Scalings

Think of some experiment that involves many interacting particles.
It is natural to adjust space and time scales to orders of magnitude where the interesting effects become visible:

Natural Hamiltonian: $H = \sum_{j=1}^{N} -\Delta_{y_j} + \sum_{j \neq k} V(y_j - y_k)$

Schrödinger: $i \frac{d}{ds} \psi = H \psi$

Rescaling of the system: Introduce $t = N^{\alpha} s$ $\quad x = N^{\beta} y$

$$H = -\sum_{j=1}^{N} \Delta_{x_j} \cdot N^{2\beta} + \sum_{j \neq k} V(N^{\beta} x_j - N^{\beta} x_k)$$

$$\Delta_y = \frac{d^2}{dy^2} = \frac{d^2}{N^{-2\beta} dx^2} = N^{2\beta} \Delta_x$$

to get $i \frac{d}{dt} \psi = \tilde{H} \psi$ we introduce $\tilde{H} = N^{-\alpha} H$

$$i \frac{d}{N^{\alpha} ds} \psi = N^{-\alpha} \left(i \frac{d}{ds} \psi \right) = \left(N^{-\alpha} H \right) \psi$$

$$\tilde{H} = -\sum_{j=1}^{N} \Delta_{x_j} N^{2\beta - \alpha} + \sum_{j \neq k} N^{-\alpha} V\left(N^{\beta} x_j - N^{\beta} x_k \right)$$

If we are interested in quantum effects we design situations where $N^{2\beta - \alpha} \sim 1$

$$2\beta - \alpha = 0$$

In addition we will often consider situations where kinetic and potential terms are of the same order. For cold Bose gases that means that the effect of

$$N^{-\alpha} V(N^{-\beta} x_j - N^{-\beta} x_k) \sim \frac{1}{N}$$

Consider situations where $\int N^{-\alpha} V(\underbrace{N^{-\beta} x}_{y}) \, d^3x \sim \frac{1}{N}$ $\quad (*)$

Case 1: V is short range, e.g. compactly supported

$(*)$ implies: $\int N^{-\alpha} V(N^{-\beta} x) \, d^3x \sim N^{-\alpha + 3\beta} \sim N^{-1}$

$\Rightarrow 3\beta - \alpha = -1 \qquad \alpha = 3\beta + 1$

$2\beta - \alpha = 0$

$2\beta - 3\beta - 1 = 0 \Rightarrow \beta = -1$

$$\tilde{H} = \sum -\Delta + \sum N^2 V(N x_j - N x_k)$$

This case (choice of α, β) is called Gross-Pitaevskii-scaling.

Case 2: $V(x) = x^{\gamma}$ $\qquad \tilde{H} = \sum -\Delta + \sum N^{-\alpha} \left(N^{-\beta} x \right)^{\gamma}$

$\sum N^{-\alpha - \beta\gamma} x^{\gamma}$

As above choose $N^{-\alpha - \beta\gamma} \sim N^{-1}$ $\Rightarrow -\alpha - \beta\gamma = -1$

$\Rightarrow -2\beta - \beta\gamma = -1$

$\Rightarrow \beta = \frac{1}{+2+\gamma}$

We will discuss scalings for Fermions later...

Grönwall

We consider a case where for some positive "time"-dependent function $f: \mathbb{R}_0^+ \to \mathbb{R}^+$ the following inequality holds: $\dot{f} \leq Cf + \varepsilon$ for $C, \varepsilon > 0$

Lemma of Grönwall: Under these assumptions it holds:

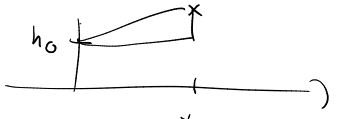
$$f(t) \leq e^{Ct} f_0 + (e^{Ct} - 1)\varepsilon \qquad \forall\, t \geq 0$$
$$\leq e^{Ct}(f_0 + \varepsilon)$$

Proof: Let $g := f_t + \frac{\varepsilon}{C} \Rightarrow \dot{g} = \dot{f} \leq Cf + \varepsilon = Cg$

$$h := e^{-Ct} g \Rightarrow \dot{h} = e^{-Ct}\dot{g} - Ce^{-Ct} g \leq e^{-Ct} \cdot Cg - Ce^{-Ct} g = 0$$
$$\Rightarrow \dot{h} \leq 0$$

It follows that $h(t) \leq h(0) \quad \forall\, t \in \mathbb{R}_0^+$

Why? Assume for contradiction $\exists\, t_0$ s.t. $h(t_0) > h(0) \Rightarrow$ (mean value Thm)



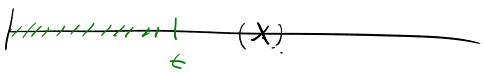
$\exists\, t_1$ s.t. $\dot{h}(t_1) > 0$ ↯

$$h \leq h_0 \Rightarrow e^{-Ct} g(t) \leq h_0 = g_0 \Rightarrow g(t) \leq e^{Ct} g_0$$
$$\Rightarrow f(t) + \frac{\varepsilon}{C} \leq e^{Ct}\left(f_0 + \frac{\varepsilon}{C}\right) \Rightarrow f(t) \leq e^{Ct} f_0 + \frac{\varepsilon}{C}(e^{Ct} - 1) \qquad \square$$

Lemma of Grönwall II: Assume that f is only differentiable from the left (or right) and continuous.

Again we wish to show that $\dot{h} \leq 0 \Rightarrow h(t) \leq h(0)$

Consider the interval $\mathbb{R}_0^+$



Define $S \subset \mathbb{R}_0^+$ via: $t \in S :\Leftrightarrow h(s) \leq h(0) \quad \forall\, s \leq t$

Because of continuity of h: $\lim_{n\to\infty} t_n = t_0 \Rightarrow \lim_{n\to\infty} h(t_n) = h(t_0)$

$\exists\, (t_n)_{n\in\mathbb{N}} \subset S$ i.e. $h(t_n) \leq h(0) \Rightarrow h(t_0) \leq h(0) \Rightarrow t_0 \in S$

Thus S is a closed set.

S is also open. Since the right sided derivative of $h(t)$ is always negative, there is an ε_t such that $h(s) \leq h(t)$

$\forall\, s \in [t; t + \varepsilon_t]$ if $h(t) \leq h(0) \Rightarrow h(s) \leq h(0)$

Since S is open, closed and non-empty $(0 \in S)$

$\Rightarrow S = \mathbb{R}_0^+$