CHAPTER 7

Classical mechanics

A physical theory is a mathematical model for how (parts of) the physical world work. Physics is about

1. inventing or discovering good theories/models,
2. collecting empirical data (experiments),
3. comparing the empirical facts about our world with our theory/model.

Classical mechanics

Classical mechanics is built on the following observations about the world:

1. **Space and time.** The physical space is three dimensional and time is one dimensional, and there is not preferred position nor time in the universe.
2. **Newton's principle of determinacy.** The positions and velocities of a mechanical system at certain time uniquely determine all it's motion (and hence future and past positions and velocities).
3. **Galileo's principle of relativity.** There exists so-called *inertial coordinate systems* such that
   (i) The laws of nature are the same at any time in all inertial coordinate systems.
   (ii) All coordinate systems in uniform rectilinear motion with respect to an inertial one are themselves inertial.

In mathematical terms these observations suggest that 1) the physical universe will be modeled in $\mathbb{R} \times \mathbb{R}^3$, where there is not preferred origin, 2) a second order (in time) differential equation will describe the whole dynamics and 3) such equations will be invariant under certain symmetries of the space and time.

Newtonian mechanics

Newton established a mathematical model for the motion of N particles (apples, planets, atoms, bullets, ...) in physical space and time. The Newtonian representation of the *physical space* is E^3 (three-dimensional affine space), $\mathbb{R}^3$ without choice of origin. The *time* in Newtonian mechanics is described by E^1 (affine line), $\mathbb{R}$ without choice of origin.

Definition 7.1 (Configuration space of N point particles)**.**

$$q \in \mathbb{R}^{3N} = \underbrace{\mathbb{R}^3 \times \ldots \times \mathbb{R}^3}_{N\text{-copies}} = \text{configuration space.}$$

$q = (q_1, q_2, \ldots, q_N)$, $q_j \in \mathbb{R}^3$ position of the j^{th} particle.

In this model, the "world" is completely specified by the position of all particles at all times, i.e. by a curve

$$\gamma : \mathbb{R} \to \mathbb{R}^{3N}, \quad t \mapsto \gamma(t)$$

in configuration space.

The physical "law" is a system of second-order ODEs for γ, Newton's law:

$$\ddot{\gamma} = M^{-1} \cdot F(t, \gamma, \dot{\gamma})$$

with M being the mass matrix, $F(t, \gamma, \dot{\gamma})$ the *force field* and $\ddot{\gamma}$ the acceleration. Only the solutions to this ODE are possible worlds, according to Newtonian mechanics. Assuming sufficient regularity of F, a unique solution is determined by specifying the positions $\gamma(t_0)$ and the velocities $\dot{\gamma}(t_0)$ at some time $t_0 \in \mathbb{R} \Rightarrow$ predictions of the theory.
The explicit specification of M and F is also part of the law.

Example 7.2 (Gravitating bodies)**.**

$$M = \begin{pmatrix} m_1 & & 0 \\ & \ddots & \\ 0 & & m_N \end{pmatrix}$$

with m_i being the mass of the i^{th} body and

$$F_j(t, q, v) = F_j(q) = G \sum_{i \neq j} \frac{m_i m_j (q_i - q_j)}{\|q_i - q_j\|^3} .$$

For $N = 2$ for example, the Newton's law is

$$\begin{pmatrix} \ddot{\gamma}_1(t) \\ \ddot{\gamma}_2(t) \end{pmatrix} = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}^{-1} \begin{pmatrix} F_1(t, \gamma(t)) \\ F_2(t, \gamma(t)) \end{pmatrix} ,$$

and reading out the components $\gamma_j(t) = (x_j(t), y_j(t), z_j(t))$, $j = 1, 2$, we are left with the system

