Overview

In class we considered different types of particles

Bosons $\qquad\qquad\qquad$ Fermions

- Fixed particles (gases)
- Creation and annihilation

The main goal was to "derive" effective equations. The main idea which guided us through heuristics was LLN.

For Bosons, for example, in a "condensate" we directly get Hartree equation.

$$i\partial_t \varphi = \left(-\Delta + V * |\varphi|^2\right)\varphi$$

since $N >>$ this is an empirical density.

Fermions vs Bosons: One might think there is a problem coming from symmetrization or anti-symmetrization since both (may) destroy independence.

Bosons: $\prod \varphi$ symmetrization does nothing, it is still i.i.d.

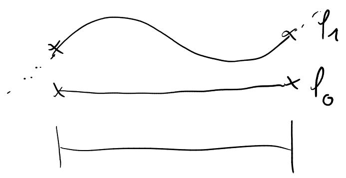
$\varphi_1(x_1)\cdot\varphi_2(x_2)\cdots\varphi_N(x_n)$ symmetrization destroys independence.

Since we mostly considered the first case, and since we can control the size of fluctuations also for anti-symmetrized products: no problem!

Fermions: The remarkable fact is that the dependence one gets from anti-symmetrization significantly reduce fluctuations.

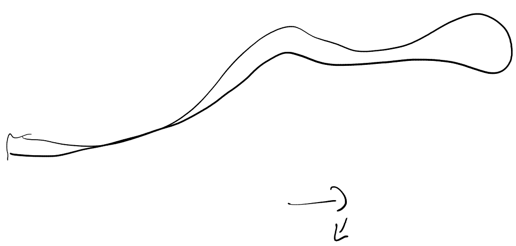
So LLN works even better than in the case of Bosons!

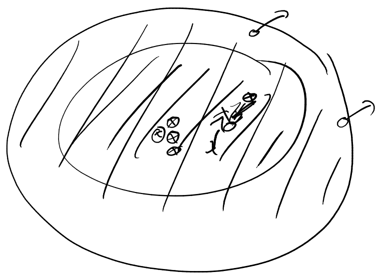
The leading order of fluctuations is suppressed at small temp.

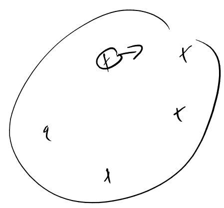


$$\Psi_{cl} = \varphi_0(x_1)\,\varphi_1(x_2) \qquad \Psi_B = (\Psi_{cl})_{sym}$$

$$\Psi_F = (\Psi_{cl})_{asym},$$







One idea to prove the validity of the Hartree equation is to count the number of "bad" particles.

$$\alpha := \langle \psi_t, q_1^{\varphi_t} \psi \rangle \in [0,1] \qquad p_1 = |\varphi(x_1)\rangle\langle\varphi(x_1)| \qquad q_1 = 1 - p_1$$

$$\frac{d}{dt}\alpha_t \leq C\left(\alpha_t + \frac{1}{N}\right) \qquad \text{for} \qquad H = \sum -\Delta_j + \frac{1}{N}\sum V(x_j - x_k)$$

$$|\dot{\alpha}| = \underbrace{\left|\langle \psi_t, [H - H^{mf}, q_1]\psi_t \rangle\right|}_{\mathbb{E}(\ell)}$$

$\uparrow$ # of bad

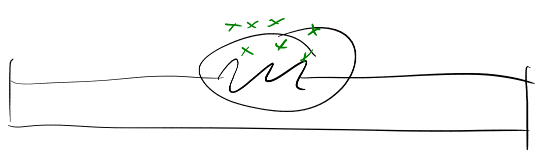
I cancellation of leading order

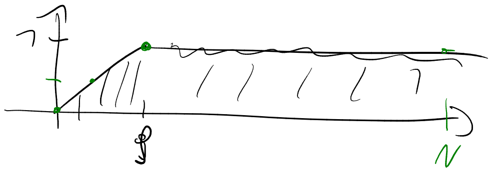
II $\leq C\alpha + C\cdot\frac{1}{N}$

III $\leq C\alpha$

$$\mathbb{E}(\ell^2) \overset{!}{\leq} \qquad \mathbb{E}(\cancel{m}(\ell))$$

φ





$$\frac{d}{dt}\langle \psi, \hat{m}\psi \rangle \cancel{\leq \langle \psi, \hat{m}\psi \rangle}$$

I, II III depending on $\hat{m}$

Bogoliubov

Extract the next order: pair correlation

$$\prod \varphi_0 \longrightarrow \prod \varphi_{dt} + \left(\varepsilon(x_1)\prod_{j=2}^{N}\varphi_{dt}(x_j)\right)_{sym} \qquad \mathrm{I} = 0$$

$$\prod\left(\varphi_{dt} + \frac{1}{N}\varepsilon\right)$$

II creates the next leading order: pairs

$$\prod \varphi_t + \left(p(x_1,x_2)\prod \varphi_t\right)_{sym} + \lambda\, p(x_1,x_2)\,p(x_3,x_4) \ldots$$

Second quantization

First: Take a system of fixed particle # N and reformulate.

$$H = \sum -\Delta_j + \underbrace{\frac{1}{N}\sum V(x_j - x_k)}_{\downarrow} \qquad \langle \psi, H\psi \rangle$$

$$\sum a_m^* a_\ell V_{\ell m k j} a_k a_j \qquad V_{\ell m k j} \; \langle \varphi_m(x_1) \langle \varphi_\ell(x_2) | V(x_1 - x_2) | \varphi_k(x_1) | \varphi_j(x_2) \rangle$$

In cases where practically all particles are in state φ_+ we can in good approximation replace $a(\varphi_+)$ and $a^*(\varphi_+)$ by $\sqrt{N}$.