Derivation of Maxwell's equations from QED

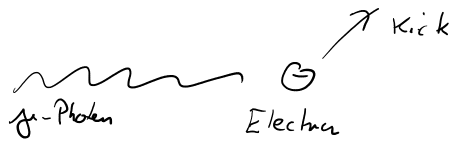
The goal is to derive a classical or semiclassical theory from QM.

The argument will obviously only work in some limiting regime. We know that there are systems, where QED can not be approximated by a classical theory.

Many photons of small energy involved

We expect semiclassical behavior. (replace the effect coming from photons by its expectation value)

Few photons of high energy involved



we do not expect semiclassical beh.

In this chapter we will consider the Pauli-Fierz system for many charges

Micro:
$$H = \sum_{j=1}^{N} (i\nabla + \lambda A)^2 + \sum_{j \neq k} \frac{1}{N} V(x_j - x_k) + H^{Ph}$$

In this model, only the A-field in Coulomb gauge is quantized!

$$H^{Ph} = \sum_{K=1}^{2} \sum_{k} |k| \underbrace{a_K^*(k)\, a_K(k)}_{\text{Particle number}}$$

$$\Psi_0 = \underbrace{\prod_{j=1}^{N}}_{\text{product}} \varphi_0(x_j) \otimes \Omega$$

$$\hat{A} = \sum_{K=1}^{2} \int \left(e^{ikx} a_K(k) + a_K^*(k) e^{-ikx} \right) \varepsilon_K$$

We wish to show, that this is approximated by some semi-classical model. This holds true if the charges remain in a product state and the photons are for any k in a coherent state. If the latter is true, $a_K(k)$ can be replaced by a number and $a_K^*(k)$ in good approximation (for large # of photons).

What about λ? We want the back-reaction of the photons on the charges to be of leading order. This is a second order effect, so $\lambda = \frac{1}{\sqrt{N}}$.

Effective description:

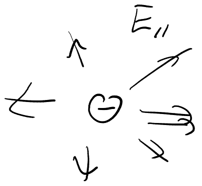
For the Bosons:
$$i \frac{d}{dt} \varphi_t = \left[(i\nabla + A)^2 + V * |\varphi|^2 \right] \varphi_t$$
Hartree

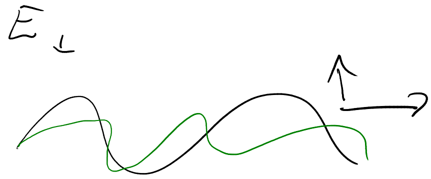
We expect A to be the vector potential (in Coulomb gauge) coming from Maxwell's equations:

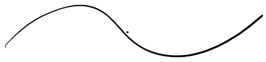
$$\nabla A = 0$$
$$\boxed{\frac{d}{dt} A = -E_\perp}$$
$$\frac{d}{dt} E_\perp = -\Delta A - j_\perp$$
$$j = \mathrm{Im}\left(\varphi^* (\nabla - iA) \varphi \right)$$

These coupled equations are called "Hartree maxwell"

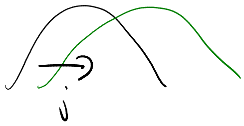
$E_\perp$ is the transversal part of the E-field



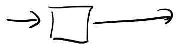




j is the probability current of the particles



$\frac{d}{dt} \rho_t = -\nabla j$ cont. equation



For many charges j will be close to the empirical current of particles! The particles are charged, so it makes sense that we have an influence on the electromagnetic fields.

Strategy of proof:

Concerning the Bosons we will again use the counting functionals from above. For the Photons, that may change their number under time evolution, we will look at the "Variance of $\hat{A}$": $\langle \psi_{t}, (\hat{A} - A)^2 \psi_t \rangle$

This would tell us two things: 1) Fluctuations of $\hat{A}$ around its mean are small. This is important for semiclassical behavior

2) A is in fact close to the expectation value of $\hat{A}$.

For technical reasons we will in addition consider the $\langle \psi_t (\hat{E}_\perp - E)^2 \psi_t \rangle$

We verify Grönwall's inequality for the sum of the tree.

$$\alpha := \langle \psi_t\, q_1^{\varphi_t} \psi \rangle + \langle \psi_t (\lambda\hat{A} - \lambda A)^2 \psi_t \rangle + \langle \psi_t (\lambda \hat{E} - \lambda E)^2 \psi_t \rangle$$