The Dirac Sea, Antiparticles and Position Operators

“QFT or…QPT?”

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III – "N(t)" Dirac Fermions

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(ii) ***Transparent*** *Approach*

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(ii) ***Heuristic*** *Dirac* ***Sea***

(iii) ***Electron-Positron*** *Model*

(iv) ***Dirac Sea*** *Model*

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(i) ***"Field"*** *as Primitive Ontology*

(ii) ***"Particle"*** *as Primitive Ontology*

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(i) A ***Simple*** *Particular Case*

(ii.1) ***General*** *Case: A Minimal* ***Process***

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- *Method A:* ***Lift sector 1 position*** *opertors*
- *Method B:* ***Use*** *the* ***Field Operators***

III – "N(t)" Dirac Fermions

(e) On the Natural e-/e+ Position Operator

*(i) Towards a **Sufficient Condition** for an **e-/e+ NOVM***

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*(iii) Example : $P_{nat}(\cdot)$ in **Finite Volume***

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IV – Outlook

(a) Ideal N(t)-Particle Treatment

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*(i) **External EM fields** re-signify e-/e+*

*(ii) **Lorentz transformations** re-signify e-/e+*

*(iii) A **solution:** make **Dirac Sea** picture primitive*

V – Bibliography

I – A “SINGLE” DIRAC FERMION

a) The Legend of Dirac's Epiphany

- Dirac, back in 1928 **blending QM & relativity**

- **Relativistic** E-p relation

$$E^2 = m_0^2 c^4 + (\vec{p}c)^2$$

If **plane-waves** $\psi(\vec{x},t) = Ae^{\frac{i}{\hbar}(\vec{p}\cdot\vec{x}-Et)}$ **should satisfy** it: the **heuristic rule** that works in non-rel. SE is

$$E \mapsto i\hbar\frac{\partial}{\partial t}; \quad \vec{p} \mapsto -i\hbar\vec{\nabla}$$

2nd ord. **Klein-Gordon eqt.**

$$-\hbar^2\frac{\partial^2}{\partial t^2}\psi(t,\vec{x}) = \left(m_0^2c^4 - \hbar^2c^2\nabla^2\right)\psi(t,\vec{x})$$

Indeed, if you plug plane-wave you get get E-p rel

- Has **finite propagation speed** (Propagation Locality -PL)
- Has **continuity eqt.** but **"prob." density is negative**
- Moreover, **all QM was 1st order in time**!

No prblm. in **Bohmian** mechs: it is a **signed flow's compont.** [2, §7.3.8], but **game break** for **orthdx.** QM

- So take relat. for "E="

$$E = \pm\sqrt{m_0^2c^4 + (\vec{p}c)^2}$$

$$i\hbar\frac{\partial}{\partial t}\psi(t,\vec{x}) = \pm\sqrt{m_0^2c^4 - \hbar^2c^2\nabla^2}\;\psi(t,\vec{x})$$

1st ord. **Klein Gordon eqt**

Pseudo-diff op. (go to Fourier space, apply multiplicat op. & back)

- **Looks Schröd.-like, has positive prob… but** worse: **not a PDE** and ∞ **porp speed** (no PL)!

- But **Dirac insisted,**

"Can we have it all: **1st order in time, linear differential,** but **when squared** correct relation $E^2 = m_0^2c^4 + (\vec{p}c)^2$?"

for this we **need "E=" relation**

for this we must **not have sqrt**

if it was E^4 then not only would there be real roots, but also complex, but thinking in relativity we wouldn't see that: i.e., **a real number expression can be obtained not only from real numbers!**

- Is there a some **"number" system** where the **square of** a **polynomial expression "E=" gives** $E^2 = m_0^2c^4 + (\vec{p}c)^2$**?**

Try linear expression: i.e., **coefficients** $\alpha_1, \alpha_2, \alpha_3, \beta$ of the **alien "number" system**, s.th.

$$E = \beta\, m_0c^2 + (\alpha_1 p_1 + \alpha_2 p_2 + \alpha_3 p_3)c \longrightarrow E^2 = m_0^2c^4 + (\vec{p}c)^2$$

so, we **assume** it is **an algebra**

true iff

$$\begin{cases} \alpha_j^2 = \beta^2 = Id \\ \alpha_j\alpha_k + \alpha_k\alpha_j = 0 \quad \forall j \neq k \\ \beta\alpha_j + \alpha_j\beta = 0 \end{cases}$$

- **Contradictory** system **in** $\mathbb{R}$ and $\mathbb{C}$
- **Also in** $\mathbb{C}^k$ (with elemnt.-wise pdct)
- But **possible in** $\mathbb{C}^k \times \mathbb{C}^k$ (with matrix multipl.)!
  - for **all odd *k* incompatible** system
  - for ***k*=2 incompatible....but....compatible for *k*=4** *(!)*

BINGO!

a) The Legend of Dirac's Epiphany

$$\begin{cases} \alpha_j^2 = \beta^2 = Id \\ \alpha_j\alpha_k + \alpha_k\alpha_j = 0 \quad \forall j \neq k \\ \beta\alpha_j + \alpha_j\beta = 0 \end{cases}$$

Compatible for *k=4* in different ways! ⟶ Differt. represents.

For **example**, $\alpha_j := \begin{pmatrix} \mathbf{0} & \sigma_j \\ \sigma_j & \mathbf{0} \end{pmatrix}$ $\quad \beta := \begin{pmatrix} Id_{2\times 2} & \mathbf{0} \\ \mathbf{0} & -Id_{2\times 2} \end{pmatrix}$

$j \in \{1,2,3\}$

- So, **apply "quantization heuristic"** $\left\{ E \mapsto i\hbar\frac{\partial}{\partial t};\ \vec{p} \mapsto -i\hbar\vec{\nabla} \right\}$

$$E = \beta\, m_0c^2 + (\alpha_1p_1 + \alpha_2p_2 + \alpha_3p_3)c$$
$$= \beta\, m_0c^2 + \vec{\alpha}\cdot\vec{p}$$

⟶

The Dirac Equation

$$i\hbar\frac{\partial}{\partial t}\psi(t,\vec{x}) = \left(c\vec{\alpha}\cdot(-i\hbar\vec{\nabla}) + \beta mc^2\right)\psi(t\vec{x})$$

- So, **need** a **"vector valued" wavefunction** too:

  a **four** compont. **spinor** $\quad \psi \in \mathcal{H} = L^2(\mathbb{R}^3, \mathbb{C}^4)$

- In **non-relativistic limit** one gets the **Pauli equation!**

  ⟶ *Explaining "why" spinors are needed in QM!*

- Defining its **Hamiltonian**

$$H_0 := c\vec{\alpha}\cdot(-i\hbar\vec{\nabla}) + \beta mc^2$$

$$i\hbar\frac{\mathrm{d}}{\mathrm{d}t}\psi_t = H_0\psi_t$$

Still a Schrödinger equation!

The **Bible**: B. Thaller, *The Dirac equation*, (Springer-Verlag, 1992)

b) The "Free" Dirac Hamiltonian

$$H_0 = c\vec{\alpha}\cdot\underbrace{(-i\hbar\vec{\nabla})}_{\hat{\vec{p}}} + \beta mc^2 \quad \text{on} \quad \mathcal{H} = L^2(\mathbb{R}^3, \mathbb{C}^4)$$

- Relats. to get "E-p relat." when squared

$$\begin{cases} \{\alpha_i, \alpha_k\} = 2\delta_{ik}\, Id_{4\times4}; & i,k \in \{1,2,3\} \\ \{\alpha_i, \beta\} = \mathbf{0}; & i \in \{1,2,3\} \\ \beta^2 = Id \end{cases}$$

$$\sigma_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ \sigma_2 := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},\ \sigma_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

→ soluts. **not unique** → but reltd by $\mathbb{C}^4$ **isomor.**

- **Dirac representation**

$$\alpha_j := \begin{pmatrix} \mathbf{0} & \sigma_j \\ \sigma_j & \mathbf{0} \end{pmatrix} \quad j \in \{1,2,3\}$$

$$\beta := \begin{pmatrix} Id_{2\times2} & \mathbf{0} \\ \mathbf{0} & -Id_{2\times2} \end{pmatrix}$$

- **Weyl representation**

$$\alpha_j := \begin{pmatrix} \sigma_j & \mathbf{0} \\ \mathbf{0} & -\sigma_j \end{pmatrix} \quad j \in \{1,2,3\}$$

$$\beta := \begin{pmatrix} \mathbf{0} & Id_{2\times2} \\ Id_{2\times2} & \mathbf{0} \end{pmatrix}$$

- **Change of Representation map**

$$T_{W\leftarrow D} = \frac{1}{2}\begin{pmatrix} Id_{2\times2} & Id_{2\times2} \\ Id_{2\times2} & -Id_{2\times2} \end{pmatrix}$$

e.g.,

Dirac **spinor** to Weyl

$$\psi_W = T_{W\leftarrow D}\psi_D$$

Dirac **β** to Weyl β

$$\beta_D = T^{-1}_{W\leftarrow D}\beta_W T_{W\leftarrow D}$$

etc.

b) The "Free" Dirac Hamiltonian

- $\mathcal{F}: L^2(\mathbb{R}^3,\mathbb{C}^4) \to L^2(\mathbb{R}^3,\mathbb{C}^4)$ **=** unique unitary **extension** of **component-wise Fourier transf**. on Schwarz fcts

$$H_0 = c\vec{\alpha}\cdot(-i\hbar\vec{\nabla}) + \beta mc^2 \longrightarrow (\mathcal{F}H_0\mathcal{F}^{-1})(\vec{p}) = \begin{pmatrix} mc^2\ Id_{2\times 2} & c\vec{\sigma}\cdot\vec{p} \\ c\vec{\sigma}\cdot\vec{p} & -mc^2\ Id_{2\times 2}\end{pmatrix}$$

multiplication op. by a **Hermitian 4x4 matrix per** $\vec{p}\in\mathbb{R}^3$ → **4 eigenvals.** per $\vec{p}\in\mathbb{R}^3$:

$$\mathcal{E}_{1,2}(\vec{p}) = \sqrt{c^2|\vec{p}|^2 + m^2c^4} =: \mathcal{E}(|\vec{p}|)$$

$$\mathcal{E}_{3,4}(\vec{p}) = -\sqrt{c^2|\vec{p}|^2 + m^2c^4}$$

"E-p relt.!"

diagonalization matrix $u(\vec{p})$

$$u(\vec{p})\ (\mathcal{F}H_0\mathcal{F}^{-1})(\vec{p})\ u(\vec{p})^{-1} = \mathcal{E}(|\vec{p}|)\beta \longleftrightarrow (\underbrace{u\mathcal{F}}_{W}H_0(u\mathcal{F})^{-1})(\vec{p}) = \mathcal{E}(|\vec{p}|)\beta$$

- W is **unitary converting** H_0 into a **multiplication op. in** $\vec{p}$ arg. and a **diagonal matrix** in **spin arg.**

$$H_0 = W^{-1}\mathcal{M}_{\mathcal{E}}\,\beta\,W$$

full spectral "diagonalization"

- In $\phi\in WL^2(\mathbb{R}^3,\mathbb{C}^4)$, where H_0 **diagonal**
  - → **upper two compts E>0**
  - → **lower two E<0**

$$P_+ := W^{-1}\begin{pmatrix} Id_{2\times 2} & 0 \\ 0 & 0\end{pmatrix}W \quad \text{"E>0 part"}$$

$$P_- := W^{-1}\begin{pmatrix} 0 & 0 \\ 0 & Id_{2\times 2}\end{pmatrix}W \quad \text{"E<0 part"}$$

are orthogonal projectors!

b) The "Free" Dirac Hamiltonian

- $\begin{cases} \mathcal{H}_+ := P_+\mathcal{H} \\ \mathcal{H}_- := P_-\mathcal{H} \end{cases}$

$$\mathcal{H} = L^2(\mathbb{R}^3, \mathbb{C}^4) = \mathcal{H}_+ \oplus \mathcal{H}_-$$

All **states have splitting in E>0 and E<0** parts

Curiosity: $U_{FW} := \mathcal{F}^{-1}W = \mathcal{F}^{-1}u\mathcal{F}$

$$U_{FW}H_0U_{FW}^{-1} = \begin{pmatrix} \sqrt{-c^2\Delta + m^2c^4} & \mathbf{0} \\ \mathbf{0} & -\sqrt{-c^2\Delta + m^2c^4} \end{pmatrix}$$

So **Dirac eq.** is **unitarily equivalt.** to 2 **decoupled spinor KG Gordon eqts!** (of E>0 and E<0)

- **Dirac eqt.** in form: **Schr. eqt.** on $\mathcal{H} = L^2(\mathbb{R}^3, \mathbb{C}^4)$

$$i\hbar\frac{\mathrm{d}}{\mathrm{d}t}\psi_t = \left(c\vec{\alpha}\cdot(-i\hbar\vec{\nabla}) + \beta mc^2\right)\psi_t \quad \longleftrightarrow$$

in relativistic form (we won't use it here)

$$\left(\gamma^\mu \cdot i\hbar\partial_\mu - mc\right)\psi(ct, \vec{x}) = 0$$

Gamma matrices

$$\gamma^0 := \beta \qquad \gamma^j := \gamma^0\alpha^j \quad j \in \{1,2,3\}$$

Theorem 1.1. in [1]

- $H_0 = c\vec{\alpha}\cdot(-i\hbar\vec{\nabla}) + \beta mc^2$ is self-adjoint in the first Sobolev space, i.e., $(H_0, D(H_0))$ is a **self-adjoint** (unbounded) operator on

$$D(H_0) = H^1(\mathbb{R}^3)^4 \subset L^2(\mathbb{R}^3, \mathbb{C}^4).$$

- Its **spectrum** is purely absolutely continious and given by

$$\sigma(H_0) = (-\infty, -mc^2] \cup [mc^2, \infty).$$

By **Stone's Theorem**, ∃ **SCOPUG**

$U_t =$ "$e^{-\frac{i}{\hbar}H_0 t}$" with

$$i\hbar\frac{\mathrm{d}}{\mathrm{d}t}U_t = H_0$$

Solution map (propagator) of Dirac Schr. eqt.!

b) The "Free" Dirac Hamiltonian: Plane wave "solutions" – formal eigenstates

- Distributionally, a **plane wave** $f_{\vec{k}}(\vec{x}) := e^{i\vec{k}\cdot\vec{x}}$ is **mapped** to a **delta** at $\vec{k} \in \mathbb{R}^3$ **by Fourier trsf.**
- Our $\mathcal{F}$ **is Fourier per compt**. in **4-spinor WFs**, **e.g.** $a_{2,\vec{k}}(\vec{x}) := e^{i\vec{k}\cdot\vec{x}} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \longrightarrow (\mathcal{F}a_{2,\vec{k}})(\vec{p}) = \delta(\vec{p}-\vec{k}) \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$
- Recall: $H_0 = \mathcal{F}^{-1}(u^{-1}\mathcal{M}_{\mathcal{E}}\,\beta\,u)\mathcal{F}$

  so, **formally**, a **plane wave times** an **eigen-spinor of** the **matrix**: $(u^{-1}\mathcal{M}_{\mathcal{E}}\,\beta\,u)(\vec{k}) = \mathcal{E}(\vec{k})\; u(\vec{k})^{-1}\beta\; u(\vec{k})$

  is an "**eigenstate" of** H_0
- $u(\vec{p})$ is already the **diagonalization matrix**, so

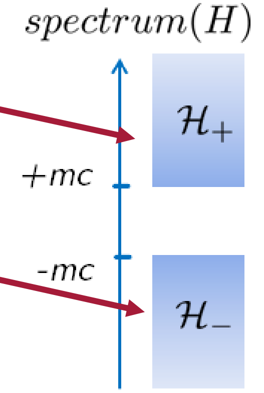
$$e_{+,1}(\vec{k}) := e^{i\vec{k}\cdot\vec{x}}\; u(\vec{k})^{-1} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad e_{-,1}(\vec{k}) := e^{i\vec{k}\cdot\vec{x}}\; u(\vec{k})^{-1} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

$$e_{+,2}(\vec{k}) := e^{i\vec{k}\cdot\vec{x}}\; u(\vec{k})^{-1} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \qquad e_{-,2}(\vec{k}) := e^{i\vec{k}\cdot\vec{x}}\; u(\vec{k})^{-1} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

"Eigenval." $\mathcal{E}(\vec{k}) = \sqrt{c^2|\vec{k}|^2 + m^2c^4}$ **"Eigenval."** $-\mathcal{E}(\vec{k})$

are **formal plane wave eigenstates** of H_0, with **"momentum"** $\vec{k} \in \mathbb{R}^3$

- **two** with **degenerate posit**. "energy" $\mathcal{E}(\vec{k})$
- **two** with **deg**. **neg**. $-\mathcal{E}(\vec{k})$



Degenerate! So any 4 LI spinor like $\begin{pmatrix} \alpha \\ \beta \\ 0 \\ 0 \end{pmatrix}$ $\begin{pmatrix} 0 \\ 0 \\ \gamma \\ \delta \end{pmatrix}$ **are also "eigenst."!**

b) The "Free" Dirac Hamiltonian: Plane wave "solutions" – formal eigenstates

- Since **degenerate in subspaces of dim 2**, we can **choose any** other **basis** of $\mathbb{C}^2$
- Given $\vec{k} \in \mathbb{R}^3$ let $v^{\uparrow}(\vec{k}), v^{\downarrow}(\vec{k}) \in \mathbb{C}^2$ be **eigenstates** of **2-spinors in direction** $\frac{\vec{k}}{|\vec{k}|}$

$$\begin{cases} \left(\vec{\sigma} \cdot \frac{\vec{k}}{|\vec{k}|}\right) v^{\uparrow}(\vec{k}) = v^{\uparrow}(\vec{k}) \\ \left(\vec{\sigma} \cdot \frac{\vec{k}}{|\vec{k}|}\right) v^{\downarrow}(\vec{k}) = -v^{\downarrow}(\vec{k}) \end{cases}$$

- **Define** then

$$w_+^{\uparrow/\downarrow}(\vec{p}) := u(\vec{p})^{-1} \begin{pmatrix} v^{\uparrow/\downarrow}(\vec{p}) \\ 0 \\ 0 \end{pmatrix}$$

$$w_-^{\uparrow/\downarrow}(\vec{p}) := u(\vec{p})^{-1} \begin{pmatrix} 0 \\ 0 \\ v^{\uparrow/\downarrow}(\vec{p}) \end{pmatrix}$$

$$v_+^{\uparrow}(\vec{k}) := e^{i\vec{k}\cdot\vec{x}} w_+^{\uparrow}(\vec{k}) \qquad v_-^{\uparrow}(\vec{k}) := e^{i\vec{k}\cdot\vec{x}} w_-^{\uparrow}(\vec{k})$$

$$v_+^{\downarrow}(\vec{k}) := e^{i\vec{k}\cdot\vec{x}} w_+^{\downarrow}(\vec{k}) \qquad v_-^{\downarrow}(\vec{k}) := e^{i\vec{k}\cdot\vec{x}} w_-^{\downarrow}(\vec{k})$$

"Eigenvalue" $E(\vec{k})$ "Eigenvalue" $-E(\vec{k})$

They are **formal** H_0 **eigenstates** of but also of **helicity op.!**

$$\left(S \cdot \frac{\vec{k}}{|\vec{k}|}\right) = \frac{\hbar}{2} \begin{pmatrix} \left(\vec{\sigma} \cdot \frac{\vec{k}}{|\vec{k}|}\right) & \mathbf{0} \\ \mathbf{0} & \left(\vec{\sigma} \cdot \frac{\vec{k}}{|\vec{k}|}\right) \end{pmatrix}$$

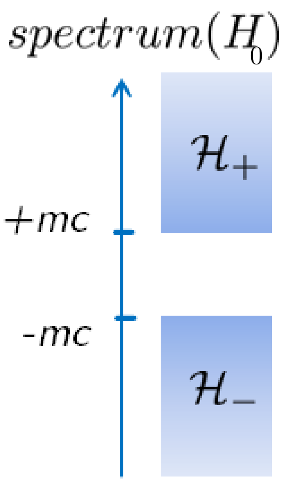
- **Formal plane wave eignsts. rigorously usable to form wavepackets!** $\forall f \in L^1(\mathbb{R}^3, \mathbb{C}^4) \cap L^2(\mathbb{R}^3, \mathbb{C}^4)$

$$\psi(t, \vec{x}) = \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} e^{-\frac{i}{\hbar}\mathcal{E}(\vec{p})t} v_+^{\uparrow}(\vec{p}) f(\vec{p}) d^3p$$

→ **is a solution of free time evolution that remains within positive E and "up" helicity spectral subspace**

Physicist's QFT treatments start from this "expansion"!

