Exercise Sheet 2: Topology I

1. Let X be a set and $(Y, \|\cdot\|_Y)$ a normed vector space. Consider the vector space

$$V \doteq \{f : X \to Y \mid \sup_{x \in X} \|f(x)\|_Y\}\,.$$

Prove that $\|f\|_\infty \doteq \sup_{x\in X} \|f(x)\|_Y$ is a norm on V.

2. Let (X, d) be a metric space.

(i) Prove that $B_r(x)$ is open for any $r > 0$ and $x \in X$.

(ii) Prove that $\emptyset$ and X are open.

(iii) Prove that if $Y, Z \subset X$ are open, then $Y \cap Z$ and $Y \cup Z$ are open.

(iv) Consider now the metric

$$d(x, y) \doteq \begin{cases} 1 & x \neq y \\ 0 & x = y \end{cases}.$$

Prove that any subset $Y \subset X$ is open.

3. Prove that

(i) all sequences in a Hausdorff space have at most one limit.

(ii) all metric spaces are Hausdorff.

4. Prove that every convergent sequence in a metric space is a Cauchy sequence.