$$\begin{aligned}
\ddot{x}_1(t) &= Gm_2 \frac{x_2(t) - x_1(t)}{\|\gamma_1(t) - \gamma_2(t)\|^3} & \ddot{x}_2(t) &= Gm_1 \frac{x_1(t) - x_2(t)}{\|\gamma_1(t) - \gamma_2(t)\|^3} \\
\ddot{y}_1(t) &= Gm_2 \frac{y_2(t) - y_1(t)}{\|\gamma_1(t) - \gamma_2(t)\|^3} & \ddot{y}_2(t) &= Gm_1 \frac{y_1(t) - y_2(t)}{\|\gamma_1(t) - \gamma_2(t)\|^3} \\
\ddot{z}_1(t) &= Gm_2 \frac{z_2(t) - z_1(t)}{\|\gamma_1(t) - \gamma_2(t)\|^3} & \ddot{z}_2(t) &= Gm_1 \frac{z_1(t) - z_2(t)}{\|\gamma_1(t) - \gamma_2(t)\|^3}
\end{aligned}$$

of six second order ODEs. One finds Kepler's ellipses as special solutions, meaning Kepler's laws follow from Newtonian gravitation.

The gravitational force is an example of a *conservative* force field, i.e. a force $F : \mathbb{R}^{3N} \to \mathbb{R}^{3N}$ that is the negative gradient of a scalar function $V : \mathbb{R}^{3N} \to \mathbb{R}$, the so-called *potential*

$$F = -\nabla V\,.$$

For conservative forces, Newtonian mechanics display *"conservation of energy"*. This means the function

$$\begin{aligned}
E &: \mathbb{R}^{3N} \times \mathbb{R}^{3N} \to \mathbb{R} \\
E(q, v) &= \sum_{j=1}^{N} \frac{m_j}{2} \|v_j\|^2 + V(q)
\end{aligned}$$

is constant along solutions of $\ddot{\gamma} = -M^{-1} \nabla V(\gamma)$, i.e.

$$E(\gamma(t), \dot{\gamma}(t)) = E(\gamma(t_0), \dot{\gamma}(t_0)) \qquad \forall t \in \mathbb{R}$$

In other words, the solutions of the Newtonian evolution stay on level sets of the function E!

If V is translation invariant, i.e.

$$V(q_1 + a, q_2 + a, \ldots, q_N + a) = V(q_1, \ldots, q_N) \qquad \forall a \in \mathbb{R}^3$$

then also the *total momentum*:

$$P(q, v) = P(v) = \sum_{j=1}^{N} m_j v_j \in \mathbb{R}^3$$

is conserved. If V is invariant under rotations of $\mathbb{R}^3$, i.e.

$$V(Rq_1, \ldots, Rq_N) = V(q_1, \ldots, q_N)$$

then *angular momentum*

$$L(q, v) = \sum_{j=1}^{N} m_j q_j \times v_j$$

is conserved. This observation of symmetries leading to the conservation of functions in q and p is more than by accident but follows the so-called conservation laws. As is the case for any 2nd-order ODE, one can write Newton's equation as a first-order ODE on $\mathbb{R}^{6N}$ leading to the concept of *Hamiltonian mechanics*.

Lagrangian mechanics

Another very popular and useful formalism is the *Lagrangian formulation* of classical mechanics as a variational problem: A Lagrangian function is a function

$$\mathcal{L} : \mathbb{R}^{3N} \times \mathbb{R}^{3N} \to \mathbb{R}, \quad (q, v) \mapsto \mathcal{L}(q, v)$$

(e.g. $\mathcal{L}(q, v) = \sum\limits_{j=1}^{N} \frac{m_j}{2} \|v_j\|^2 - V(q)$). Let

$$\Gamma = \{\gamma : C^2([0, T], \mathbb{R}^{3N})\}$$

the space of C^2-paths in configuration-space on time interval $[0, T]$. The action of such a path is

$$S(\gamma) = \int_0^T \mathcal{L}(\gamma(t), \dot{\gamma}(t))\, dt$$

$$S : \Gamma \to \mathbb{R}$$

Then the principle of least action asserts that the physically possible paths are those for which S (when adding appropriate constraints) is critical, i.e.

$$D(S - \lambda \cdot H)|_\gamma = 0 \qquad \text{Euler-Lagrange equation} \tag{7.1}$$