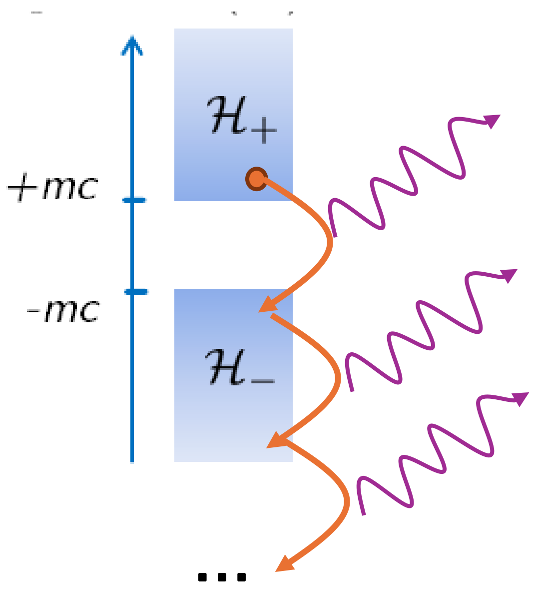
b) The "Free" Dirac Hamiltonian



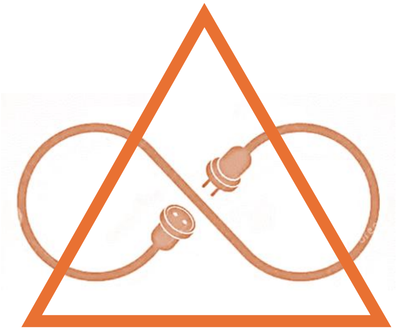
- $\sigma(H_0) = (-\infty, -mc^2] \cup [mc^2, \infty)$ **call it "Energy" spectrum**
- Def. **spectral PVM of** H_0 **as** $P_{H_0}(\cdot)$ → **Projectors** to E>0, E<0 subspaces **match the ones we found**! → $P_{H_0}((0,+\infty)) = P_+$, $P_{H_0}((-\infty,0)) = P_-$
- Now meaningful to call → $\mathcal{H}_+ := P_+\mathcal{H}$ **Positive E subspace** & $\mathcal{H}_- := P_-\mathcal{H}$ **Negative E subspace** → $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$

- But **then**...a **Dirac-particle can have unboundedly low energy**... (*no GS*)
- **Heuristic* fear:**

If **coupled with light** → "**energy exchange**" could **decay e-** **to infinite negative energy**, → **infinite "light energy"!** → **Not observed in experiment!** ↕ **Neither a banned gap in e- E!**

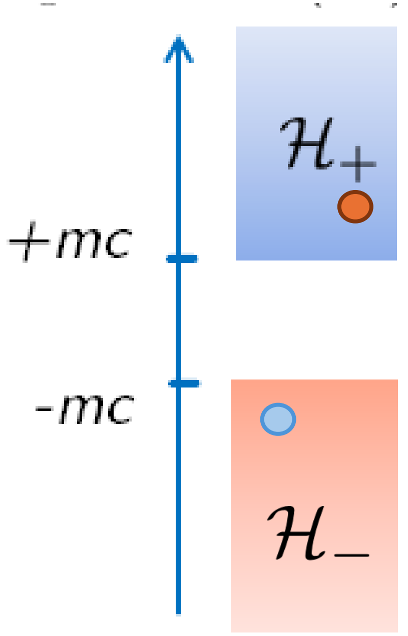


**Technically, need multiparticle theory to check if true!*



c) And yet Another Epiphany...

- **Dirac's** idea: (Back to 1929) — if $\mathcal{H}_-$ **already "filled" with e-s** → "by **Pauli exclusion** principle", **extra e- can only be in** $\mathcal{H}_+$ *(where "there is GS")* → **The Dirac Sea** ("filled $\mathcal{H}_-$")
- **Background** e- **sea** "**uniform**" → "**invisible**" **for us** → **only** "the **drops over the sea**" "**visible**": the **observed e-s!**
- **But if so...**
  - A **hole** in the **sea** would **behave as a positive charge particle with same mass!** ⟷ a..."**positron**"? let's denote it **e+** perhaps?
  - If we **"inject" energy** to apparent **nothingness promoting an e- from sea, we would see** as if **e- & hole get created** from **vacuum**! ⟷ "matter" **creation from "vacuum"**?
  - An **e- could vanish** to vacuum **if matched with a hole!** ⟷ **e-/e+ annihilation?**



- So, **matter** particles can be **"created" out of "apparent nothingness"?**

  Loose talk... ***To formalize this need N...or better N(t) particle theory!*** — *Birth of QFT for fermions!*

- **Yet,** the **qualitative conseqs.** were **experimentally verified after** Dirac **proposed** them! → So, **either a quasi-particle or a particle, but e+ exists!**

  e.g., Anderson discovers an e- with opposed mass in 1932...

d) An External EM field – Charge Conjugation

- In **search** of the **positron's equation** using the 1 Dirac particle
- Consider the **conjugate-unitary** (or antiunitary) **automorphism** $C\psi = U_C \psi^*$

conjugat-unitary

$$C(\lambda\psi) = \lambda^* C\psi$$

$$\langle C\psi, C\phi \rangle = \langle \psi, \phi \rangle^*$$

with U_C a **unitary matrix s.th.**
$$\begin{cases} \beta U_C = -U_C \beta^* \\ \alpha_k U_C = U_C \alpha_k^* \end{cases} \longrightarrow$$
e.g., in Dirac repr.
$$U_C = i\beta\alpha_2$$

- **Dirac operator in external EM field** is

$$\boxed{H(e) = c\vec{\alpha} \cdot \left(-i\hbar\vec{\nabla} - \frac{e}{c}\vec{A}(t,\vec{x})\right) + \beta mc^2 + e\phi_{el}(ct,\vec{x})}$$

- If $\psi(t)$ **solves Dirac** eqt. **with** ***H(e)*** $\Longrightarrow$ $C\psi(t)$ **solves** Dirac eqt **with** ***H(-e)***
- $CH(e)C^{-1} = -H(-e)$ $\longrightarrow$

$$\psi \; st.th \; H(e)\psi = E\psi \Longrightarrow H(-e)C\psi = -EC\psi$$

$$\psi \in \mathcal{H}_-(e) \Longrightarrow C\psi \in \mathcal{H}_+(-e)$$

- **E<0** subspace **of H(e)** ⟷ (*identified via C*) **E>0** subspace of **H(-e)** **=** **Hamilt. of same mass particle but opposide charge**

  → **Def** this as **"antiparticle"**

- Hence call C **charge conjugation op**

d) An External EM field – Charge Conjugation

- If **EM field off** (no longer refce. to charge in H) **still a map identifying** $\mathcal{H}_-, \mathcal{H}_+$

- $|C\psi(x)|^2 = |\psi(x)|^2$ ⟶ **E<0 e-** state's **motion indistinguishable of E>0 e+**

  ⟶ all **consistent** with **interpretation**: $\psi \in \mathcal{H}_-$ **describes an "antiparticle" with E>0**

  the **positron** e+

- **So, needn't formalize** the **sea and N,N(t) Dirac-particles**? **Well...**

  If e+ interpretation accepted for $\psi \in \mathcal{H}_-$, then $\mathcal{H} = L^2(\mathbb{R}^3, \mathbb{C}^4)$ **contains superpositions** of **e-s and e+s** (E>0 and E<0), **but a single position variable**

  ⟶ **A single particle state describing a mix of two?!**

- **Idea:** To **restore single particle theory** ⟶ **restrict all to** $\mathcal{H}_+$ **via** P_+

  ⟶ **For free particle** H_0 : $[H_0, P_+] = 0$ ⟶ $[e^{-iH_0 t}, P_+] = 0$ ⟶ $\boxed{\psi(t) = e^{-iH_0 t}\psi_0 \in \mathcal{H}_+ \iff \psi_0 \in \mathcal{H}_+}$

- However $\boxed{\text{general } H(e) \text{ produces superpositions of } \mathcal{H}_-, \mathcal{H}_+ \text{ !}}$ ⟶ **so yes, N or N(t) particle theory needed!**

  interpretable as indicator of pair creation/annihilation

A **PVM** is a pair $W, \{P(B)\}_{B\in W}$

- W a **measurable** space, e.g., $W=\mathbb{R}^3$ (with Borel sigma algebra) *notat. abuse*
- $P(B):\mathscr{H}\to\mathscr{H}$ **bounded** linear **operators**, s.th.

– **projectors**
$P(B)P(B)=P(B)$

– **s.a.** ops.
$P(B)^*=P(B)$

– "**compltns**. relat."
$P(W)=Id_{\mathscr{H}}$
"$\int_W P(dw)=Id_{\mathscr{H}}$"

– **strong σ-addtvty**
for $\{B_k\}_{k\in\mathbb{N}}$ disjoint in W

$$P\left(\bigsqcup_{k\in\mathbb{N}} B_k\right)=\sum_{k=0}^{\infty}P(B_k)$$

(strong sense)

- $\mathbb{P}(\cdot):=\langle\psi,P(\cdot)\psi\rangle$ is a measure on W, with meas. 1 for whole: **a probability measure**

...But probability of what?

"Of the uniquely associated s.a. operator's observable"
(orthodox jargon)

- **Spectral Theorem:**

(i) For any PVM $W=\mathfrak{B}(\mathbb{R}),\{P(\Omega)\}_{\Omega\subset W}$ on $\mathscr{H}$, the operator $(Y,D(Y))$ s.th.,

$$Y:=\int_{\sigma(Y)}yP(dy)\quad\&\quad D(Y):=\{\psi\in\mathscr{H}\mid\int_{\sigma(Y)}y^2\langle\psi|P(dy)\psi\rangle<+\infty\}$$

with $\sigma(Y):=supp\,P$, is **self-adjoint** (s.a.).

(ii) For any s.a. operator $Y,D(Y)$ on $\mathscr{H}$, there **exists** a **unique** PVM on $\mathscr{H}$, s.th. the above holds for Y.

(iii) Given the s.a. operator Y, there exists a Hilbert space $\tilde{\mathscr{H}}_{y\,rep}$ (called **"y-representat."**) and a **unitary** op. $U:\mathscr{H}\to\tilde{\mathscr{H}}_{y\,rep}$ s.th., $Y=U^{-1}\mathcal{M}(y)U$, that $P(B)=U^{-1}\mathcal{M}(\chi_B)U$ and $\mathbb{P}(B|\psi)=\int_B|U\psi|^2(y)dy$, for $B\subseteq W$.

Recall: Positive Operator Valued Measure

A **POVM** is a pair $W, \{P(B)\}_{B\in W}$

- W a **measurable** space, e.g., $W = \mathbb{R}^3$ (with Borel sigma algebra) *notat. abuse*
- $P(B) : \mathscr{H} \to \mathscr{H}$ **bounded** linear **operators**, s.th.

~~**projectors** $P(B)P(B) = P(B)$~~

→ **Non-negative operators**

$\langle\psi|P(B)\psi\rangle \geq 0 \ \forall\psi \in \mathscr{H}$

– **s.a.** ops.

$P(B)^* = P(B)$

– "**compltns**. relat."

$P(W) = Id_{\mathscr{H}}$

"$\int_W P(dw) = Id_{\mathscr{H}}$"

– **strong σ-addtvty**

for $\{B_k\}_{k\in\mathbb{N}}$ disjoint in W

$$P\left(\bigsqcup_{k\in\mathbb{N}} B_k\right) = \sum_{k=0}^{\infty} P(B_k)$$

(strong sense)

- It is the **minimal requirement** for $\mathbb{P}(\cdot) := \langle\psi, P(\cdot)\psi\rangle$ to be a **probability measure!**
- **PVM** $\implies$ **POVM**
- Most **general quantum experiments given by POVMs! [2, §5.1.2]**

↪ **If** there is an **experiment determining positions** of **particles**, **there is** an **associated POVM**!

We will abuse language to **call a POVM with $W = \mathbb{R}^3$ a (1 particle) position "operator"**!

↪ **Not necessarily a s.a. operator or PVM (although collapse tricky then)**

e) The Difficulty with Positions for e-/e+

(i) The Standard Dirac-particle Position Operator

- $\hat{\vec{x}} = (\hat{x}_1, \hat{x}_2, \hat{x}_3)$ the **standard position operator** in $L^2(\mathbb{R}^3, \mathbb{C}^4)$ s.th. $(\hat{x}_j\psi)(\vec{x}) = \begin{pmatrix} x_j\psi_1(\vec{x}) \\ x_j\psi_2(\vec{x}) \\ x_j\psi_3(\vec{x}) \\ x_j\psi_4(\vec{x}) \end{pmatrix}$

  is **self-adjoint** in $D(\hat{x}_j) = \{\psi \in L^2(\mathbb{R}^3, \mathbb{C}^4) \,|\, |\hat{x}_j\psi|_{L^2} < \infty\}$ with **PVM** $P_{std,j}(B_j) = \mathcal{M}(\chi_{x_j \in B_j})$ $B_j \subseteq \mathbb{R}$

- **Joint PVM** of $\hat{\vec{x}}$, $B \subseteq \mathbb{R}^3$: $P_{std}(B) = \mathcal{M}(\chi_B)$ s.th. $\Big(P_{std}(B)\,\psi\Big)(\vec{x}) = \psi(\vec{x})\ \chi_B(\vec{x}) = \begin{cases} \psi(\vec{x}) & if\ \vec{x} \in B \\ 0 & if\ \vec{x} \notin B \end{cases}$

- **Yields probability measure**

$$\mathbb{P}(particle \in B) = \langle\psi|P_{std}(B)\psi\rangle = \int_B \sum_{k=1}^{4} |\psi_k(\vec{x})|^2 d^3x$$

- $\hat{\vec{x}}(t) := e^{\frac{i}{\hbar}H_0 t}\vec{x}e^{-\frac{i}{\hbar}H_0 t}$

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{\vec{x}}(t) = \frac{i}{\hbar}[H_0, \vec{x}(t)] = e^{\frac{i}{\hbar}H_0 t}c\vec{\alpha}e^{-\frac{i}{\hbar}H_0 t} =: c\vec{\alpha}(t)$$

→ **Assctd. velocity operator** $c\vec{\alpha}$

$\langle\psi(t)|A\psi(t)\rangle = \langle\psi(0)|A(t)\psi(0)\rangle$

- $\sigma(c\vec{\alpha}(t)) = \{-c, +c\}\ \forall t$

  pure discrete **spectrum**!

  → **|Expected velocity| < c**

- Recall, **Dirac-Bohm velocity is its local expectation!** (in Lorentz frame)

$$\frac{\mathrm{d}\vec{X}(t)}{\mathrm{d}t} = \frac{\psi^\dagger\, c\vec{\alpha}\,\psi}{\psi^\dagger\psi}(t, \vec{X}(t))$$

By **equivariance**, in **std. pos. repr.**, ψ **propagates finite speed** → **Propagation Locality!**

e) The Difficulty with Positions for e-/e+

(i) The Standard Dirac-particle Position Operator

- **For us Dirac-Bohm** law is **perfect**, **but historically**, **velocity op** $c\vec{\alpha}$ was surprising

In **relativistic** kinematics $\vec{v} = c^2\vec{p}/E$ → **naïve quantum guess** would have been $c^2\hat{\vec{p}}H_0^{-1}$

$c^2\hat{\vec{p}}H_0^{-1}$ **bded** op. with $[-c, c]$ pure **a.c. spectrum** & **commutes** with H_0 → **free particle const. of mot.**

- **But** for $c\vec{\alpha}$: $[H_0, c\vec{\alpha}] = 2H_0\hat{\vec{F}}$ → NOT free particle const. of mot.

