Bogoliubov maps / Bogoliubov transformations

We have learned that for any creation op $a^*(f)$ it holds:

$$\mathrm{I}: \quad U^{Bog}_{t,s}\, a^*(f)\, U^{Bog}_{s,t} = a(\eta) + a^*(\rho)$$

resp.

$$\mathrm{II}: \quad U^{Bog}_{t,s}\, a(f)\, U^{Bog}_{s,t} = a(\tilde{\eta}) + a^*(\tilde{\rho})$$

Properties: a) The equations must still hold taking the adjoint:

$$\Rightarrow \mathrm{I}^{adj}: \quad U^{Bog}_{t,s}\, a(f)\, U^{Bog}_{s,t} = a^*(\eta) + a(\rho)$$

If we summarize I and II in matrix form:

$$\begin{pmatrix} f \\ g \end{pmatrix} \to T \begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} T_{11} f + T_{12} g \\ T_{21} f + T_{22} g \end{pmatrix}$$

$$U^{Bog}_{t,s}\, a(f)\, U^{Bog}_{t,s} = a(\underbrace{T_{11}(f)}_{\tilde{\eta}}) + a^*(\underbrace{T_{21}(f)}_{\tilde{\rho}})$$

$$U^{Bog}_{t,s}\, a^*(g)\, U^{Bog}_{t,s} = a(\underbrace{T_{12}(g)}_{\eta}) + a^*(\underbrace{T_{22}(g)}_{\rho})$$

$$\Rightarrow \boxed{\begin{aligned} T_{12} &= \overline{T_{21}} \\ T_{11} &= \overline{T_{22}} \end{aligned}}$$

$$\boxed{T = \begin{pmatrix} U & W \\ \bar{W} & \bar{U} \end{pmatrix}}$$

b) Further $[a(f), a^*(f)] = 1$ for f normalized.

$$U^{Bog}_{t,s}\, [a(f), a^*(f)]\, U^{Bog}_{s,t} = 1$$

$$\parallel$$

$$[a(U(f)) + a^*(\bar{W}(f)),\ a(W(f)) + a^*(\bar{U} f)] = 1$$

$$U U^* - W^* W = 1$$

$$\langle U(f), \bar{U}(f)\rangle - \langle W(f), \bar{W}(f)\rangle = 1$$

Also $[a(f), a(f)] = 0$

$$\Rightarrow [a(U(f)) + a^*(\bar{W}(f)),\ a(U(f)) + a^*(\bar{W}(f))] = 0$$

$$\langle U(f), \bar{W}(f)\rangle - \langle U(f), \bar{W}(f)\rangle = 0 \qquad \text{etc.}$$

Definition: A linear map of the form $\begin{pmatrix} U & W \\ \bar{W} & \bar{U} \end{pmatrix}$ that satisfies the properties induced by the commutation relations is called a Bogoliubov map.

In case there is a unitary transformation S such that

$$S\, a(f)\, S^{-1} = a(Uf) + a^*(\bar{W}(f)) \qquad \text{etc.}$$

is called Bogoliubov transformation.

Theorem: If $Tr(W^* W)$ is finite, then there is a Bog. trafo for $\begin{pmatrix} U & W \\ \bar{W} & \bar{U} \end{pmatrix}$.

Beyond Bog. dynamics

Using some perturbation argument one can describe the true time evolution with many details. We do so using Duhamel expansion:

$$\| U_{t,s}\psi - \tilde{U}_{t,s}\psi \| = \| \int_s^t \underbrace{\cancel{\tfrac{d}{d\tau}} \tilde{U}_{t,\tau} U_{\tau s} \psi}\, \cancel{d\tau} \| \le \int_s^t \| \cancel{\tilde{U}_{t,\tau}} (\tilde{H} - H) U_{\tau,s} \psi \| d\tau$$

$$[\tilde{U}_{t,\tau} U_{\tau s}\psi]_s^t = \cancel{\tilde{U}_{tt}} U_{ts}\psi - \tilde{U}_{ts} \cancel{U_{ss}} \psi$$

Duhamel expansion:

$$\rightarrow \tilde{\psi}_t = \tilde{U}_{t,0}\psi = U_{t,0}\psi - i \int_0^t \tilde{U}_{t,s} \underbrace{(\tilde{H} - H) U_{s,0}\psi}_{\times}\, ds - \iint \dots$$

$$\langle \tilde{\psi}_t, M \tilde{\psi}_t \rangle = \langle \psi_t^{Bog}, M \psi_t^{Bog} \rangle + \langle \psi_t^{Bog}, M \left(-i \int_0^t U_{t,s}^{Bog} (\tilde{H} - H) U_{s,0}^{Bog} \psi_0 \, ds \right\rangle \dots$$

$U_{t,0}$ $\quad$ $U_{s,t} U_{t,0}$

$$\langle \Omega, \underbrace{U_{t,0}^{Bog} M U_{0t}}_{T(M)} \Omega \rangle +$$