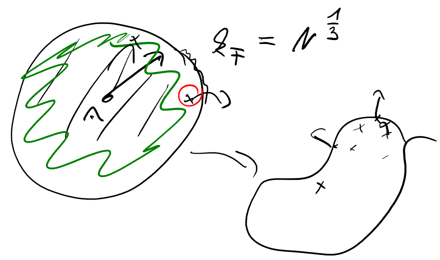
Different scalings for Fermi gases

$d = 3$

$$H = \underbrace{\sum_{j=1}^{N} -\Delta_j}_{\sim N^{\frac{5}{3}}} + \underbrace{N^{-\frac{1}{3}} \sum_{j \neq k} V(x_j - x_k)}_{\sim N^{-\frac{1}{3}} N^{2} = N^{\frac{5}{3}}}$$



$$\psi_t : \qquad i\partial_t \psi_t = H \psi_t \qquad \psi_0 = \bigwedge_{j=1}^{N} \varphi_j(x_j)$$

Remark: To see leading order effects of the interaction, the time scale we will consider is of order $N^{-\frac{1}{3}}$ (different to the Bosons were $t \sim 1$). On that time scale, the distance the particles travel is of order 1. Force acting on each particle is of order $N^{\frac{2}{3}}$, on a time scale of order $N^{-\frac{1}{3}}$ the change in momentum is of order $N^{\frac{1}{3}}$.

$$\boxed{i N^{\frac{1}{3}} \frac{d}{ds} \psi_s = H \psi_s} \quad \Big| \, N^{-\frac{2}{3}}, \ \varepsilon = N^{-\frac{1}{3}} \qquad N^{-\frac{1}{3}} s = t \qquad s = N^{\frac{1}{3}} t$$

That means that s will be of order one. It makes sense to prove Grönwall on the s-time scale.

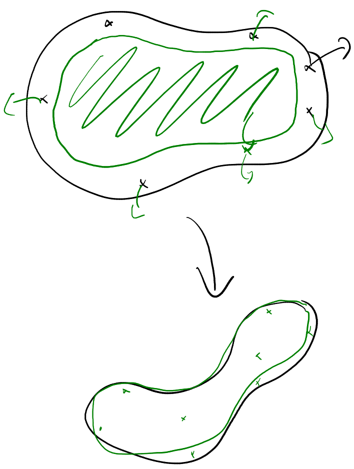
$$i \varepsilon \frac{d}{ds} \psi_s = \left(\sum_{j=1}^{N} -\Delta_j \cdot \varepsilon^2 + \frac{1}{N} \sum_{j \neq k} V(x_j - x_k) \right) \psi_s$$

Taking the limit $N \to \infty$ is like taking the mean-field limit together with a classical limit ($\hbar \to 0$) at the same time.

Remark: Of leading order the system will in fact behave classically (Vlasov eq.) (This will not be proven in this class).

Defining an appropriate α

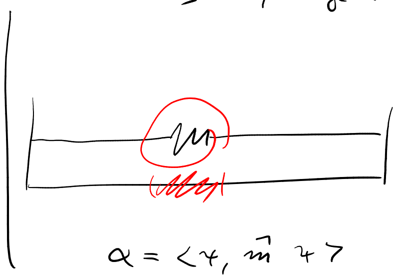
Since the excitations happen close to the surface, which contains only of order $N^{\frac{2}{3}}$ particles it makes sense to control the number of bad particles compared to $N^{\frac{2}{3}}$.



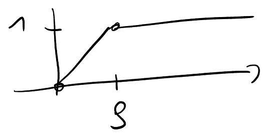
$$\alpha := \langle \psi, \hat{m} \psi \rangle \quad \text{with}$$

$$m(k) = \min \left\{ \frac{k}{N^{2/3}}, 1 \right\}$$

Bosons of large volume



$$\alpha = \langle \psi, \hat{m} \psi \rangle$$



$$m(k) = \min\left\{\frac{k}{\ell}, 1\right\}$$

Another situation:

There are systems where effective equations (Hartree, Hartree-Fock) are used, where the leading order behaviour is not classical.
For example theoretical chemists use Hartree-Fock to simulate chemical reactions of large molecules (proteins).

To avoid, that the leading order is classical, we have to drop the high density assumption, that means that the volume of the system increases.
Let us assume that particles interact via Coulomb (without singularity)

Density is of order one, volume of order N

$$V*\rho \text{ is, for Coulomb, of order } N^{\frac{2}{3}}$$

Therefore it (seems to) make sense to consider $H = \sum -\Delta + N^{-\frac{2}{3}} \sum V(x_j - x_k)$
But here one has to be careful: The leading order of $V*\rho$ is a constant and only changes the phase of the local orbitals.
We want the derivative of $V*\rho$ to be of order one!

$$\lambda \underbrace{\nabla V * \rho}_{\sim N^{\frac{1}{3}}} \sim O(1) \qquad \text{so we choose} \qquad \boxed{\lambda = N^{-\frac{1}{3}}} \leftarrow$$

depends on decay behaviour of ∇V.

In this setting it makes sense to consider a time scale of order one.

Trick: The leading order effect can be dealt with using a Gauge transformation.

Define
$$i\frac{d}{dt}\psi_t = H\psi_t$$
and
$$\chi_t := e^{it\sum V(x_j - x_k)}\psi_t$$

Schrödinger equation for χ_t:
$$i\frac{d}{dt}\chi_t = -e^{it\sum V}\cdot \sum V(x_j - x_k)\psi_t + e^{\sim} H\psi_t$$
$$= e^{\sim}\left(H - \sum V\right)\psi_t = e^{\sim}\left(\sum -\Delta\right)\psi_t$$

$$[-\Delta_j, e^{it\sum V}]f = -\Delta_j e^{it\sum V} f + e^{it\sum V}\Delta_j f$$
$$= -2it\sum \underset{e^{it\sum V}}{\underset{\uparrow}{\nabla_j V}}\, \nabla_j f - \cancel{e^{\sim}\Delta_j f} + \cancel{e^{\sim}\Delta_j f} - (it\sum V)^2 e^{\sim} f - (it\sum \Delta_j V) f$$

$$\Rightarrow \quad i\frac{d}{dt}\chi_t = \sum -\Delta + \dots = \sum_{j=1}^{N}\Big(i\nabla_j + \sum_{k\neq j} t\nabla_j V(x_j - x_k)\Big)^2 \chi_t$$