$\hat{\vec{F}} := c\vec{\alpha} - c^2\hat{\vec{p}}H_0^{-1}$ **diffce. betw. std and naïve vel. op.**

$$c\vec{\alpha}(t) = c^2\hat{\vec{p}}H_0^{-1} + e^{2\frac{i}{\hbar}H_0 t}\hat{\vec{F}}(t)$$

Expected velocity will **oscillate around conserved** mean value of **classically expected** $c^2\hat{\vec{p}}H_0^{-1}$

} **Schrödinger called it *Zitterbewegung***

- Likewise $$\hat{\vec{x}}(t) = \underbrace{\hat{\vec{x}} + c^2\hat{\vec{p}}H_0^{-1}t}_{\text{Classical free particle traj.}} + \underbrace{\frac{1}{2iH_0}(e^{2iH_0 t} - 1)\hat{\vec{F}}}_{\textit{Oscillation}}$$

- For **us no problem, except** that *Zitterb.* is **symptom of E>0 & E<0 mixing!**

$$\hat{\vec{F}}P_\pm = P_\mp\hat{\vec{F}}$$

F maps $\mathcal{H}_\pm \to \mathcal{H}_\mp$

If $\psi \in \mathcal{H}_\pm$ → **no Zitterb.**

$$\langle\psi(t)|\hat{\vec{x}}\,\psi(t)\rangle = \langle\psi|\hat{\vec{x}}\,\psi\rangle + \langle\psi|\,c^2\hat{\vec{p}}H_0^{-1}\,\psi\rangle\,t$$

e) The Difficulty with Positions for e-/e+

- More **explicitly**: *Zitterbebew* F is **symptom** of $\mathcal{H}_-, \mathcal{H}_+$ **mixing by std. position op** $\hat{\vec{x}}$

$$P_+ \hat{\vec{x}} P_- + P_- \hat{\vec{x}} P_+ = \frac{1}{2iH_0} \hat{\vec{F}} \longrightarrow \text{even if } \psi \in \mathcal{H}_+ \text{, in general } x_j \psi \notin \mathcal{H}_+$$

(ii) The Projected e-/e+ Position Operator

- If **not want** posit. operator **mixing** subspaces, **force it**

$$\boxed{\hat{\vec{x}}_{proj} := P_+ \hat{\vec{x}} P_+ + P_- \hat{\vec{x}} P_-} = \hat{\vec{x}} - \frac{1}{2iH_0} \hat{\vec{F}}$$

- Still **s.a.** in $D(\hat{\vec{x}})$
- **Canonical commutat**. with $\hat{\vec{p}}$
- Ofc. no zitterbeweg. $\hat{\vec{x}}_{proj}(t) = \hat{\vec{x}} + c^2 \hat{\vec{p}} H_0^{-1} t$

- Then to **consider** $\mathcal{H}_+, \mathcal{H}_-$ (or $C\mathcal{H}_-$) as **independent e-/e+ Hilbert spaces** , **need operators** that **map** $\mathcal{H}_\pm \to \mathcal{H}_\pm$ so the **obvious single e-/e+ position operators are** restrictions of $\hat{\vec{x}}_{obv}$

$$\hat{\vec{x}}_{proj|\mathcal{H}_\pm} = P_\pm \hat{\vec{x}} P_\pm \longrightarrow \hat{\vec{x}}_{e\pm} : \mathcal{H}_\pm \to \mathcal{H}_\pm \qquad \boxed{\hat{\vec{x}}_{e\pm} = P_\pm \hat{\vec{x}} \; inclus_{\mathcal{H}}}$$

BUT:

- The **different directions do not commute!**

  So no joint measure on $\mathbb{R}^3$ subsets!

$$[\hat{x}_{proj,\ell}, \hat{x}_{proj,k}] = -i \frac{c^2}{H_0^2} S$$

Immediately discarded! Else finding particle in $\Omega \subset \mathbb{R}^3$ meaningless!

For most $B_1 \times B_2 \times B_3 \subset \mathbb{R}^3$: **localize in one dir.** $x_j \in B_j \longrightarrow$ not in others

(iii) The Distinguished e-/e+ Position Operator

General Characterization of a s.a. position operator (with commuting directions) [1, p.29]

- If $\{P(B)\}_{B\subseteq\mathbb{R}^3}$ is
  1. **a PVM on** $\mathbb{R}^3$ subsets
  2. s.th., for any $B\subseteq\mathbb{R}^3, \vec{a}\in\mathbb{R}^3, R\in SO(3)$,

  $$P(RB+\vec{a}) = U(\vec{a},R)P(B)U(\vec{a},R)^*$$

  for some unitary representation of the **Euclidean group** in the Hilbert space of the particle.

  Then, $\{P(B)\}_{B\subset\mathbb{R}^3}$ is the PVM of the **unique "position" operator** $\hat{\vec{q}} = (\hat{q}_1, \hat{q}_2, \hat{q}_3)$, defined by

  $$q_j := \int_{-\infty}^{\infty} \lambda P_{q_j}(d\lambda)$$

  with $P_{q_j}(\lambda) := P(\{\vec{x}\in\mathbb{R}^3 | x_j \leq \lambda\})$.

- Conversely, for **any position operator with commuting directions** we can define $P(B)$ **via spectral theorem (joint diagonalization)** as $\mathcal{M}(\chi_B)$ in the Hilbert space of its "position" representation.

e) The Difficulty with Positions for e-/e+

(iii) The Distinguished e-/e+ Position Operator

Recall **FW representation** was

$U_{FW} := \mathcal{F}^{-1}W = \mathcal{F}^{-1}u\mathcal{F}$ s.th

$U_{FW}H_0U_{FW}^{-1} = \beta\sqrt{-c^2\Delta + m^2c^4}$

Theorem 1.5 in [1]

Let $\hat{\vec{q}} = (\hat{q}_1, \hat{q}_2, \hat{q}_3)$ be s.a. commuting ops. in $\mathcal{H} = L^2(\mathbb{R}^3, \mathbb{C}^4)$ with joint spectral PVM $\{P(B)\}_{B\subset\mathbb{R}^3}$ satisfying 1,2 of characterization. There is a **unique** such $\hat{\vec{q}}$ also satisfying

3. $P(B)$ **leaves** $\mathcal{H}_+$ **and** $\mathcal{H}_-$ **invariant,** i.e., $P(B)\mathcal{H}_\pm \subseteq \mathcal{H}_\pm$.

It is called the **Newton-Wigner**(-Wightman) position operator $\hat{\vec{x}}_{NW}$ and it is $\hat{\vec{x}}_{NW} := U_{FW}^{-1}\hat{\vec{x}}U_{FW}$.

(i.e., std. pos. op. of Foldy-Wouthuysen (FW) space brought to Dirac space)

- No Zitterb.

$$\hat{\vec{x}}_{NW}(t) = \hat{\vec{x}}_{NW} + c^2\hat{\vec{p}}H_0^{-1}t$$

- $\psi \in \mathcal{H}_\pm$ **localized in NW** position **in** $B \subseteq \mathbb{R}^3$ **can be** $\psi(\vec{x}) \neq 0$ *in* $\vec{x} \notin B$ *(i.e., not localized in std. pos. op)*

BUT:

Theorem 1.6 in [1]: $\forall\psi(0) \in \mathcal{H}$ & for all open $B \subset \mathbb{R}^3$ and $\varepsilon > 0$, $\exists t \in (0,\varepsilon)$, s.th. $\langle\psi(t)|P_{NW}(B)\psi(t)\rangle \neq 0$.

so, if **bded** $B_0 \subset \mathbb{R}^3$ **&** $\psi_0 \in range\, P_{NW}(B_0)$ → **at t=0 all prob. localized in** B_0

For **arbitrarily far** $B \subset \mathbb{R}^3$ there is ≠ **0 prob. at arbitrarily small *t*>0**

→ **there is prob. that particle goes from B_0 to B arbitrarily fast!**

In NW posit. repr.
∞ propagt. speed!
No Propagt. locality!

e) The Difficulty with Positions for e-/e+

(iv) The Obvious e-/e+ Position "Operator" (POVM)

- **If start from pdf** not opt. **obvious demand**:

given $\psi \in \mathcal{H}_\pm$ we **want prob.** to **find e-/e+ in** $B \subset \mathbb{R}^3$ is $\int_B \sum_k |\psi_k(\vec{x})|^2 d^3x$ (∗)

Same as using **std. post**. op's PVM $P_{std}(B) = \mathcal{M}(\chi_B)$

→ **Ensures** a **joint** $\mathbb{R}^3$ probability, **PL** and meaningful for $\mathcal{H}_+, \mathcal{H}_-$ (or $C\mathcal{H}_-$) **alone**!

- This is the **obvious position measure not mixing** $\mathcal{H}_+, \mathcal{H}_-$ (or $C\mathcal{H}_-$): $P_{std}(\cdot)$ **projected to** $\mathcal{H}_\pm$

$$P_{obv}(B) := P_+\mathcal{M}(\chi_B)P_+ + P_-\mathcal{M}(\chi_B)P_-$$

so $P_{obv}(B)\psi = P_\pm\mathcal{M}(\chi_B)\psi$ for $\psi \in \mathcal{H}_\pm$

→ $P_{obv,e\pm}(B) : \mathcal{H}_\pm \to \mathcal{H}_\pm$

$$P_{obv,e\pm}(B) = P_\pm\mathcal{M}(\chi_B)\, inclus_{\mathcal{H}}$$

associatd. prob. measure $\langle\psi|P_{obv}(B)\psi\rangle$ yields exactly (∗)

Almost a But:

- **No longer** are a **PVM** : $P_{obv}(B)^2 \neq P_{obv}(B)$

→ **No s.a. posit. op. behind!** (also by thm.)

→ But **yes** a **POVM**! **Valid too**! *(Although collapse unclear)*

The true But:

- ***Corollary 1.7 in [1]:*** $\forall\psi \in \mathcal{H}_\pm \quad supp(\psi) := \text{closure of } \{\vec{x} \in \mathbb{R}^3 | \psi(\vec{x}) \neq 0\} = \mathbb{R}^3.$

So **not possible** to **localize to** any **compact set!!!** ← $\nexists\, \psi \in \mathcal{H}_\pm$ with prob 1 in $B \neq \mathbb{R}^3$

e) The Difficulty with Positions for e-/e+

(iv) The Obvious e-/e+ Position "Operator" (POVM)

Following [11]

- $f \in \mathcal{H}_\pm$ **vanishing** in **open** set ⟶ **vanishes everywhere** [1, Cor. 1.7.] ⟶ ρ_{obv} **never compact** support!
  - **No localized post-measurement state** ⟶ Violation of Interaction Locality

- Let $\psi_{pre} \in \mathscr{H}$ only ≠ 0 in **(1,0) sector** –according to $P_{obv}(\cdot)$ –

↳ if **detector** finds **particle** in **compact** $A \subset \mathbb{R}^3$ at t=0 : **collapse** must happen $\psi_{pre} \mapsto \psi_{post}$

But post-meas ψ_{post} **can't** be **localized** in ***any* compact** ⟶ **has tails arbitrarily far from A** $\forall t > 0$

↳ **non-zero probab.** to **immediately find particle arbitrarily far away!**

↳ But **according to** $P_{obv}(\cdot)$ **wavefunctions** (and Bohmian trajs) **propagate at v<c!**

Conclusion: With positive prob. **particle was not in A at t=0** ⟶ detector triggered by spacelike away particle

↲ ***Violation of interaction locality!***

e) The Difficulty with Positions for e-/e+ – In Summary

(i) $\hat{\vec{x}}, \{P_{std}(B)\}_{B\subseteq\mathbb{R}^3}$ **Standard position operator** in $\mathcal{H} = L^2(\mathbb{R}^3, \mathbb{C}^4)$ ⟶ **s.a.,** ∃ joint-PVM and PL,

- **Perfect** for **Dirac-particles** (not caring if E>0 & E<0 superpositions)
- **Perfect for Dirac Sea particles…but unless we formalize it…*“res de rien”***
- **Bad for e-/e+** picture, since **mixes** $\mathcal{H}_+, \mathcal{H}_-$ (*Zitterbeweg.*)

(ii) $\hat{\vec{x}}_{proj}, \{P_{proj,j}(B_j)\}_{B_j\subseteq\mathbb{R}}$ **project.** of **std.** op. to $\mathcal{H}_+, \mathcal{H}_-$ ⟶ s.a. ops but **no joint diagonalizt.** ⟶ **no meas. on $\mathbb{R}^3$!**

(iii) $\hat{\vec{x}}_{NW}, \{P_{NW}(B)\}_{B_j\subseteq\mathbb{R}^3}$ **unique posit**. op. **presving**. $\mathcal{H}_+, \mathcal{H}_-$ (i.e., safely restrictable) & **with measure on $\mathbb{R}^3$!**

But ∞ Propagation Speed for prob.! Violates propagation locality!

(iv) $\{P_{obv}(B)\}_{B\subseteq\mathbb{R}^3}$ **projection of** $P_{std}(\cdot)$ **to** $\mathcal{H}_+, \mathcal{H}_-$

Conclusion: e-/e+ picture doomed in 1 particle pict

- Familiar $|\psi|^2(x)$ pdf
- It is **POVM**, so **no associated operator nor collapse rule**
- But, $\nexists\, \psi \in \mathcal{H}_\pm$ with localized density: **always possible to find particle anywhere**

*they say posit. **localization** need so **much E** that **pair creation** involved:* N(t) picture needed for e-/e+ again

II – “N” DIRAC FERMIONS

a) Anti-symmetrized States

- Let **1-particle HS** be $\mathcal{H}_1 = L^2(\mathbb{R}^3, \mathbb{C}^s)$

induced **IP** $\langle \psi, \phi \rangle_{\mathcal{H}_1^{\otimes N}} := \int_{\mathbb{R}^{3N}} \sum_{k=1}^{s} \psi_{s_k}^* \phi_{s_k}(\vec{x_1}, ..., \vec{x_N}) d^N x$

- Def. **anti-symmetrization** of states $\{\varphi_j\}_{j=1}^N \subset \mathcal{H}_1$ as the vector of $\mathcal{H}_1^{\otimes N} = L^2((\mathbb{R}^3)^N, (\mathbb{C}^s)^{\otimes N})$ s.th.

$$Anti\Big(\varphi_1 \otimes \cdots \otimes \varphi_N\Big)(x_1, ..., x_N) := \frac{1}{\sqrt{n!}} \sum_{\sigma \in S_N} sgn(\sigma)\ \varphi_1(x_{\sigma(1)}) \cdots \varphi_N(x_{\sigma(N)})$$

$x_j \equiv (\vec{x}_j, s_j) :\ \vec{x}_j \in \mathbb{R}^3,$
$s_j \in \{1, 2, 3, 4\}\ \forall j$

- It is the **part of the tensor product** of the states **that yields** a $\mathbf{-1} \cdot$ if we **permute two variables**.
- The **symmetrization map** $Sym(\cdot)$ is the **same but changin**g $\mathrm{sgn}(\sigma) \mapsto +1$
- It is **normalized** in the **IP of** $\mathcal{H}_1^{\otimes N}$ ⟶ if φ_j ON $\Big\langle Anti(\varphi_1 \otimes \cdots \otimes \varphi_N), Anti(\varphi_1 \otimes \cdots \otimes \varphi_N)\Big\rangle = 1$
- We denote it by **wedge product**: $\varphi_1 \wedge \cdots \wedge \varphi_N := Anti\Big(\varphi_1 \otimes \cdots \otimes \varphi_N\Big)$
- "Trick" to compute it:

$$(\varphi_1 \wedge \cdots \varphi_N)(x_1, ..., x_N) \equiv \sqrt{N!} \begin{vmatrix} \varphi_1(x_1) & \varphi_1(x_2) & \cdots & \varphi_1(x_N) \\ \varphi_2(x_1) & \varphi_2(x_2) & \cdots & \varphi_2(x_N) \\ \vdots & \vdots & \ddots & \vdots \\ \varphi_N(x_1) & \varphi_N(x_2) & \cdots & \varphi_N(x_N) \end{vmatrix}$$

so-called **Slater determinant**

a) Anti-symmetrized States

- **Key property:** $\varphi_1 \wedge \cdots \wedge \varphi_j \wedge \varphi_{j+1} \wedge \cdots \wedge \varphi_N = -\varphi_1 \wedge \cdots \wedge \varphi_{j+1} \wedge \varphi_j \wedge \cdots \wedge \varphi_N$

  → If at least **two** φ_j, φ_k are **linearly dependent:** $\varphi_1 \wedge \cdots \wedge \varphi_N = 0$

- Hence, a **non-zero wedge product of N-vectors** must **define** an **N dim subspace** of $\mathcal{H}_1$

- In fact, **given** an **N dim** closed subspace $V \subset \mathcal{H}_1$**, for any two ONB of *V*,**

$$\{\psi_1, ..., \psi_N\} \subset V \quad \& \quad \{\varphi_1, ..., \varphi_N\} \subset V \longrightarrow \boxed{\psi_1 \wedge \cdots \wedge \psi_N = e^{i\theta} \varphi_1 \wedge \cdots \wedge \varphi_N}$$

for some $\theta \in [-\pi, \pi)$

The Pauli Exclusion Principle

- $$\boxed{\varphi_1 \wedge \cdots \wedge \varphi_N = 0 \quad \text{if} \quad \varphi_k = \varphi_j \ for \ j \neq j}$$

→ **Naïve interpretation**:

"Two fermions cannot occupy the same state"

This is **very loose talk**! A **particle does NOT "possess a state"** in a multiparticle **entangled** wavefunction! (say, an anti-symmetrized one)

It is NOT a tensor product of states!

(Moreover, they are **indistinguishable particles**...)

