-free

Example: $\hat V(k) = \hat V(-k)$, non zero only for one value k.

Space: torus, 1d, length 2π. $\quad \mathcal{B} = \{ \frac{e^{ikx}}{\sqrt{2\pi}} ;\ k \in \mathbb{Z} \}$

$$H^{\text{Bog}} = \sum_{k \in \mathbb{Z}} k^2 a_k^* a_k + \sum_{\ell, k \neq 0} a_k^* a_\ell^* V_{k\ell 00} + \text{c.c.} + \sum_{\ell k \neq 0} a_k^* a_\ell V_{k0\ell 0} \qquad a_k = a(e^{ikx})$$

$$\sum_{i \ell k j} a_k^* a_\ell^* a_i a_j V_{k\ell ij}$$

(vacuum: all particles in 0-mode)

$$V_{k\ell 00} = \langle e^{ikx} e^{i\ell y}, V(x-y)\, 1 \rangle \cdot \frac{1}{(2\pi)^2}$$

$$= 0 \text{ for } k \neq -\ell \qquad \text{(momentum conservation)}$$

In our case (by assumption) only one $\pm k$ is non zero, thus

$$H^{\text{Bog}} = \sum_{k \neq 0} k^2 a_k^* a_k + a_k^* a_{-k}^* \hat V(k) + \text{c.c.} + a_k^* a_k \hat V(k) + a_{-k}^* a_{-k} \hat V(k)$$

There is an eigenfunction of H^{Bog}:

$$H^{\text{Bog}} \psi = E \psi \qquad \psi = \lambda_0 \Omega + \lambda_1 (1,1) + \lambda_2 (2,2) \dots \lambda_n (n,n)$$

one particle with momentum k
" " " " $-k$

$$E \lambda_0 = \lambda_1 \hat V(k)$$

$$H^{\text{Bog}} \psi = \lambda_0 \hat V (1,1) + \lambda_1 (1,1)\, 2k^2 + \lambda_1 (2,2) \cdot 2 \hat V(k) + \lambda_1 \Omega \cdot \hat V(k) + \lambda_1 (1,1) \hat V(k) \cdot 2$$
$$+ \lambda_2\, 4 k^2 (2,2) + \lambda_2 \sqrt{4}\, \hat V(k)\, (1,1)$$

$(1,1)$: $\quad \lambda_0 \hat V + \lambda_1 2 k^2 + 2 \hat V \lambda_1 + \lambda_2 \hat V \frac{\sqrt{4}}{n}$ etc. $= E \lambda_1 = \lambda_1 \cdot \frac{\lambda_1}{\lambda_0} \hat V(k)$

One can solve for all values λ_n iteratively and gets quasi-free states.

Theorem: Let φ, η be arbitrary one-particle wave-function.
Let H^{Bog} be some Bog. Hamiltonian, i.e. an Hamiltonian that is quadratic in creation and annihilation operators.
Then there is a linear map $T: L^2(\mathbb{R}^3) \times L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$

such that $$U^{Bog}_{0,t} \left(a(\varphi) + a^*(\eta) \right) U^{Bog}_{t,0} = a\left(T\binom{\varphi}{\eta}_1 \right) + a^*\left(T\binom{\varphi}{\eta}_2 \right)$$

Remark: This makes it easy to answer questions like:
$$\langle \psi_t^{Bog}, a^*(\varphi) a(\varphi) \psi_t^{Bog} \rangle$$
what is the expected number of particles is state φ after time evolution.

$$= \langle \psi_0, \underbrace{U^{Bog}_{t,0} \, a^* \, U^{Bog}_{0,t}} \underbrace{U^{Bog}_{t,0} \, a(\varphi) \, U^{Bog}_{0,t}} \psi_0 \rangle$$

Proof: Let us look at an time interval of order dt.
Up to errors of order dt^2 we have:
$$U^{Bog}_{0,dt} \, a(\varphi) \, U^{Bog}_{dt,0} = \left(1 - i\,dt\, H^{Bog} \right) a(\varphi) \left(1 + i\,dt\, H^{Bog} \right) =$$
$$= a(\varphi) + i\,dt \left[a(\varphi), H^{Bog} \right] + o(dt)^2$$

One can already see, that th order dt-term is <u>linear</u> in creation and annihilation operators!
Since the order dt^2-terms are transported unitarily, they vanish.
$\Rightarrow$ $U^{Bog} a(\varphi) U^{Bog}$ stays linear in creation + annihilation operators.
(The same argument holds for a^*).
Let us calculate $[a(\varphi), H^{Bog}]$ in more details for the simplified example

$$\left[a_k, \; k^2 a_k^* a_k + \underline{\hat{V} a_k^* a_{-k}^*} + \hat{V} a_k a_{-k} \atop + k^2 a_{-k}^* a_{-k} \right] = \boxed{k^2 \cdot 1 \, a_k} + \underline{\hat{V} \cdot 1 \, a_{-k}^*}$$