As

$$\begin{aligned} DS|_\gamma h = {} & D_v\mathcal{L}|_{(\gamma(T),\dot{\gamma}(T))} \cdot h(T) - D_v\mathcal{L}|_{(\gamma(0),\dot{\gamma}(0))} \cdot h(0) \\ & + \int_0^T \left\{ D_q\mathcal{L}|_{(\gamma(t),\dot{\gamma}(t))} - \left(\frac{d}{dt} D_v\mathcal{L}_{(\gamma(t),\dot{\gamma}(t))} \right) \right\} h(t)\, dt \end{aligned}$$

a part of Eq. (7.1) is often (when h is only contained at single points)

$$D_q\mathcal{L}|_{(\gamma(t),\dot{\gamma}(t))} - \frac{d}{dt} D_v\mathcal{L}|_{(\gamma(t),\dot{\gamma}(t)} = 0 \qquad \forall t$$

For $\mathcal{L} = \sum \frac{m_j}{2} \|v_j\|^2 - V(q)$ these are exactly Newton's equations.

Hamiltonian mechanics

Another approach is the one of *Hamiltonian mechanics*. The phase space of N particles in $\mathbb{R}^3$ is

$$P = \mathbb{R}^{6N}, \ x \in P\,,$$

where

$$x = (\underbrace{q_1, \ldots, q_N}_{\text{positions}}, \underbrace{p_1, \ldots, p_N}_{\text{momenta}})$$

(in general, P is a symplectic space or manifold). The canonical *symplectic form* on $P = \mathbb{R}^{6N}$ is

$$J : \mathbb{R}^{6N} \times \mathbb{R}^{6N} \to \mathbb{R}, \quad (x_1, x_2) \mapsto \langle x_1 | I x_2 \rangle$$

with

$$I = \begin{pmatrix} 0 & id_{\mathbb{R}^{3N}} \\ -id_{\mathbb{R}^{3N}} & 0 \end{pmatrix}, \quad I^T = -I$$

The law of motion is the first-order ODE on P where the vector field is the symplectic gradient of a function. $H : P \to \mathbb{R}$, the *Hamiltonian*:

$$\dot{\alpha} = I \nabla H(\alpha), \qquad \alpha : \mathbb{R} \to P = \mathbb{R}^{6N}$$

With $\alpha(t) = (Q(t), P(t))$ this reads

$$\begin{aligned} \begin{pmatrix} \dot{Q}(t) \\ \dot{P}(t) \end{pmatrix} &= \begin{pmatrix} 0 & id \\ -id & 0 \end{pmatrix} \cdot \begin{pmatrix} \nabla_q H(Q(t), P(t)) \\ \nabla_p H(Q(t), P(t)) \end{pmatrix} \\ &= \begin{pmatrix} \nabla_p H(Q(t), P(t)) \\ -\nabla_q H(Q(t), P(t)) \end{pmatrix} \end{aligned}$$

For $H(q,p) = \sum\limits_{j=1}^{N} \frac{1}{2m_j} \|p_j\|^2 + V(q)$ one finds again Newton's equation.
Let $\Phi : \mathbb{R} \times P \to P, \quad (t, x) \mapsto \alpha_x(t)$ be the flow of a Hamiltonian system. Then one has

1. conservation of energy: $H \circ \Phi_t = H \quad \forall t \in \mathbb{R}$
2. conservation of phase space volume (Liouville's theorem):

   $$\Phi_t^* \lambda = \lambda \qquad (\text{i.e. } \lambda(\Phi_t(A)) = \lambda(A) \quad \forall A \in \mathcal{B}(P))$$

   with λ the Lebesgue measure, respectively Liouville measure.

Exercises

1. Check that the Newtonian potential

$$V(q) = -\frac{G}{2} \sum_{i \neq j} \frac{m_j m_i}{\|q_i - q_j\|}$$

leads to the Newton's gravitational force field.

2. Show that under the time-evolution of an N-particle system obeying Newton's law for a given conservative force $F(q, v) = -\nabla V(q, v)$, the energy

$$E(q, v) = \frac{1}{2} \sum_{i=1}^{N} m_i \|v_j\|^2 + V(q)$$

is conserved.

3. Consider a pendulum of length L and denote by θ the angle formed with the vertical axis. Denote by g the norm of the gravitational field (assume it to be constant). Use Newton's law to derive the equation of motion

$$\frac{d^2\theta}{dt^2} + \frac{g}{L} \sin\theta = 0 \,.$$