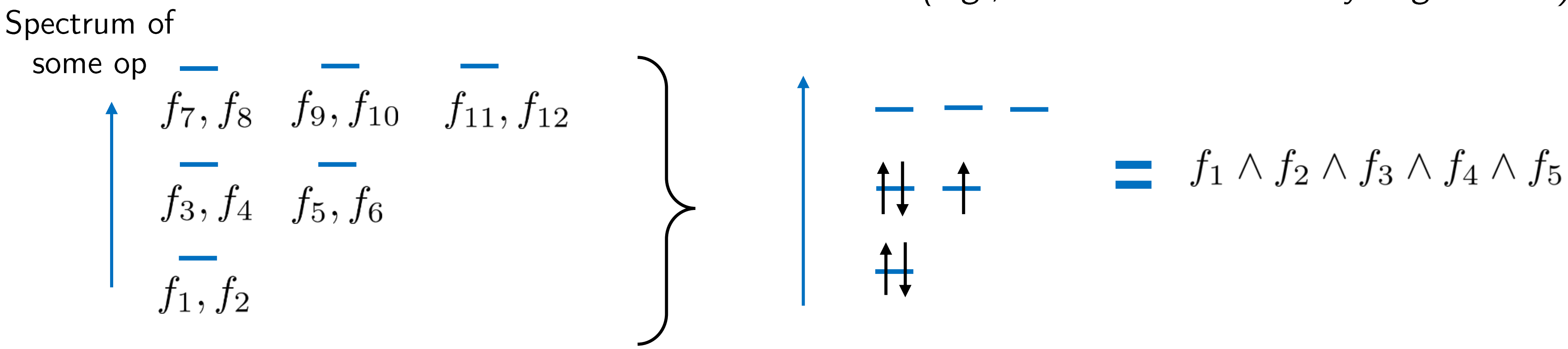
→ Although true th. **antisymzd**. **state** always **0 in "diagonals"!**

$$\forall \psi \in \ range(Anti_N)$$
$$\psi(x_1, ..., x_N) = 0 \quad \text{if } x_j = x_k \text{ for some } j \neq k$$

→ 0 **prob.** density of **2 fermions in same space&spin point!**

a) Anti-symmetrized States

- This is what **chemistry Electron diagrams** meant! Given $\{f_j\}_{j\in\mathbb{N}} \subset \mathcal{H}_1$ "**eigen-basis" of some operator** *(e.g., Hamiltonian of the Hydrogen atom)*



The N Fermion Hilbert Space

- $Anti(\cdot) : \mathcal{H}_1^{\otimes n} \to \mathcal{H}_1^{\otimes n}$ is an **orthogonal projector**, so **its image**, the space of **anti-symmetrized N particle WFs is a Hilbert space** → $\mathscr{F}^{(n)} := Anti(\mathcal{H}_1^{\otimes n})$ **=** **the *N* particle fermionic Hilbert space**

- Then given $\{f_j\}_{j\in\mathbb{N}} \subset \mathcal{H}_1$ **ONB** *(for bosonic use $Sym(\cdot)$)*

$$\{f_{j_1} \wedge \cdots \wedge f_{j_N} \mid j_k \in \mathbb{N} \text{ and } j_1 < \cdots < j_N\} \subset Anti(\mathcal{H}_1^{\otimes N})$$ **is an ONB of** $\mathscr{F}^{(n)}$

b) Usual N Dirac-Particle treatment

$\mathcal{H}_1 = L^2(\mathbb{R}^3, \mathbb{C}^4) \longrightarrow \mathscr{F}^{(N)} := Anti(\mathcal{H}_1^{\otimes N}) \longrightarrow$ **(single time) N particle state** $\psi : \mathbb{R} \to \mathscr{F}^{(n)}$ ruled by

$$i\hbar \frac{d\psi(t)}{dt} = H^{(N)}\psi(t)$$

ODE on $\mathscr{F}^{(N)}$

N free Dirac Particles

- If Hamiltonian for 1-free particle $H_0 = c\vec{\alpha} \cdot (-i\hbar\vec{\nabla}) + \beta mc^2$

$$H_0^{(N)} := \sum_{j=1}^{N} Id \otimes \cdots \otimes Id \otimes \underset{j}{H_0} \otimes Id \otimes \cdots Id = \sum_{j=1}^{N} \left[c\vec{\alpha}_j \cdot (-i\hbar\vec{\nabla}_j) + \beta_j m_j c^2\right]$$

$\alpha_j^k := Id \otimes \cdots \otimes \alpha^k \otimes \cdots \otimes Id$ etc.

N Dirac Particles in an External/Internal Potential

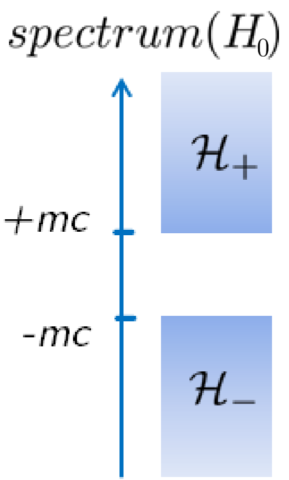
- $H^{(n)} := H_0^{(n)} + \hat{V}$ with $(\hat{V}\psi)(t, x_1, ..., x_N) = V(t, x_1, ..., x_N)\psi(t, x_1, ..., x_N)$ *a (matrix valued) multiplication op.*

- **As a PDE** $$i\partial_t\psi = \sum_{j=1}^{N}\left[\sum_{k=1}^{3} c\alpha_j^k(-i\hbar\partial_{x_{j,k}}) + \beta_j m_j c^2\right]\psi + V \cdot \psi$$

- **Obvious Born Rule:** $$\rho(\vec{x}_1, ..., \vec{x}_N \mid \psi, t) = \sum_{k=1}^{4} |\psi_{s_k}|^2(t, \vec{x}_1, ..., \vec{x}_n)$$

c) The Dirac Sea

- **Recall** that **1-particle** H_0 **is not lower bounded**! → Even **less** $H_0^{(N)}$**!** → **Can we** already **"rigorify" the Dirac Sea?**



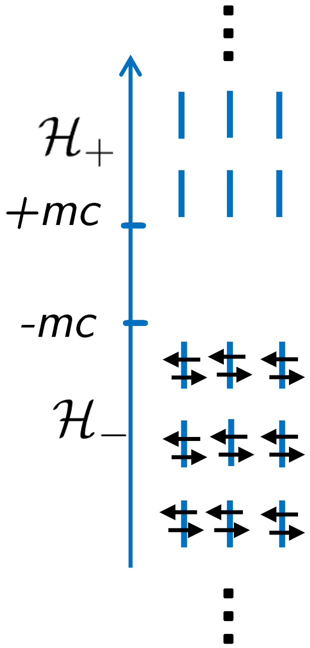
Energy eigenstates "continuous ONB"...meh

$$\mathcal{H}_1 = \mathcal{H}_+ \oplus \mathcal{H}_-$$

$$\{h_j\}_{j\in\mathbb{N}} \subset \mathcal{H}_+$$

$$\{g_k\}_{k\in\mathbb{N}} \subset \mathcal{H}_-$$

Take actual **ONB with same splitting**



- Recall, we conceptualized it **as negative energy subspace fully "occupied", so consider formally the ∞ wedge prodct**

$$|\Omega\rangle := g_1 \wedge g_2 \wedge \cdots \wedge g_n \wedge \ldots$$

It would be **independent of chosen basis (up to phase)**

- Then **heuristically** we see an **identificat.**:

$g_k \wedge |\Omega\rangle = 0$ ⟷ **"No more E<0** allowed **in sea"**

$h_1 \wedge h_2 \wedge |\Omega\rangle$ ⟷ "**Two E>0 e-'s** in $h_1 \wedge h_2$ state"

$g_1 \vee |\Omega\rangle := g_2 \wedge g_3 \wedge \cdots$ ⟷ "**Lack of E<0 e-** in g_1 state" = "a **charge>0 particle in state** Cg_1" (a positron!)

$g_2 \vee (h_2 \vee |\Omega\rangle) := h_2 \wedge g_1 \wedge \hat{g}_2 \wedge g_3 \wedge \cdots$ ⟷ "**e-/e+** in the state $h_2 \wedge Cg_2$"

etc.

Apparently, we could talk **about states relative to the sea-level!** So + rigorous to use the net balance via:

$$\mathscr{F}^{(N,M)} := Anti(\mathcal{H}_+^{\otimes N}) \otimes Anti(C\mathcal{H}_-^{\otimes M})$$

N e- and M e+ Hilbert Space

Dirac's Sea is Hilbert's Hotel!

- So, we **avoided** the **"rigorification"** of **infinite wedge products through** the **e-/e+ "net balance" picture**

- **BUT:**

Dirac Sea contains ∞ **particles** (countably or not):

→ **even if states** described **wrt sea-level** and we **thought** of the **sea having** a **"fixed infinite number"** of **particles**, **by Hilbert's Hotel,** the **number of fermions** (say N e- and M e+) **over the surface** can **change in the time evolution!**

- That is, **additional E<0 electrons** can **flow to E>0 states** → **generate e-/hole pair**

  or E>0 electrons can **find hole in** E<0 **sea** → **e-/hole annihilation**

- Indeed, **although** H_0 **preserves splitting** $\mathcal{H}_1 = \mathcal{H}_+ \oplus \mathcal{H}_-$ and **hence conserves N,M,**

  slightest EM field **causes flow of probability between** $\mathcal{H}_+ \; and \; \mathcal{H}_-$ → **N,M cannot be fixed!**

1st reason to go to N(t)

Still, if Universe Finite and UV cutoff true...Dirac Sea would avoid N(t) (i.e., QFT) for fermions...

- If **Universe finite** 3-volume, $(0,L)^3$ → **only commensurate freqs.** in **Fourier expansions contribute**

  → $e^{i\vec{p}\cdot\vec{x}}$ with $\vec{p} \in \left(\frac{2\pi}{L}\right)\mathbb{Z}^3$ now **countable!**

  → **& Plane waves become actual states!**

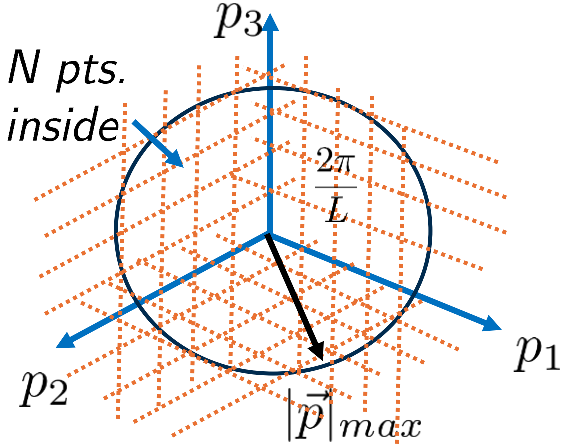
$$\left\{\frac{1}{L^{3/2}} e^{i\vec{p}\cdot\vec{x}}\right\}_{\vec{p}\in\left(\frac{2\pi}{L}\right)\mathbb{Z}^3} \subset L^2((0,L)^3,\mathbb{C})$$

is an ONB!

Dirac-particle E eigenstates are "legal" now!

→ $\left\{u_{\pm}^{(j)} e^{i\vec{p}\cdot\vec{x}}\right\}_{j\in\{1,2\},\vec{p}\in\left(\frac{2\pi}{L}\right)\mathbb{Z}^3}$

- \+ If **UV cutoff**, i.e., **impose** $|\vec{p}|_{max}$ → **Finite** (4N>>1) **energy eigenstates!** → 2N in $\mathcal{H}_-$ + 2N in $\mathcal{H}_+$



- **Take 2N E<0 eigstates** $\{g_k\}_{k=1}^{N} \subset \mathcal{H}_-$ → **Dirac Sea state is rigorous** $|\Omega\rangle := g_1 \wedge \cdots \wedge g_{2N}$

- Then, given **usual n particle deterministic Bohmian law of motion**, everything solved!

Creation and **annihilation of particles, antiparticles etc., merely effective: fundamentally ∃ persistent fermion ontology!!!**

- **All fermions** in **Standard Model** modellable **this way** → **NO need of N(t) or QFT for Fermions!**

  unless we wish it for an effective theory relative to surface of Sea since still 2N>>1...

See [4,5]

d) Ideal N Dirac-Particle treatment: Multi-time wavefunctions

- **Limitations** of **usual N particle approach:**
  - **Single-time but N space** vars. → if **Lorentz** transf. $(t, x_j) \mapsto (t'_j, x'_j)$ → **N different times!**

    Wave-vector NOT Lorentz covart.! (Its Lorentz transf. is **undefined**!)
  - **Single-time N-particle Dirac eqt. not manifestly Lorentz invariant!**
  - **Born rule** pdf $\rho(\vec{x}_1, ..., \vec{x}_N \mid \varphi, t) = \sum_{k=1}^{s} |\varphi_{s_k}|^2(t, \vec{x}_1, ..., \vec{x}_n)$ **only on particular sptime hypersurface**

Not even Lorentz-frame independent!

- As seen in prev. seminars, **instead define** a **wave-spinor** ψ **over** whole $\mathbb{R}^{4N}$ or **all space-like configs:**

$$\mathcal{S}_N := \{(x_1, ..., x_N) \in (\mathbb{R}^4)^N \mid x_i, x_j \; spacelike \; sep\} \longrightarrow \boxed{\psi_{r_1,...,r_N}(t_1, \vec{x}_1, ..., t_N, \vec{x})}$$

recover **particular case:** $\varphi(t, \vec{x}_1, ..., \vec{x}_N) = \psi(t, \vec{x}_1, ..., t, \vec{x}_N)$

- For **each space-like hypersurface** $\Sigma \subset \mathbb{R}^4$, **multi-time** ψ **yields** ϕ_Σ **in Hilbert space over Σ**

$$(\phi_\Sigma)^{(n)}(x_1, ..., x_n) := \psi^{(n)}(x_1, ..., x_n) \; \forall x_j \in \Sigma$$

Allows +**general Born rules, N particls. in curved spctme etc.**

- **Multi-time evolut. eqts**

$$i\hbar \frac{\partial}{\partial t_j}\psi = H_j \psi \quad \text{with} \quad H = \sum_{j=1}^{N} H_j$$

BUT recall, Consistency conditions prohibit interaction potentials! Only (BCs or) create/annih. ops allowed!...N(t) needed!
2nd reason to go to N(t)

III – “N(t)” DIRAC FERMIONS

Let's Demystify Second Quantization and QFT! – Via an Example : Quantized Schrödinger Field

(i) *Opaque* (but Usual) Approach

1. Take the **Lagrangian** density: $$\mathscr{L} = i\psi^*\partial_t\psi - \frac{(\nabla\psi^*\cdot\nabla\psi)}{2m}$$

2. Its **Euler-Lagrange** equation: $$i\hbar\partial_t\psi(\vec{x},t) = -\frac{\Delta}{2m}\psi(\vec{x},t)$$ **(Schrödinger eqt.)**

3. Field **Hamiltonian** (integral of Hamiltonian density associated to $\mathscr{L}$): $$\mathbb{H} = \int \psi^*(\vec{x},t)\left(\frac{-\Delta}{2m}\right)\psi(\vec{x},t)d^3x$$

4. *(Deus ex machina...)* Impose *the* **Canonical Quantization**:

$\mathbb{C}$ -Field $\psi(\vec{x},t)$ → Operator-Field $\hat{\Psi}(\vec{x},t)$

Motivated by "First" Quantization

$$\left\{\begin{aligned} \left[\hat{x}_k, \hat{p}_j\right] &= i\delta_{kj} \\ \left[\hat{x}_k, \hat{x}_j\right] &= 0 \\ \left[\hat{p}_k, \hat{p}_j\right] &= 0 \end{aligned}\right\}$$

"Second" Quantization via

$$\pi^{\mathscr{L}} := \frac{\partial\mathscr{L}}{\partial(\partial_t\psi)} = i\psi^*$$

$$\left\{\begin{aligned} \left[\hat{\Psi}(\vec{x},t), \hat{\Psi}^\dagger(\vec{x}',t)\right] &= \delta(\vec{x}-\vec{x}') \\ \left[\hat{\Psi}(\vec{x},t), \hat{\Psi}(\vec{x}',t)\right] &= 0 \\ \left[\hat{\Psi}^\dagger(\vec{x},t), \hat{\Psi}^\dagger(\vec{x}',t)\right] &= 0 \end{aligned}\right.$$

Operators on some "**Abstract**" Hilbert space $\mathscr{H}$

(*i*) *Opaque* (but Usual) Approach

"Classical" Field

(in "Schrödinger Picture")

$$\begin{cases} i\hbar\partial_t\psi(\vec{x},t) = -\dfrac{\Delta}{2m}\psi(\vec{x},t) \\ \psi(\vec{x},0) = \dfrac{1}{(2\pi)^{3/2}}\displaystyle\int e^{i\vec{x}\cdot\vec{p}}\,a(\vec{p})d^3p \end{cases}$$

$$\psi(\vec{x},t) = \frac{1}{(2\pi)^{3/2}}\int e^{-i\left(\frac{|\vec{p}|^2}{2m}t-\vec{x}\cdot\vec{p}\right)}a(\vec{p})d^3p$$

⇓

"Quantized" Field

(in "Heisenberg Picture")

$$\begin{cases} i\hbar\partial_t\hat{\Psi}(\vec{x},t) = -\dfrac{\Delta}{2m}\hat{\Psi}(\vec{x},t) \\ \hat{\Psi}(\vec{x},0) = \dfrac{1}{(2\pi)^{3/2}}\displaystyle\int e^{i\vec{x}\cdot\vec{p}}\,\hat{a}(\vec{p})d^3p \end{cases}$$

$$\hat{\Psi}(\vec{x},t) = \frac{1}{(2\pi)^{3/2}}\int e^{-i\left(\frac{|\vec{p}|^2}{2m}t-\vec{x}\cdot\vec{p}\right)}\hat{a}(\vec{p})d^3p$$

⇒

Annihilation and Creation Operators

$\hat{a}(\vec{p})$ and $\hat{a}^\dagger(\vec{p})$

$$\begin{cases} \left[\hat{a}(\vec{p}),\hat{a}^\dagger(\vec{p}\,')\right] = \delta(\vec{p}-\vec{p}\,') \\ \left[\hat{a}(\vec{p}),\hat{a}(\vec{p}\,')\right] = 0 \\ \left[\hat{a}^\dagger(\vec{p}),\hat{a}^\dagger(\vec{p}\,')\right] = 0 \end{cases}$$

Hamiltonian Operator

$$\mathbb{H} = \int \hat{\Psi}^\dagger(\vec{x},t)\left(\frac{-\Delta}{2m}\right)\hat{\Psi}(\vec{x},t)d^3x$$

$$= \int \frac{|\vec{p}|^2}{2m}\hat{a}^\dagger(\vec{p})\hat{a}(\vec{p})d^3p$$

- $\dfrac{\mathrm{d}}{\mathrm{d}t}\hat{\Psi}(\vec{x},t) = i\left[\mathbb{H},\hat{\Psi}(\vec{x},t)\right]$

(i) *Opaque* (but Usual) Approach

Number Operator

$$\hat{N} = \int \hat{a}^\dagger(\vec{p})\hat{a}(\vec{p})d^3p$$

Why "particle" interpret.?

Enter "Vacuum" : $|\Omega\rangle \in \mathscr{H}$, a state such that

- **"Ground State" of** $\mathbb{H}$
- $\hat{a}(\vec{p})|\Omega\rangle = 0$ **"No particle" to destroy** → $\hat{N}|\Omega\rangle = 0$
- $\hat{a}^\dagger(\vec{p})|\Omega\rangle =: |\vec{p}\rangle$ **"Create" a Particle of momentum p** → $\hat{N}|\vec{p}\rangle = 1$
- $\hat{a}^\dagger(\vec{p}_1)\cdots\hat{a}^\dagger(\vec{p}_n)|\Omega\rangle =: |\vec{p}_1, ..., \vec{p}_n\rangle$ **"Create" n Particles** → $\hat{N}|\vec{p}_1, ..., \vec{p}_n\rangle = n$
- $\hat{a}(\vec{p}_j)|\vec{p}_1, ..., \vec{p}_n\rangle = |\vec{p}_1, ..., \vec{p}_{j-1}, \vec{p}_{j+1}, ..., \vec{p}_n\rangle$ **"Destroy" a Particles**

- **Most general "1 particle state" i.e.** $\hat{N}|1\rangle = 1$ → $|1\rangle = \int \varphi(\vec{p})|\vec{p}\rangle d^3p$
- **Most general "n particle state" i.e.** $\hat{N}|n\rangle = n$ → $|n\rangle = \int S_+\varphi(\vec{p}_1, ..., \vec{p}_n)|\vec{p}_1\cdots\vec{p}_n\rangle d^3p_1\cdots d^3p_n$

(i) *Opaque* (but Usual) Approach

("The contact with reality!" –you think…but)

And finally, the **"position eigenstates"** $|\vec{x}_1, ..., \vec{x}_n, t\rangle := \hat{\Psi}^\dagger(\vec{x}_1, t) \cdots \hat{\Psi}^\dagger(\vec{x}_n, t)|\Omega\rangle$

- " So $\hat{\Psi}^{(\dagger)}(\vec{x}, t)$ creates/destroys a particle in $(\vec{x}, t)$ "

- And you are told*: "But do not take the particle picture very seriously, a **particle** is just an **excitation of the field**. And either way, **during interactions** there is **no** consistent way to talk about **particles**"

- Or when asked about **what does all this have to do with reality?:** "wait until **cross sections** and scattering matrices, those will be the observable predictions, the rest is **merely a metaphor**".

- Scattering arrives and: "In an interaction, **particles take all possible paths** and all possible Feynman diagrams happen at once, but still do not think about particles, this is just a metaphor…."

"well… **shut up and calculate**!"

And you still wonder…

- What about the Born rule?
- Why are we ignoring positions?
- Can we really talk about experiments only in momentum terms?

*By an imaginary orthodox QFT expert

a) How QFT ≡ QM with Variable Particle Number

Via Example : Quantized Schrödinger Field

(*ii*) *Transparent* Approach

- **1** Schrödinger **particle Hilbert Space** $\mathcal{H}_1 := L^2(\mathbb{R}^3, \mathbb{C})$

- **Fock Space:** Hilb. sp. of **all numbers** of (identical) **particles**

$$\mathscr{H} := \mathscr{F}_{sym}\Big(\mathcal{H}_1\Big) := \bigoplus_{n=0}^{\infty} Sym\big(\mathcal{H}_1^{\otimes n}\big) = \bigoplus_{n=0}^{\infty} Sym\Big(\underbrace{L^2(\mathbb{R}^3,\mathbb{C}) \otimes \cdots \otimes L^2(\mathbb{R}^3,\mathbb{C})}_{n \text{ times}}\Big)$$

$$\psi \in \mathscr{H} \quad \text{is s.t.} \quad \psi = (\psi^{(0)}, \psi^{(1)}, \psi^{(2)}, \ldots) \quad \text{with } \psi^{(n)} \in Sym\big(L^2(\mathbb{R}^3,\mathbb{C})^{\otimes n}\big) \ \text{ and } \ \langle \psi, \psi \rangle < +\infty$$

Sectors by particle number

Inner Product: $\langle \psi, \phi \rangle := \sum_{n=0}^{\infty} \langle \psi^{(n)}, \phi^{(n)} \rangle_{\mathcal{H}_1^{\otimes n}}$

- **Obvious position probability density** in *n*-th sector

$$\rho_{\psi}^{(n)}(\vec{x}_1, \ldots, \vec{x}_n) := |\psi^{(n)}(\vec{x}_1, \ldots, \vec{x}_n)|^2 \longrightarrow \mathbb{P}(n \text{ particles} \mid \psi) = \int_{\mathbb{R}^{3n}} |\psi^{(n)}(\vec{x}_1, \ldots, \vec{x}_n)|^2 d^{3n}x$$

(*ii*) *Transparent* Approach

- **Obvious definition** of **creation** and **annihilation** operators (no "antiparticles" ⟶ match with **field operators**)

$f \in \mathcal{H}_1$, $\psi \in \mathscr{H}$:

$$(\Psi^\dagger(f)\psi)^{(n)}(\vec{x}_1, \dots, \vec{x}_n) := Sym\Big(f(\vec{x}_1)\psi^{(n-1)}(\vec{x}_2, \dots, \vec{x}_n)\Big) \quad \text{with } (\Psi^\dagger(f)\psi)^{(0)} \equiv 0$$

Add particle in state *f* *(in each sector)* and symmetrize

$$(\Psi(f)\psi)^{(n)}(\vec{x}_1, \dots, \vec{x}_n) := \sqrt{n+1}\int_{\mathbb{R}^3} f^*(\vec{x}_{n+1})\psi^{(n+1)}(\vec{x}_1, \vec{x}_2, \dots; \vec{x}_{n+1})d^3x_{n+1}$$

"Trace-out" particle in state *f* (from each sector) i.e., partial inner product

- Naturally **canonical commutation relations** emerge

$$\begin{cases} \left[\Psi(f), \Psi^\dagger(g)\right] = \langle f, g\rangle Id \\ \left[\Psi(f), \Psi(g)\right] = 0 \\ \left[\Psi^\dagger(f), \Psi^\dagger(g)\right] = 0 \end{cases} \qquad \forall f, g \in \mathcal{H}_1$$

(*ii*) *Transparent* Approach

- **Obvious definition** of **number** operator

$$\psi \in \mathscr{H}$$

$$\hat{N}\psi := \left(0, 1\,\psi^{(1)}, 2\,\psi^{(2)}, 3\,\psi^{(3)}, ...\right)$$

Obvious definition of **vacuum state**

$$|\Omega\rangle := (1, 0, 0, ...) \longrightarrow \hat{N}|\Omega\rangle = 0$$

Property: $\Psi(f)|\Omega\rangle = 0 \quad \forall f \in \mathcal{H}_1$

Example : $\psi = (0, 0, \psi^{(2)}, 0, ...) \longrightarrow \hat{N}\psi = 2\psi$ (most general –exactly– 2 particle state)

only $\neq 0$ *prob. for 2-particles*

- **Time Evolution** of state $\psi = (\psi^{(0)}, \psi^{(1)}, \psi^{(2)}, ...) \in \mathscr{H}$:

Given "**Fock-Hamiltonian**" $\mathbb{H} : \mathscr{H} \to \mathscr{H}$ s.th.

$$(\mathbb{H}\psi)^{(n)}(x_1, ..., x_n) = F(\psi^{(1)}, \psi^{(2)}, \psi^{(3)}, ...)$$

(possible dependence on other sectors!)

"Fock-Schrödinger" Eqt

$$\begin{cases} i\hbar\partial_t\psi_t = \mathbb{H}\psi_t \\ \psi_t = \psi_0 \in \mathscr{H} \end{cases}$$

Ideally: $\exists\, \mathbb{U}_t : \mathscr{H} \to \mathscr{H}$ SCOPUG

$$\boxed{\psi_t = \mathbb{U}_t\psi_0}$$

(*ii*) *Transparent* Approach

What are $\mathbb{H}$ or $\mathbb{U}_t$ for the "free quantized Schrödinger field"?

Method 1: Lift H ("second quantize" the Hamiltonian)

- For **free time** evolution, we know the **1-particle evolution is**

$$i\hbar\partial_t\psi^{(1)} = H\psi^{(1)} \qquad \text{with} \qquad H := \frac{-\Delta}{2m}$$

- So the **obvious lift/implementation** to all sectors is so-called "**second quantized**" H

$$i\hbar\partial_t\psi^{(n)} = (\underbrace{\Gamma(H)}_{\mathbb{H}}\psi)^{(n)} \qquad \text{with} \quad \Gamma(H)^{(n)} := \sum_{j=1}^{n} \underbrace{Id\otimes\cdots\otimes Id\otimes(-\frac{\Delta}{2m})}_{j}\otimes Id\otimes\cdots Id$$

$$= \text{"}\sum_{j=1}^{n}\frac{-\Delta_j}{2m}\text{"}$$

- Up to technicalities, in general,

$$\mathbb{H} = \sum_{i,j\in\mathbb{N}} \Psi^*(f_i)\langle f_i, Hf_j\rangle\Psi(f_j) \qquad \text{for any } \{f_j\}_{j\in\mathbb{N}} \text{ ONB of } \mathcal{H}_1$$

(*ii*) *Transparent* Approach

Method 2: Lift U_t ("second quantize" propagator)

- If 1-particle propagator SCOPUG $U_t : \mathcal{H}_1 \to \mathcal{H}_1$ (e.g., $U_t = e^{-\frac{i}{\hbar}Ht}$)

  In operator norm!

  time evolving field operator "Heisenberg picture"

  $$\Psi_t(f) := \Psi(e^{\frac{i}{\hbar}Ht} f) \longrightarrow i\hbar \frac{\mathrm{d}}{\mathrm{d}t}\Psi_t(f) = \Psi_t(Hf) \quad \forall f \in \mathcal{H}_1$$

  annih(/create) at state f of time t but taken to t=0 –bec. in Heisenb. it's all relative to "now"-states

- If $\exists\, \mathbb{U}_t : \mathscr{H} \to \mathscr{H}$ s.th

  $$\Psi_t(f) = \mathbb{U}_t^\dagger \Psi(f) \mathbb{U}_t$$ → is **lift** of $\mathbb{U}_t$ to all sectors U_t

  It is the same to (rhs) evolve state to t, create/annih particle at f and unevolve to t=0, than (lhs) create/annih particle at how f (state of time t) was at t=0

- Possible to choose $\mathbb{U}_t|\Omega\rangle = |\Omega\rangle \quad \forall t \in \mathbb{R}$

- $\langle f, g\rangle = \langle e^{\frac{i}{\hbar}Ht} f,\ e^{\frac{i}{\hbar}Ht} g\rangle$ ⟶ $\Psi_t(f)$ still satisfy **CAR** $\forall t$

- Fock space SCOPUG $\mathbb{U}_0 = Id; \quad \mathbb{U}_t\mathbb{U}_s = \mathbb{U}_{t+s}\ \forall t, s \in \mathbb{R}$

  $$\mathbb{H} := i\hbar \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \mathbb{U}_t \quad on \quad \mathfrak{D} = \left\{ \phi \in \mathscr{F} \;\middle|\; \lim_{\tau \to 0} \frac{1}{\tau}(\mathbb{U}_t - Id)\phi \ exists \right\}$$

  $$\boxed{\mathbb{U}_t = e^{-\frac{i}{\hbar}\mathbb{H}t}}$$

Opaque Approach		*Transparent Approach*
$\hat{\Psi}(\vec{x},t)$ and $\hat{a}(\vec{p})$	⟷	" $\Psi_t(\delta_{\vec{x}})$ " and " $\Psi(e^{i\vec{p}\cdot(\cdot)})$ "
$\int_{\mathbb{R}^3} \hat{\Psi}(x,t) f(x) dx$	⟷	$\Psi_t(f)$
$\hat{N} = \int \hat{\Psi}^\dagger(\vec{x},0)\hat{\Psi}(\vec{x},0)d^3x = \int a^\dagger(\vec{p})a(\vec{p})dp$	⟷	$\hat{N} = \sum_{j\in\mathbb{N}} \Psi^\dagger(f_j)\Psi(f_j)$ with $\{f_j\}_{j\in\mathbb{N}}$ ONB

"." because
$\delta_{\vec{x}} \notin \mathcal{H}_1$
$e^{i\vec{p}\cdot(\cdot)} \notin \mathcal{H}_1$

Define $|\vec{x}_1, ..., \vec{x}_n, t\rangle := \hat{\Psi}^\dagger(\vec{x}_1,t)\cdots\hat{\Psi}^\dagger(\vec{x}_n,t)|\Omega\rangle$

$\langle\vec{x}_1, ..., \vec{x}_n, t|\psi\rangle = \langle\Omega|\hat{\Psi}(\vec{x}_n,t)\cdots\hat{\Psi}(\vec{x}_1,t)|\psi\rangle$	⟷	$\psi^{(n)}(\vec{x}_1, ..., \vec{x}_n, t)$
$\langle\vec{x}_1, t|\vec{p}\rangle = \langle\vec{x}_1, t|\hat{a}^\dagger(\vec{p})|\Omega\rangle$	⟷	$e^{i(\frac{|\vec{p}|^2}{2m}t - \vec{p}\cdot\vec{x})}$
$\langle\vec{x}_1, ..., \vec{x}_n, t|\mathbb{H}|\psi\rangle$	⟷	$(\mathbb{H}\psi)^{(n)}(\vec{x}_1, ..., \vec{x}_n, t)$
$\mathbb{H} = \int \hat{\Psi}^\dagger(\vec{x},0)\left(\frac{-\Delta}{2m}\right)\hat{\Psi}(\vec{x},0)d^3x = \int \frac{|\vec{p}|^2}{2m}\hat{a}^\dagger(\vec{p})\hat{a}(\vec{p})d^3p$	⟷	$\mathbb{H} =$ " $\int \Psi^\dagger(\delta_{\vec{x}'})\langle\vec{x}', Hx\rangle\Psi(\delta_{\vec{x}})d^3x d^3x'$ " $=$ " $\int \Psi^\dagger(e^{-i\vec{p}\cdot\vec{x}'})\langle\vec{p}', Hp\rangle\Psi(e^{-i\vec{p}\cdot\vec{x}})d^3p d^3p'$ "

a) How QFT ≡ QM with Variable Particle Number

- So yes, **second quantization** (and QFT) turned out to be merely the consideration of **wave-functions that can have different numbers of identical particle variables** and are **allowed to interact** between such **different particle-number sectors**

- **Naturally**, one **considers literal creation and annihilation ops.** to **"add" or "remove" states to/from all sectors at once** ⟶ call them **"field operators"**

- If the **addition/removal of states**
  - **symmetrizes** (and sectors contain symmetric wavefunctions) –i.e., *boson* QFT
    ⟶ **field ops.** satisfy **canonical commutation relations (CCR)**
    } *just saw an example*
  - **antisymmetrizes** –i.e., *fermion* QFT
    ⟶ field ops. satisfy **canonical anti-commutation relations (CAR)**
    } *we will check an example now!*

  - As [2] would say: "***Second quantization...big name, little substance...***"

(Although to be fair, in a field ontology, it can indeed be seen as a literal second quantization...we'll see later)

a) How QFT ≡ QM with Variable Particle Number

The Occupation Number Representation

From [2, Exercise 6.10]

- Let 1-particle Hilbert space $\mathcal{H}_1$ with ONB $\left\{|j\rangle\right\}_{j\in J} \subset \mathcal{H}_1 \quad J \subseteq \mathbb{N}\cup\{0\}$
- For **Symmetric** species (**bosons**)
  - A "list with **occupation** number **per state**" is a map $n : J \to \mathbb{N}\cup\{0\}$ with $\sum_j n_j = N < \infty$ *number of "occupiers"*

    e.g., $n = (3,7,1,0,0,\ldots)$ *3 particles in state 0, 7 in state 1, 1 in state 2 etc.*
  - Then define (e.g., $|3710000\cdots\rangle$)

$$|n\rangle := Symm\Big(\bigotimes_{\substack{j\in J\\ n_j\neq 0}} |j\rangle^{\otimes n_j}\Big) \in Symm(\mathcal{H}_1^{\otimes N}) \subset \mathscr{F}_{sym}(\mathcal{H}_1)$$

→ $\{|n\rangle\}_n$ **form an ONB of** $\mathscr{F}_{sym}(\mathcal{H}_1)$

↳ the **field op.** of $\mathscr{F}_{sym}(\mathcal{H}_1)$ is defined s.th. equivalently

$$|n\rangle = \prod_{\substack{j\in J\\ n_j\neq 0}} \Psi^\dagger(|j\rangle)^{n_j}|\Omega\rangle$$

- **Similarly** for **fermions**, but **restricting "occupation" to 1 or 0 per state** and applying $Anti(\cdot)$ instead of $Symm(\cdot)$

Although for fermions the wedge product notation is more practical!

(b) Quantized Dirac Field

(i) Dirac-Particle Model

We will assume that

$$x_j \equiv (\vec{x}_j, s_j); \;\; \vec{x}_j \in \mathbb{R}^3, s_j \in \{1,2,3,4\}$$

unless context says otherwise

- **1 Dirac-particle Hilbert Space** $\mathcal{H}_1 := L^2(\mathbb{R}^3, \mathbb{C}^4)$

- **Fock Space:** Hilb. sp. of **all numbers** of (identical) **particles**

$$\mathscr{H} := \mathscr{F}_{anti}\Big(\mathcal{H}_1\Big) := \bigoplus_{n=0}^{\infty} Anti\big(\mathcal{H}_1^{\otimes n}\big) = \bigoplus_{n=0}^{\infty} Anti\Big(\underbrace{L^2(\mathbb{R}^3,\mathbb{C}^4)\otimes\cdots\otimes L^2(\mathbb{R}^3,\mathbb{C}^4)}_{n \text{ times}}\Big)$$

$$\psi \in \mathscr{H} \quad \text{is s.t.} \quad \psi = (\psi^{(0)}, \psi^{(1)}, \psi^{(2)}, \ldots) \quad \text{with } \psi^{(n)} \in Anti\big(L^2(\mathbb{R}^3,\mathbb{C}^4)^{\otimes n}\big) \text{ and } \langle \psi, \psi \rangle < +\infty$$

$$\langle \psi, \phi \rangle := \sum_{n=0}^{\infty} \langle \psi^{(n)}, \phi^{(n)} \rangle_{\mathcal{H}_1^{\otimes n}} = \int_{\mathbb{R}^{3n}} \sum_{s_1,\ldots,s_n=1}^{4} \psi^{(n)*}_{s_1,\ldots,s_n}(\vec{x}_1,\ldots,\vec{x}_n)\phi^{(n)}_{s_1,\ldots,s_n}(\vec{x}_1,\ldots,\vec{x}_n) d^{3n}x$$

- **Obvious position probability density** in n-th sector

$$\rho^{(n)}_{\psi}(\vec{x}_1,\ldots,\vec{x}_n) := \sum_{s_1,\ldots,s_n=1}^{4} |\psi_{s_1,\ldots,s_n}(\vec{x}_1,\ldots,\vec{x}_n)^{(n)}|^2$$

- **Define creation/annihilation operators,**

Add particle in state *f* *(in each sector)* and anti-symmetrize

$$\psi \in \mathscr{H},\quad f \in \mathcal{H}_1 \quad \begin{cases} (\Psi^\dagger(f)\psi)^{(n)}(x_1,...,x_n) := Anti\Big(f(x_1)\psi^{(n-1)}(x_2,...,x_n)\Big) = (f\wedge\psi^{(n-1)})(x_1,...,x_n) \\ (\Psi(f)\psi)^{(n)}(x_1,...,x_n) := \sqrt{n+1}\int_{\mathbb{R}^3} f^\dagger(x)\psi^{(n+1)}(x,x_1,x_2,...,x_n)d^3x = (f\vee\psi^{(n+1)})(x_1,...,x_n) \end{cases}$$

"Trace-out" particle in state *f* (from each sector) i.e., partial inner product

- Naturally **Canonical Anti-commutation** Relations emerge $\forall f_1, f_2 \in \mathcal{H}_1$

$$\begin{cases} \left\{\Psi(f_1),\Psi^\dagger(f_1)\right\} = \langle f_1, f_2\rangle Id \\ \left\{\Psi(f_1),\Psi(f_2)\right\} = 0 \\ \left\{\Psi^\dagger(f_1),\Psi^\dagger(f_2)\right\} = 0 \end{cases}$$

- **Indeed adjoint** $(\Psi(f))^* = \Psi^\dagger(f) \quad \forall f \in \mathcal{H}_+$

- **Bounded operators**

$$||\Psi^*(f)\psi||^2 + ||\Psi(f)\psi||^2 = \langle\psi, \{\Psi(f),\Psi^*(f)\}\psi\rangle = ||f||^2||\psi||^2$$

- **Number op** $\hat{N}\psi := \left(0, 1\,\psi^{(1)}, 2\,\psi^{(2)}, ...\right)$

$$|0\rangle := (1,0,0,...) \qquad \hat{N}|0\rangle = 0$$

- **Vacuum behaviour**

$$\Psi(f)|0\rangle = 0 \in \mathscr{H} \;; \quad (\Psi^*(f)|0\rangle)^{(n)} = \begin{cases} f & for\ n=1 \\ 0 & else \end{cases}$$

- **An ONB of** $\mathscr{H}$ **:** Take **any** $\{f_j\}_{j\in\mathbb{N}} \subset L^2(\mathbb{R}^3, \mathbb{C}^4)$ **1-particle ONB:**

$$\Psi^\dagger(f_{j_1})\cdots\Psi^\dagger(f_{j_n})|0\rangle = f_{j_1}\wedge\cdots\wedge f_{j_n} \qquad j_1<\cdots<j_n$$

is **ONB** of $Anti(\mathcal{H}_1^{\otimes n})$ → union $\forall n$ = ONB of $\mathscr{H}$

For a general state $\psi\in\mathscr{H}$ → $$\psi^{(n)} = \sum_{1\le j_1<\cdots<j_n<\infty} c_{j_1,\ldots,j_n}\, f_{j1}\wedge\cdots\wedge f_{jn}$$

$$\psi = \left(c^{(0)},\ \sum_{j_1} c^{(1)}_{j1} f_{j1},\ \sum_{j_1<j_2} c^{(2)}_{j_1,j_2} f_{j1}\wedge f_{j2},\ \ldots\right)$$

- **Naturally, Pauli Exclusion**

in n-th sector

$$\Psi(f)^\dagger\Psi(f)^\dagger\psi = \frac{1}{2}\{\Psi(f),\Psi(f)\}\psi = 0 \quad \longleftarrow \quad (\Psi(f)^\dagger\Psi(f)^\dagger\psi)^{(n)} = \underbrace{f\wedge f}\wedge\psi^{(n-2)} = 0$$

Recall: two LD states yield 0 if antisymtzd.

- **Number operator in terms of** $\Psi(\cdot)$ → $\hat{N} = \sum_{j=1}^{\infty} \Psi^\dagger(f_j)\Psi(f_j)$ or **formally** $\hat{N} = \text{“}\int_{\mathbb{R}^3} \Psi^\dagger(x)\Psi(x)d^3x\text{”}$

  $\{f_j\}_{j\in\mathbb{N}} \subset L^2(\mathbb{R}^3, \mathbb{C}^4)$ **any 1-particle ONB**

- Why?

  For a **general *(n)*-sector basis vector**:

$f_{k_1} \wedge \cdots \wedge f_{k_n}$ → [$\Psi(f_j)$ "annihilate"]

- if $j \notin \{k_1, ..., k_n\}$ → 0 → 0
- if $j \in \{k_1, ..., k_n\}$ → $(-1)^j f_{k_1} \wedge \cdots \wedge \widehat{f_j} \wedge \cdots \wedge f_{k_n}$ → [$\Psi^\dagger(f_j)$] → $f_{k_1} \wedge \cdots \wedge f_{k_n}$ same as input!

- So,

$$\sum_{j=1}^{\infty} \Psi^\dagger(f_j)\Psi(f_j)(f_{k_1} \wedge \cdots \wedge f_{k_n}) = n\ f_{k_1} \wedge \cdots \wedge f_{k_n}$$

$\Psi^\dagger(f_j)\Psi(f_j)$ acts *n* times as "identity" + the rest "0"

- Because in general,

$$\psi^{(n)} = \sum_{1\le j_1<\cdots<j_n<\infty} c_{j_1,\ldots,j_n}\ f_{j1} \wedge \cdots \wedge f_{jn}$$

→
$$\sum_{j=1}^{\infty} \Psi^\dagger(f_j)\Psi(f_j)\psi^{(n)} = n\ \psi^{(n)} = (\hat{N}\psi)^{(n)}$$

□

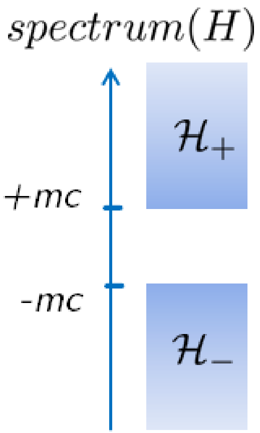
- 1-particle Hamiltonian $H = -i\hbar c\vec{\alpha}\cdot\vec{\nabla} + \beta mc^2$ is self-adjoint (in Sobolev space)

→ **lifted Hamiltonian** $\mathbb{H} = \Gamma(H)$ $\qquad \Gamma(H)^{(n)} := \sum_{j=1}^{n} Id \otimes \cdots \otimes Id \otimes \underbrace{H}_{j} \otimes Id \otimes \cdots Id$

- 1-particle propagator SCOPUG $U_t = e^{-\frac{i}{\hbar}Ht}$ field op. evolution $\Psi_t(f) := \Psi(e^{\frac{i}{\hbar}Ht} f)$

→ **lifted Propagator** $\mathbb{U}_t$ s.th. $\Psi_t(f) = \mathbb{U}_t^\dagger \Psi(f)\mathbb{U}_t$ $\qquad \mathbb{H} := i\hbar \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \mathbb{U}_t$

- But recall that 1-particle H **is not lower bounded**!



Even **less** $\mathbb{H}$ **!**

$$L^2(\mathbb{R}^3, \mathbb{C}^4) = \mathcal{H}_+ \oplus \mathcal{H}_-$$

Take **ONB** $\{g_k\}_{k\in\mathbb{N}} \subset \mathcal{H}_-$

→

"Energy" e.g., $\mathbb{H}$ expectat. ↑

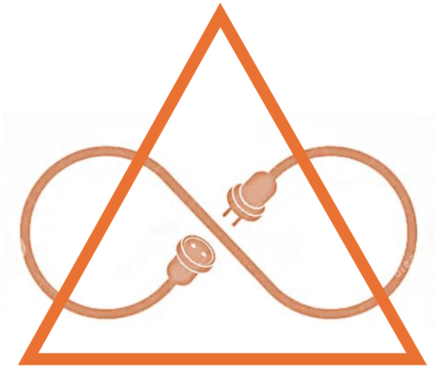
$g_1 \wedge g_2 \wedge g_3$

$g_1 \wedge g_2 \wedge g_3 \wedge g_4$

$g_1 \wedge g_2 \wedge g_3 \wedge g_4 \wedge g_5$

$\vdots$

Unboundedly low energy possible!

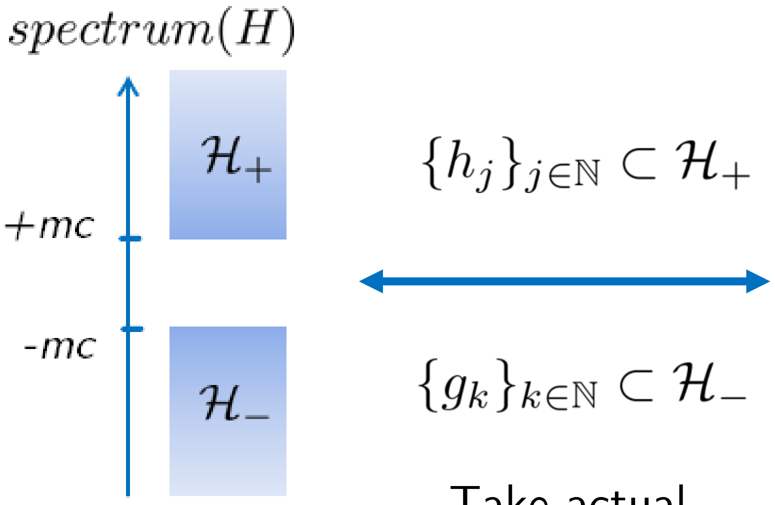


So "∞ **Negative energy problem" is worse** now!!! Not only **probability can flow to unboundedly negative energy states**, but **unboundedly many negative energy particle-sectors can get occupied/superposed** (particles created)!

Remedies :

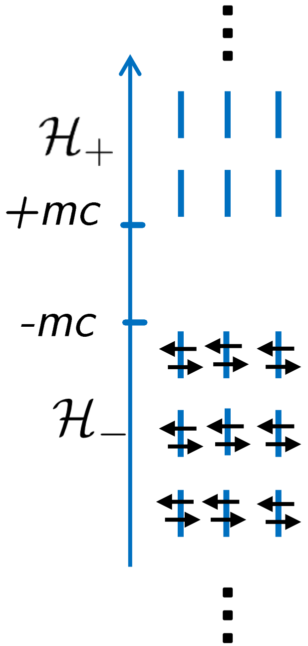
1. **Heuristic Dirac Sea** *("rigorification" attempt 2)*
2. **Electron-Positron Splitting**
3. **No, no, Dirac Sea, now seriously** *(third time's the charm!)*

(ii) Heuristic Dirac Sea



Energy eigenstates "continuous ONB"...meh

Take actual **ONB with same splitting**



- Define **formally** the state **occupying whole** $\mathcal{H}_-$ *(all E<0 subspace "filled" with e-s)*

$$|\Omega\rangle := \prod_{j=1}^{\infty} \Psi^\dagger(g_j)|0\rangle \;=\; g_1 \wedge g_2 \wedge \cdots \wedge g_n \wedge \ldots$$

It would be **independent of chosen basis**, so formally

$$|\Omega\rangle = \text{“} \prod_{E_j<-mc} \Psi^\dagger(|E_j\rangle)|0\rangle\text{”}$$

It would be **the GS of** $\mathbb{H}$ **!**

Then call $|\Omega\rangle$ the **Sea State**

How would the **Sea State behave?**

$$|\Omega\rangle := \prod_{j=1}^{\infty} \Psi^\dagger(g_j)|0\rangle = g_1 \wedge g_2 \wedge \cdots$$

- Annihilate electron of negative "energy" → creates a "hole" in the sea

$$\Psi(g_k)|\Omega\rangle = (-1)^k \underbrace{\Psi(g_k)\Psi^\dagger(g_k)}_{Id} \prod_{j=1, j\neq k}^{+\infty} \Psi^\dagger(g_j)|0\rangle = (-1)^k\, g_1 \wedge g_2 \wedge \cdots \wedge \widehat{g_k} \wedge \ldots$$

- Create electron of negative "energy" → by Pauli exclusion zero

$$\Psi^\dagger(g_k)|\Omega\rangle = (-1)^k \Psi^\dagger(g_k)\Psi^\dagger(g_k) \prod_{j=1, j\neq k}^{+\infty} \Psi^\dagger(g_j)|0\rangle = (-1)^k\, g_1 \wedge g_2 \wedge \cdots \underbrace{\wedge g_k \wedge g_k \wedge}_{} \ldots = 0$$

- Annihilate electron of positive "energy" → there was none → zero

$$\Psi(h_k)|\Omega\rangle = \Psi(h_k) \prod_{j=1, j\neq k}^{+\infty} \Psi^\dagger(g_j)|0\rangle = 0 \qquad \langle h_k, g_j\rangle = 0 \quad \forall j, k$$

- Create electron of positive "energy" → Ok

$$\Psi^\dagger(h_k)|\Omega\rangle = \Psi^\dagger(h_k) \prod_{j=1, j\neq k}^{+\infty} \Psi^\dagger(g_j)|0\rangle = h_k \wedge g_1 \wedge g_2 \wedge \cdots$$

- Feels **like creation** & **annih**. of **holes** and **E>0 e-s**, with $|\Omega\rangle$ as **vacuum** and **for holes being** create./annih. switched

- $\Psi(g_k)|\Omega\rangle = g_1 \wedge g_2 \wedge \cdots \wedge \widehat{g_k} \wedge \ldots$
- $\Psi^\dagger(g_k)|\Omega\rangle = 0$

→ **Create hole in** surface = Only **finite E<0 states away from surface** → **Enough** if **keep track of absent E<0 states**

- $\Psi^\dagger(h_k)|\Omega\rangle = h_k \wedge g_1 \wedge g_2 \wedge \cdots$
- $\Psi(h_k)|\Omega\rangle = 0$

→ **Create e-** over surface = Only **finite E>0 states away from surface** → **Enough if keep track of present E>0 states**

Keep track of net balance of particles

- So, def. **create/annih operators taking only positive or negative energy states** s.th.

$f \in L^2(\mathbb{R}^3, \mathbb{C}^4)$

$f = P_+ f + P_- f$

$$a : \mathcal{H}_+ \to Bded\ ops\ (\mathscr{H})$$
$$a^\dagger(P_+ f) := \Psi^\dagger(P_+ f)$$
$$a(P_+ f) := \Psi(P_+ f)$$

Extra electron operator

$$b : \mathcal{H}_- \to Bded\ ops\ (\mathscr{H})$$
$$b^\dagger(P_- f) := \Psi(P_- f)$$
$$b(P_- f) := \Psi^\dagger(P_- f)$$

Hole operator

$$\Psi^\dagger(f) = a^\dagger(P_+ f) + b(P_- f)$$
$$\Psi(f) = a(P_+ f) + b^\dagger(P_- f)$$

Field operators can be rewritten as

- For $a(\cdot)\,\&\,b(\cdot)$, $|\Omega\rangle$ is indeed a vacuum state!

$$\Psi(g_k)|\Omega\rangle = g_1 \wedge g_2 \wedge \cdots \wedge \widehat{g_k} \wedge \ldots \longleftrightarrow b^\dagger(g_k)|\Omega\rangle = \widehat{g_k}$$

$$\Psi^\dagger(g_k)|\Omega\rangle = 0 \longleftrightarrow b(g_k)|\Omega\rangle = 0$$

$$\Psi^\dagger(f_k)|\Omega\rangle = f_k \wedge g_1 \wedge g_2 \wedge \cdots \longleftrightarrow a^\dagger(f_k)|\Omega\rangle = f_k$$

$$\Psi(f_k)|\Omega\rangle = 0 \longleftrightarrow a(f_k)|\Omega\rangle = 0$$

Then we would call $|\Omega\rangle$ the **(effective) vacuum** (instead of Dirac sea)

$$\mathscr{F}_{anti}(\mathcal{H}_+ \oplus \mathcal{H}_-) \qquad \mathscr{F}_{anti}(\mathcal{H}_+) \otimes \mathscr{F}_{anti}(\mathcal{H}_-)$$

← **Only net balance wrt the sea!**

- $a(\cdot), b(\cdot)$ **trivially satisfy anti-commutation** relations

$\forall f_1, f_2 \in \mathcal{H}_1$

$$\begin{cases} \left\{\Psi(f_1), \Psi^\dagger(f_1)\right\} = \langle f_1, f_2\rangle Id \\ \left\{\Psi(f_1), \Psi(f_2)\right\} = 0 \\ \left\{\Psi^\dagger(f_1), \Psi^\dagger(f_2)\right\} = 0 \end{cases}$$

⟷

$\forall h_1, h_2 \in \mathcal{H}_+$

$$\begin{cases} \left\{a(h_1), a^\dagger(h_2)\right\} = \langle h_1, h_2\rangle Id \\ \left\{a(h_1), a(h_2)\right\} = 0 \\ \left\{a^\dagger(h_1), a^\dagger(h_2)\right\} = 0 \end{cases}$$

$\forall g_1, g_2 \in \mathcal{H}_-$

$$\begin{cases} \left\{b(g_1), b^\dagger(g_2)\right\} = \langle g_1, g_2\rangle^* Id \\ \left\{b(g_1), b(g_2)\right\} = 0 \\ \left\{b^\dagger(g_1), b^\dagger(g_2)\right\} = 0 \end{cases}$$

- Recall, $\hat{N} = \sum_{j=1}^{\infty} \Psi^\dagger(f_j)\Psi(f_j)$ for **any 1-particle ONB** $\{f_j\}_{j\in\mathbb{N}} \subset L^2(\mathbb{R}^3, \mathbb{C}^4)$

- Choose ONBs, $\begin{cases} \{h_j\}_{j\in\mathbb{N}} \subset \mathcal{H}_+ \\ \{g_k\}_{k\in\mathbb{N}} \subset \mathcal{H}_- \end{cases}$ → $\{h_j, g_k\}_{j,k\in\mathbb{N}} \subset L^2(\mathbb{R}^3, \mathbb{C}^4)$ **is 1-particle ONB**

$$\hat{N} = \sum_{j=1}^{\infty} \Psi^\dagger(h_j)\Psi(h_j) + \sum_{k=1}^{\infty} \Psi^\dagger(g_k)\Psi(g_k)$$

So, **counts E>0 states** in **wedge** $\hat{N}_{E>0}$ — **counts E<0 states** in **wedge** $\hat{N}_{E<0}$

But,
recall $\Psi^\dagger(f_j)\Psi(f_j)$ acts as Id on wedge if $f_j \in$ wedge, else **0**

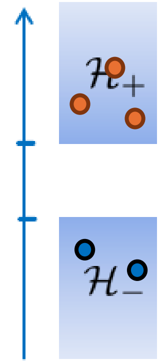
$$|\Omega\rangle := g_1 \wedge g_2 \wedge \cdots \longrightarrow \hat{N}|\Omega\rangle = +\infty|\Omega\rangle$$

- **Need to "re-normalize" number operator to Sea-level**

↓

2 ways → **Convenient for Dirac sea pictr.**
→ **Convenient for e-/e+ picture**

In terms of the adapted ops $\hat{N} = \sum_{j=1}^{\infty} a^\dagger(h_j)a(h_j) + \underbrace{\sum_{k=1}^{\infty} b(g_k)b^\dagger(g_k)}_{\textit{almost!}}$



Number operator relative to sea-level 1: **count excess electrons wrt Sea = (E>0 electrons − holes)**

$$\hat{N}_{\substack{excess\ e^-\\ wrt\ \Omega}} = \sum_{j=1}^{\infty}\Psi^\dagger(h_j)\Psi(h_j) - \sum_{k=1}^{\infty}(\underbrace{\|g_k\|^2}_{1} - \Psi^\dagger(g_k)\Psi(g_k)) \overset{(CAR)}{=} \sum_{j=1}^{\infty}\Psi^\dagger(h_j)\Psi(h_j) - \sum_{k=1}^{\infty}\Psi(g_k)\Psi^\dagger(g_k)$$

$$\hat{N}_{\substack{excess\ e^-\\ wrt\ \Omega}}|\Omega\rangle = 0$$

It worked!

Take out contribt. of Sea e-s before add ∞

In terms of e-/e+ ops

$\hat{N}_{e-}$ − $\hat{N}_{e+}$

Counts e-s — Counts holes

$$\hat{N}_{\substack{excess\ e-\\ wrt\ \Omega}} = \sum_{j=1}^{\infty} a^\dagger(h_j)a(h_j) - \sum_{k=1}^{\infty} b^\dagger(g_k)b(g_k)$$

- **But…it has negative eigenvalues if less electrons than Sea e.g.,**

$$\psi = g_3 \wedge \cdots \wedge g_n \wedge \cdots = \text{“}\widehat{g_1} \wedge \widehat{g_2}\text{”}$$

$$\hat{N}_{\substack{excess\ e^-\\ wrt\ \Omega}}\psi = -2\psi$$

It **counts charge wrt** $|\Omega\rangle$ sea!

That's why it is aka **charge operator** *(well, -e times it)*

$$\hat{Q} := q\hat{N}_{\substack{excess\ e^-\\ wrt\ \Omega}}$$

q:=-e fermion charge

Number operator relative to sea-level 2: **count number of E>0 electrons + holes (positrons)**

$$\hat{N}_{\substack{E>0\ e-\\ \&\ holes}} = \sum_{j=1}^{\infty} a^\dagger(h_j)a(h_j) + \sum_{k=1}^{\infty} b^\dagger(g_k)b(g_k)$$

Canonical number operator in e-e+ perspective! It is $\hat{N}_{e-} + \hat{N}_{e+}$

In terms of a general ONB $\{f_j\}_{j\in\mathbb{N}} \subset L^2(\mathbb{R}^3,\mathbb{C}^4)$

- $$\hat{N}_{\substack{excess\ e-\\ wrt\ \Omega}} = \sum_{j=1}^{\infty}\left(\Psi^\dagger(f_j)\Psi(f_j) - ||P_- f_j||^2\right) = \sum_{j=1}^{\infty}\Big(a^\dagger(P_+ f_j)a(P_+ f_j) - b^\dagger(P_- f_j)b(P_- f_j)\Big) =: q^{-1}\hat{Q}$$

Take out contribt. of Sea e-s before add ∞ **=** $:\Psi^\dagger(f_j)\Psi(f_j):$ **=** **They call it *"normal ordering"***

Count how many Sea e- not present

- $$\hat{N}_{\substack{E>0\ e-\\ \&holes}} = \sum_{j=1}^{\infty}\left(\Psi^\dagger(P_+ f_j)\Psi(P_+ f_j) + ||P_- f_j||^2 - \Psi^\dagger(P_- f_j)\Psi(P_- f_j)\right)$$

$$= \sum_{j=1}^{\infty} a^\dagger(P_+ f_j)a(P_+ f_j) \ + \ b^\dagger(P_- f_j)b(P_- f_j) =: N_{e^- e^+}$$

For this one **no "normal ordering"-like shorth. notation**, so **no simple form** in **terms** of $\Psi^{(\dagger)}(\cdot)$ *only in e-/e+ indiv. field ops.*

These allow us to **partition Fock space in an alternative way**!

Using **charge sector or electron-positron sectors**! $\mathscr{F}_{anti}(\mathcal{H}_+) \otimes \mathscr{F}_{anti}(\mathcal{H}_-)$

Even more! In doing so, they give us **alternative "position" representations**!

(i) Dirac-Particle Model ⟷ (iii) Electron-Positron Model

We made the link through heuristic Dirac sea

Dirac-Particle Quantization ⟶ **e-/e+ Quantization**

$\mathscr{F}_{anti}(\mathcal{H}_+ \oplus \mathcal{H}_-)$ $\mathscr{F}_{anti}(\mathcal{H}_+) \otimes \mathscr{F}_{anti}(\mathcal{H}_-)$

$\Psi(\cdot)$ $a(\cdot),\ b(\cdot)$

But...link is actually rigorous!

First way to check ⟶ **Exercise 6.11. [2]**

Second (more explicit) way: sect. (iv)

(a) Find a bijection between $\mathscr{F}(Q_1 \cup Q_2)$ and $\mathscr{F}(Q_1) \times \mathscr{F}(Q_2)$ for arbitrary disjoint sets Q_1, Q_2.

(b) Find a unitary isomorphism between $\mathscr{F}_{(anti)symm}(\mathcal{H}_1 \oplus \mathcal{H}_2)$ and $\mathscr{F}_{(anti)symm}(\mathcal{H}_1) \otimes \mathscr{F}_{(anti)symm}(\mathcal{H}_2)$ for arbitrary Hilbert spaces $\mathcal{H}_1, \mathcal{H}_2$.

Dirac-Particle   **e-/e+ picture**

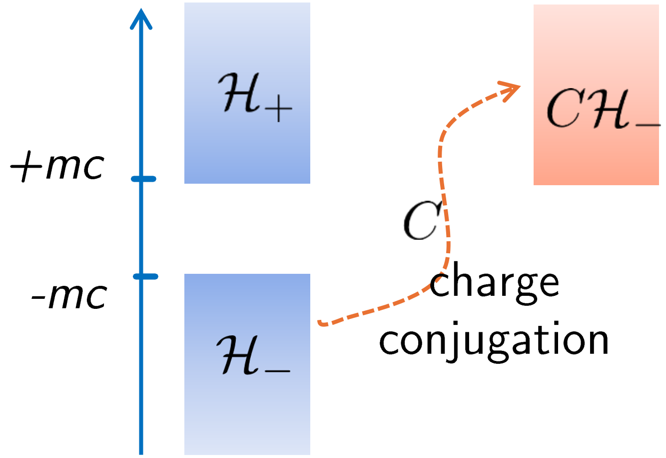
$$\mathscr{F}_{Dirac} = \mathscr{F}_{anti}(\underbrace{\mathcal{H}_+ \oplus \mathcal{H}_-}_{L^2(\mathbb{R}^3, \mathbb{C}^4)}) \quad \overset{\exists U \text{ unitary}}{\longleftrightarrow} \quad \mathscr{F}_{anti}(\mathcal{H}_+) \otimes \mathscr{F}_{anti}(\mathcal{H}_-) =: \mathscr{F}_{e-e+}$$

So, any "Dirac sea" (closed subspace of $\mathcal{H}$ with ∞ dim & codim) with can be made into a 2 species theory rigorously!

1 Free Dirac Particle

$$\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$$

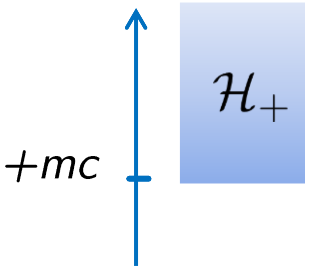
$spectrum(H)$



1 Free Electron

$$\mathcal{H}_+ \subset L^2(\mathbb{R}^3, \mathbb{C}^4)$$

$spect(H_{e-})$

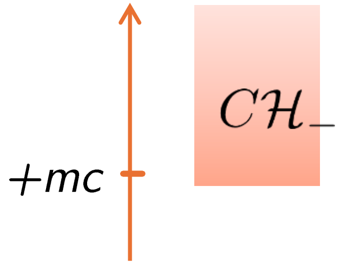


$$H_{e-} := H_{|\mathcal{H}_+}$$

1 Free Positron

$$C\mathcal{H}_- \subset L^2(\mathbb{R}^3, \mathbb{C}^4)$$

$spect(H_{e+})$



$$H_{e+} := -CHC^{-1}_{|C\mathcal{H}_-} \ (= -H(-e))$$

For rigor, use charge conjug. C

- **Fock Space of e-** $\mathscr{F}_{e-} = \mathscr{F}_{anti}\left(\mathcal{H}_+\right) := \bigoplus_{n=0}^{\infty} Anti(\mathcal{H}_+^{\otimes n})$
- **Fock Space of e+** $\mathscr{F}_{e+} = \mathscr{F}_{anti}\left(C\mathcal{H}_-\right) := \bigoplus_{\bar{n}=0}^{\infty} Anti((C\mathcal{H}_-)^{\otimes \bar{n}})$

- **Fock space of e-&e+**

$$\mathscr{H} := \mathscr{F}_{e+} \otimes \mathscr{F}_{e-}$$

- **Hilbert space** $\mathscr{H} := \mathscr{F}_{e+} \otimes \mathscr{F}_{e-}$

$\psi \in \mathscr{H}$ is s.t. $\psi = (\psi^{(0,0)}, \psi^{(1,0)}, \psi^{(0,1)}, \psi^{(1,1)}, ...)$ with $\psi^{(n,\bar{n})} \in Anti((\mathcal{H}_+)^{\otimes n}) \otimes Anti((C\mathcal{H}_-)^{\otimes \bar{n}})$

2 indices per sector!

- $\langle \psi, \phi \rangle := \sum_{n,\bar{n}=0}^{\infty} \langle \psi^{(n,\bar{n})}, \phi^{(n,\bar{n})} \rangle_{\mathcal{H}_+^{\otimes n} \otimes C\mathcal{H}_-^{\otimes \bar{n}}} =$

$$= \int_{\mathbb{R}^{3(n+\bar{n})}} \sum_{s_1,...,s_n,z_1,...,z_{\bar{n}}=1}^{4} \overline{\psi}^{(n,\bar{n})}_{s_1,...,z_{\bar{n}}}(\vec{x}_1, ..., \vec{x}_n, \vec{y}_1, ..., \vec{y}_{\bar{n}}) \phi^{(n,\bar{n})}_{s_1,...,z_{\bar{n}}}(\vec{x}_1, ..., \vec{x}_n, \vec{y}_1 ..., \vec{y}_{\bar{n}}) d^{3n}x d^{3\bar{n}}y$$

- **Vacuum** $|\Omega\rangle := (1, 0, 0, ...) \in \mathscr{H}$

- Obvious **position probability density** $\mathbb{P}(\text{there are } n\ e^-,\ \bar{n}\ e^+ \,|\, \psi) = \langle \psi^{(n,\bar{n})}, \psi^{(n,\bar{n})} \rangle_{\mathcal{H}_+^{\otimes n} \otimes C\mathcal{H}_-^{\otimes \bar{n}}}$

$$\rho_\psi^{(n,\bar{n})}(\vec{x}_1, ..., \vec{x}_n, \vec{y}_1, ..., \vec{y}_{\bar{n}}) = \sum_{s_1,...,s_n,z_1,...,z_{\bar{n}}=1}^{4} |\psi^{(n,\bar{n})}_{s_1,...,z_{\bar{n}}}(\vec{x}_1, ..., \vec{x}_n, \vec{y}_1 ..., \vec{y}_{\bar{n}})|^2$$

- **Define ELECTRON creation/annihilation operators,** $a, a^\dagger$

Acts as Id *on* $\mathscr{F}_{e^+}$

$f \in \mathcal{H}_+$

$$(a^\dagger(f)\psi)^{(n,\bar{n})}(x_1,...,x_n,y_1,...,y_{\bar{n}}) := Anti\Big(f(x_1)\psi^{(n-1,\bar{n})}(x_2,...,x_n,y_1,...,y_{\bar{n}})\Big)$$

Add particle in state *f* *(in each sector)* and anti-symmetrize

$$(a(f)\psi)^{(n,\bar{n})}(x_1,...,x_n,y_1,...,y_{\bar{n}}) := \sqrt{n+1}\int_{\mathbb{R}^3} f^\dagger(x)\psi^{(n+1,\bar{n})}(x,x_1,...,x_n,y_1,...,y_{\bar{n}})d^3x$$

"Trace-out" particle in state *f* (from each sector) i.e., partial inner product

- Naturally **the CAR** emerge

$\forall f,g \in \mathcal{H}_+$

$$\begin{cases} \left\{a(f), a^\dagger(g)\right\} = \langle f, g\rangle Id \\ \left\{a(f), a(g)\right\} = 0 \\ \left\{a^\dagger(f), a^\dagger(g)\right\} = 0 \end{cases}$$

- **Indeed adjoint** $(a(f))^* = a^\dagger(f) \quad \forall f \in \mathcal{H}_+$

- **Bounded operators**

$$||a^*(f)\psi||^2 + ||a(f)\psi||^2 = \langle\psi, \{a(f), a^*(f)\}\psi\rangle = ||f||^2||\psi||^2$$

- **Vacuum behaviour**

$a(f)|\Omega\rangle = 0 \in \mathscr{H}$

$$(a^*(f)|\Omega\rangle)^{(n,\bar{n})} = \begin{cases} f & for\ (n,\bar{n}) = (1,0) \\ 0 & else \end{cases}$$

- **Pauli Exclusion**

$$a(f)^2\psi = \frac{1}{2}\{a(f), a(f)\}\psi = 0$$

- **Define POSITRON creation/annihilation operators,** $b, b^\dagger$

Acts as Id *on* $\mathscr{F}_{e^-}$

$g \in \mathcal{H}_-$

$$(b^\dagger(g)\psi)^{(n,\bar{n})}(x_1,...,x_n,y_1,...,y_{\bar{n}}) := (-1)^n \, Anti\Big((Cg)(y_1)\psi^{(n,\bar{n}-1)}(x_1,...,x_n,y_2,...,y_{\bar{n}})\Big)$$

Add particle in state Cg *(in each sector)* and anti-symmetrize

$$(b(g)\psi)^{(n,\bar{n})}(x_1,...,x_n,y_1,...,y_{\bar{n}}) := (-1)^n\sqrt{n+1}\int_{\mathbb{R}^3} Cg(y)^\dagger\psi^{(n,\bar{n}+1)}(x_1,...,x_n,y,y_1,...,y_{\bar{n}})d^3y$$

"Trace-out" particle in state Cg (from each sector) i.e., partial inner product

- Naturally **the CAR** emerge

$\forall f, g \in \mathcal{H}_-$

$$\begin{cases} \left\{b(f), b^\dagger(g)\right\} = \langle Cf, Cg\rangle Id = \langle f, g\rangle^* Id \\ \left\{b(f), b(g)\right\} = 0 \\ \left\{b^\dagger(f), b^\dagger(g)\right\} = 0 \end{cases}$$

- **Indeed adjoint**

$$(b(g))^* = b^\dagger(g) \quad \forall g \in \mathcal{H}_-$$

- **Bounded operators**

$$||b^*(g)\psi||^2 + ||b(g)\psi||^2 = \langle\psi, \{b(g), b^*(g)\}\psi\rangle = ||g||^2||\psi||^2$$

- **Vacuum behaviour**

$$b(g)|\Omega\rangle = 0 \in \mathscr{H}$$

$$(b^*(g)|\Omega\rangle)^{(n,\bar{n})} = \begin{cases} Cg & for\ (n,\bar{n}) = (0,1) \\ 0 & else \end{cases}$$

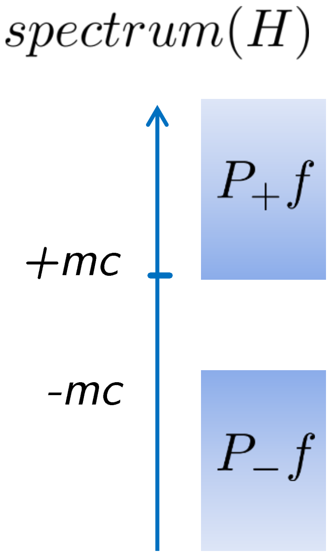
- **Pauli Exclusion**

$$b(g)^2\psi = \frac{1}{2}\{b(g), b(g)\}\psi = 0$$

- **Vacuum** is **unique up to a constant** $a(f)\psi = 0,\ b(g)\psi = 0 \quad \forall f \in \mathcal{H}_+, g \in \mathcal{H}_-$

$$\Longrightarrow \psi = \alpha|\Omega\rangle \quad \alpha \in \mathbb{C}$$

- Projection to spectral subsp. $L^2(\mathbb{R}^3, \mathbb{C}^4) = \mathcal{H}_+ \oplus \mathcal{H}_- \qquad P_\pm : L^2(\mathbb{R}^3, \mathbb{C}^4) \to \mathcal{H}_\pm$



- **Field operators of joint QFT**

$$\Psi(f) := a(P_+ f) + b^\dagger(P_- f)$$

$$\Psi^\dagger(f) := a^\dagger(P_+ f) + b(P_- f)$$

$$\Psi(P_- f) = b^\dagger(P_- f)$$

→ **Annihilation** of "electron" with **negative energy** = **Creation** of a "**hole**" in Dirac sea / particle of same mass but –charge i.e., **positron**

$$\Psi(P_+ f) = a(P_+ f)$$

→ **Annihilation** of "electron" with **positive energy** = **Annihilation** of **electron**

- **Canonical anti-commutation** Relations (CAR)

$$\begin{cases} \left\{\Psi(f), \Psi^\dagger(g)\right\} = \langle f, g\rangle Id \\ \left\{\Psi(f), \Psi(g)\right\} = 0 \\ \left\{\Psi^\dagger(f), \Psi^\dagger(g)\right\} = 0 \end{cases}$$

$$\forall f, g \in L^2(\mathbb{R}^3, \mathbb{C}^4)$$

- **Indeed adjoint**

$$(\Psi(f))^* = \Psi^\dagger(f) \quad \forall f \in L^2(\mathbb{R}^3, \mathbb{C}^4)$$

- **Bounded operators**

$$||\Psi(f)||_{op} = ||\Psi^\dagger(f)||_{op} = ||f||$$

- An **ONB** of $\mathscr{H}$: Take any $\{f_j\}_{j\in\mathbb{N}} \subset \mathcal{H}_+, \{g_k\}_{k\in\mathbb{N}} \subset \mathcal{H}_-$ ONB, then

$$a^*(f_{j_1})\cdots a^*(f_{j_n})b^*(g_{k_1})\cdots b^*(g_{k_{\bar n}})|\Omega\rangle = f_{j_1}\wedge\cdots\wedge f_{j_n} \otimes (Cg_{k_1})\wedge\cdots\wedge(Cg_{k_{\bar n}}) \quad \Big| \begin{array}{l} j_1<\cdots<j_n \\ k_1<\cdots<k_{\bar n}\end{array}$$

is **ONB** of $\mathscr{F}^{(n)}_{e-}\otimes\mathscr{F}^{(\bar n)}_{e+}$ ⟶ union $\forall n,\bar n$ = ONB of $\mathscr{H}$

On their dense span, define

- **Number operator** $$\hat N = \sum_{j=1}^{\infty}(a^\dagger(f_j)a(f_j) + b^\dagger(g_j)b(g_j))$$

- **Charge operator** $$\hat Q = q\sum_{j=1}^{\infty}(a^\dagger(f_j)a(f_j) - b^\dagger(g_j)b(g_j))$$ $q := -e$

- Both **independent** of **chosen basis**
- Above ONB are **eigenvectors** of both **eigenvalues *n+m* and *n-m*** *resp.*
- **Purely discrete spectrum** $\mathbb{N},\mathbb{Z}$ resp. so **unbounded** ops
- **Particle number sector** $(n,\bar n)$ = $\hat N$ **Eigenspace of eigenv.** $(n,\bar n)$
- **Charge number sector** (q) = $\hat Q$ **Eigenspace of eigenv.** (q)

Lifts of
$$H \mapsto \mathbb{H},\quad Id_{L^2(\mathbb{R}^3,\mathbb{C}^4)} \mapsto \hat Q,\quad sgnH \mapsto \hat N$$
generate ⟶
$$e^{iHt} \mapsto e^{i\mathbb{H}t} \equiv \mathbb{U}_t,\quad \mathcal{M}_{e^{i\theta}} \mapsto exp(i\theta\hat Q)$$

Why is any other irreducible representation of the CAR equivalent to our position representation?

Abstract "C^*algebra" of field ops. Ψ (abstraction of CAR) → A **representation** of the abstract CAR is

- Some **Hilbert space** $\mathfrak{H}$
- **Antilinear** map (field op.'s repr.) $\Phi : \mathcal{H}_1 \to Bded\ ops(\mathfrak{H})$

  s.th. $\forall f, g \in \mathcal{H}_1$

  $$\left\{\Psi(f), \Psi^\dagger(g)\right\} = \langle f, g\rangle Id \qquad \begin{cases} \left\{\Psi(f), \Psi(g)\right\} = 0 \\ \left\{\Psi^\dagger(f), \Psi^\dagger(g)\right\} = 0 \end{cases}$$

Then def. **annihilation** ops. of each species →

$$d(g) := \Phi^*(g)\ \forall g \in \mathcal{H}_- \quad \text{and} \quad c(f) := \Phi(f)\ \forall f \in \mathcal{H}_+$$

- If $\exists \Lambda \in \mathfrak{H}$ s.th. $c(f)\Lambda = d(g)\Lambda = 0 \quad \forall f \in \mathcal{H}_+, g \in \mathcal{H}_-$ → $\Lambda =:$ **vacuum of the representation**

Example: **Our position Fock** space $\mathscr{H}$ with $\Psi(\cdot)$

→ It is a **distinguished representation!** Because

Theorem 10.2 in [1]:

There is **no proper subspace of** $\mathscr{H}$ which **is invariant wrt all field operators** $\Psi(f)$ and $\Psi^\dagger(f)$, $f \in \mathscr{H}$, i.e., the position Fock representation is **irreducible**.

Why is any other irreducible representation of the CAR equivalent to our position representation?

***Theorem** 10.2 in [1]:*

Let $\Phi(f)$ and $\Phi^*(f)$ be **any other representation** of the CAR which is **irreducible** on some Hilbert space $\mathfrak{H}$. Define

$$c(f) := \Phi(f), \quad d(g) := \Phi^*(g) \quad \forall f \in \mathcal{H}_+, g \in \mathcal{H}_-.$$

The representation Φ has a vacuum vector $\Lambda \in \mathfrak{H}$ s.th.

$$c(f)\Lambda = 0 = d(g)\Lambda \quad \forall f \in \mathcal{H}_+, g \in \mathcal{H}_-$$

if and only if there is a unitary $\mathscr{U} : \mathscr{F} \to \mathfrak{H}$ s.th.

$$\Phi(f) = \mathscr{U}\Psi(f)\mathscr{U}^* \quad \forall f \in \mathcal{H}_1$$

where $\Psi(f)$ are "our" field operators.

Therefore, no point in using an "abstract representation" of the QFT (as in orthodox lectures)

We lose nothing for committing to the "position representation"!

- **Lift of free unitary time evolution**

$$H = -i\hbar c\vec{\alpha}\cdot\vec{\nabla} + \beta mc^2 \longrightarrow U_t = e^{-\frac{i}{\hbar}Ht} \longrightarrow \Psi_t(f) := \Psi(e^{\frac{i}{\hbar}Ht}f)$$

$\Psi_t, \Psi_t^\dagger$ act **irreducibly** on $\mathscr{H}$

$\Psi_t, \Psi_t^\dagger$ still satisfy **CAR** since $\langle f, g\rangle = \langle U_t f, U_t g\rangle$

$U_t\mathcal{H}_\pm = \mathcal{H}_\pm \longrightarrow \forall\theta\in[-\pi,\pi),\ \ \Lambda := e^{i\theta}$ is vacuum of $\Psi_t, \Psi_t^\dagger$

By last theorem,

$\exists\mathbb{U}_t : \mathscr{H}\to\mathscr{H}$

s.t.

$\Psi_t(f) = \mathbb{U}_t^\dagger\Psi(f)\mathbb{U}_t$

Choice unique by $\mathbb{U}_t|\Omega\rangle = |\Omega\rangle\ \forall t \longrightarrow \{\mathbb{U}_t\}_t$ is SCOPUG [1, sect 10.2.2]

- Generator $\mathbb{H} := i\hbar\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\mathbb{U}_t$ *on* $\mathfrak{D} = \left\{\phi\in\mathscr{F}\ \middle|\ \lim_{\tau\to 0}\frac{1}{\tau}(\mathbb{U}_t - Id)\phi \text{ exists}\right\}$

$$\mathbb{H} = \sum_{i,j\in\mathbb{N}} a^*(f_i)\langle f_i, Hf_j\rangle a(f_j) - b^*(g_i)\langle g_i, Hg_j\rangle b(g_j) \quad \text{for any } \{f_j\}_{j\in\mathbb{N}}\subset\mathcal{H}_+,\ \{g_j\}_{j\in\mathbb{N}}\subset\mathcal{H}_-\ \text{ONB}$$

(Semi-bounded below because $-H$ is in $\mathcal{H}_-$)

(iv) Dirac Sea Model

We made the link through (ii) heuristic Dirac sea

(i) Dirac Particle Quantization → **(iii) e-/e+ Quantization**

$$\mathscr{F}_{Dirac} = \Gamma_{anti}(\mathcal{H}_+ \oplus \mathcal{H}_-)$$

But…link is actually rigorous!

$$\mathscr{F}_{e-e+} = \Gamma_{anti}(\mathcal{H}_+) \otimes \Gamma_{anti}(\mathcal{H}_-)$$

$\Psi(\cdot)$

$a(\cdot),\ b(\cdot)$

Moreover, the Dirac Sea model itself can be made rigorous too!!!

J. **Dimock**, "*The Dirac sea*", Letters in Mathematical Physics **98**, 157 (2011).

—

D.-A. **Deckert**, D Dürr, F Merkl, and M Schottenloher, "*Time-evolution of the external field problem in quantum electrodynamics*", Journal of Mathematical Physics **51** (2010).

$$\mathcal{H}_1 = \mathcal{H}_+ \oplus \mathcal{H}_- \left\{ \begin{array}{l} \{h_1, h_2, ...\} \subset \mathcal{H}_+ \quad \text{ONB} \\ \{g_{-1}, g_{-2}, ...\} \subset \mathcal{H}_- \quad \text{ONB} \end{array} \right\} \quad \{\underbrace{..., f_{-2}, f_{-1}}_{\mathcal{H}_-}, \underbrace{f_1, f_2, ...}_{\mathcal{H}_+}\} := \{..., g_{-2}, g_{-1}, h_1, h_2, ...\}$$

$\longrightarrow$ $\{f_j\}_{j\in\mathbb{Z}\setminus 0} \subset \mathcal{H}_1$ **is ONB**

- Let sequence in $\mathbb{Z}\setminus\{0\}$

  $J = (j_1, j_2, j_3, ...)$ with $j_1 > j_2 > j_3 > \cdots$ and $j_{k+1} = j_k - 1$ for large enough k

  depth/tail of seqce is same

  } **∃ Countably many such sequences**

- Associate each seq. the **formal symbol** $f_J = f_{j_1} \wedge f_{j_2} \wedge f_{j_3} \wedge \cdots$

  a possible state of the seas e.g. $f_3 \wedge f_{-1} \wedge f_{-3} \wedge f_{-4} \wedge \cdots$

- **Define semi-infinite wedge product** space $\Lambda_\infty(\mathcal{H}_+, \mathcal{H}_-)$ as the complex vect space of all **formal** linear combinations

$$\sum_J c_J f_J \quad ; \; c_J \in \mathbb{C} \quad \text{with} \quad c_J \neq 0 \quad \textbf{only for finitely many } J$$

  Superposits. of sea states

- **Define inner product** with $\{f_J\}_J$ as **ONB** $\longrightarrow$ $\langle f_J, f_I \rangle := \delta_{JI}$ and $\left\langle \sum_J c_J f_J, \sum_I c_I f_I \right\rangle := \sum_J |c_J|^2$

- **Completion of** $\Lambda_\infty(\mathcal{H}_+, \mathcal{H}_-)$ **in associated norm is a Hilbert space** $\mathscr{H}_{Sea} := \mathscr{H}(\Lambda_\infty(\mathcal{H}_+, \mathcal{H}_-))$

- Define **wedge product** and wedge "division" on $\Lambda_\infty(\mathcal{H}_+,\mathcal{H}_-)$ extending following def. on ONB linearly

$$\Psi^\dagger(f_k)\big(f_{j_1}\wedge f_{j_2}\wedge\cdots\big) := \begin{cases}(-1)^s f_{j_1}\wedge\cdots\wedge f_{j_{s-1}}\wedge f_k\wedge f_{j_{s+1}}\wedge\cdots & \text{if } j_s<k<j_{s+1}\\ 0 & \text{if } k=j_s \text{ for some } s\in\mathbb{N}\end{cases}$$

$\forall k\in\mathbb{Z}\setminus 0$

$$\Psi(f_k)\big(f_{j_1}\wedge f_{j_2}\wedge\cdots\big) := \begin{cases}0 & \text{if } k\neq j_s \ \ \forall\, s\in\mathbb{N}\\ (-1)^{s+1} f_{j_1}\wedge\cdots\wedge f_{j_{s-1}}\wedge f_{j_{s+1}}\wedge\cdots & \text{if } k=j_s \text{ for some } s\in\mathbb{N}\end{cases}$$

Properties:

- $\Psi^\dagger(f_j)=\Psi^*(f_j)$
- $$\begin{cases}\big\{\Psi(f_i),\Psi^\dagger(f_j)\big\}=\delta_{ij}Id\\ \big\{\Psi(f_i),\Psi(f_j)\big\}=0\\ \big\{\Psi^\dagger(f_i),\Psi^\dagger(f_j)\big\}=0\end{cases}$$

$\forall i,j\in\mathbb{Z}\setminus 0$

→

- $||\Psi^\dagger(f_j)\psi||^2+||\Psi(f_i)\psi||^2=||\psi||^2 \qquad \forall\psi\in\Lambda_\infty$

→ $||\Psi(f_j)\psi||\leq||\psi||$ → $\Psi(f)$ **extends to bounded operator on** $\mathscr{H}_{Sea}$

- $|\Omega_{Sea}\rangle := f_{-1}\wedge f_{-2}\wedge f_{-3}\wedge\cdots$ distinguished by

$\Psi(f_j)|\Omega_{Sea}\rangle=0 \quad \forall j>0 \qquad \Psi^\dagger(f_j)|\Omega_{Sea}\rangle=0 \quad \forall j<0$

Theorem 1 **: Extend $\Psi(\cdot)/\Psi^\dagger(\cdot)$ to any argument** [6]

For any $g \in \mathcal{H}_1$ the sums

$$\Psi(g) := \sum_{j\in\mathbb{Z}\setminus\{0\}} \langle g, f_j\rangle \Psi(f_j) \quad \& \quad \Psi^\dagger(g) := \sum_{j\in\mathbb{Z}\setminus\{0\}} \langle f_j, g\rangle \Psi^\dagger(f_j)$$

converge in operator norm to bounded operators on $\mathscr{H}_{Sea}$ and they satisfy $\forall g, h \in \mathcal{H}_1$

$$\{\Psi(h), \Psi^\dagger(g)\} = \langle h, g\rangle \quad \& \quad \text{all other anticommutators } 0.$$

Furthermore, $||\Psi(f)||_{op} \leq ||f||$ and $||\Psi^\dagger(f)|| \leq ||f||$.

Remarks *[6]:*

- $\Psi(f)$ is anti-linear in f and $\Psi^\dagger(f)$ linear in f.
- $\Psi(h)|\Omega_{Sea}\rangle = 0 \ \forall h \in \mathcal{H}_+$ and $\Psi^\dagger(g)|\Omega_{Sea}\rangle = 0 \ \forall g \in \mathcal{H}_-$
- Linear combinations of vectors like $\prod_{j=1}^n \Psi^\dagger(h_j) \prod_{k=1}^m \Psi(g_j)|\Omega_{Sea}\rangle \quad h_j \in \mathcal{H}_+, g_k \in \mathcal{H}_-$ are dense in $\mathscr{H}_{Sea}$.

***Theorem 2* : Construction is independent of basis** [6]

The triplet $\mathscr{H}(\Lambda_\infty(\mathcal{H}_+,\mathcal{H}_-)), \Psi(\cdot), |\Omega_{Sea}\rangle$ constructed from a basis $\{f_j\}$ compatible with the splitting $\mathcal{H}_1 = \mathcal{H}_+ \oplus \mathcal{H}_-$ is **independent of the basis** in the sense that if $\mathscr{H}(\Lambda'_\infty(\mathcal{H}_+,\mathcal{H}_-)), \Psi'(\cdot), |\Omega'_{Sea}\rangle$ is a triplet constructed from another basis $\{f'_j\}_j$ compatible with the splitting, then there is a unitary operator $U : \mathscr{H}(\Lambda_\infty(\mathcal{H}_+,\mathcal{H}_-)) \to \mathscr{H}(\Lambda'_\infty(\mathcal{H}_+,\mathcal{H}_-))$ such that

$$U|\Omega_{Sea}\rangle = |\Omega'_{Sea}\rangle \quad \textit{and} \quad U\Psi(f)U^{-1} = \Psi'(f).$$

***Theorem 3* : Time evolution of field operators** [6]

Given the Dirac Hamiltonian $H = -i\nabla\cdot\alpha + \beta m$, and the field operator $\Psi_t(f) := \Psi(e^{iHt}f)$,

- $\{\Psi_t(g), \Psi_t^\dagger(f)\} = \langle g, f\rangle \ \forall g, f \in \mathscr{H}_1, \ \forall t,$
- $i\frac{\mathrm{d}}{\mathrm{d}t}\Psi_t(f) = \Psi_t(Hf) \ \forall f \in \mathscr{H}_1, \ \forall t$
- time evolution is unitarily implementable with positive energy, i.e., $\exists \mathbb{H} \geq 0$ s.t.

$$\Psi_t(f) = e^{i\mathbb{H}t}\Psi(f)e^{-i\mathbb{H}t} \quad \forall f \in \mathcal{H}_1 \ \ \forall t.$$

Define derived operators:

- **Create/annihilate e- over surface** $a^\dagger(h) := \Psi^\dagger(h) \quad \& \quad a(h) := \Psi(h) \quad \forall h \in \mathcal{H}_+ \longrightarrow \{a(h_1), a^\dagger(h_2)\} = \langle h_1, h_2 \rangle \; Id$

- **Create/annihilate hole on sea surface** $b^\dagger(g) := \Psi(g) \quad \& \quad b(g) := \Psi^\dagger(g) \quad \forall g \in \mathcal{H}_- \longrightarrow \{b(g_1), b^\dagger(g_2)\} = \langle g_1, g_2 \rangle^* \; Id$

They satisfy

- $a(h)|\Omega_{sea}\rangle = 0 \qquad b(g)|\Omega_{sea}\rangle = 0$
- $a^\dagger(h), b^\dagger(g)$ acting on $|\Omega_{sea}\rangle$ are **dense** in $\mathscr{H}_{sea}$
- In their terms : $\Psi_t(f) \equiv a(e^{iHt}P_+ f) + b^\dagger(e^{iHt}P_- f)$

Theorem 4: (iii) Dirac Sea $\longleftrightarrow$ (ii) e-e+ Fock space [6]

There is a unitary operator $U : \mathscr{H}(\Lambda_\infty(\mathcal{H}_+, \mathcal{H}_-)) \to \mathscr{F}_{anti}(\mathcal{H}_+) \otimes \mathscr{F}_{anti}(\mathcal{H}_-) =: \mathscr{F}_{e^- e^+}$ s.th.

- $U \; |\Omega_{sea}\rangle = |\Omega_{e^+ e^-}\rangle$
- $U \; a_{sea}(h) \; U^{-1} = a_{e^- e^+}(h) \quad \forall h \in \mathcal{H}_+$
- $U \; b_{sea}(g) \; U^{-1} = b_{e^- e^+}(g) \quad \forall g \in \mathcal{H}_-$
- $U \; \Psi_t^{(sea)}(f) \; U^{-1} = \Psi_t^{(e^- e^+)}(f).$

c) An Ontology for QFT?

Given a QFT, is there any way to **let there be concrete things beyond "our sight"?**

(i) "Field" as Primitive Ontology

*There is an actual **physical-space field pervading the Universe "piloted" by a PDE depending** on the **wavefunction.***

- **Example 1** (Bohm, 1952)**: Secd. qtzt of radiation field** (QFT feat.ing **photons**) equivalent to assume:
  ∃ a **determined EM vector potential** (class), hence a **determined electric and magnetic field** in our Universe that are **"piloted" by Maxwell's equations**, **with an extra term ("quantum potential")** due to the **wavefunction** (a wavef. that has as its "position variables" the **amplitudes** of expansion modes of the EM field).

- **Example 2:** (Teaser in [2])**: The Field Representation (where Field Ops. are Multiplication Ops.)**

$\mathscr{Q}_{field}$ space of field configurations, e.g., $\{\phi : \mathbb{R}^3 \to \mathbb{R} \mid \phi \textit{ is continous}\}$

Wavefunctional $\Psi \in \mathscr{H}$ & **Piloted** actual **field** $\Phi(t, \vec{x})$

A **complex amplitude per field config.** $\Psi : \mathscr{Q}_{field} \to \mathbb{C}$

A **trajectory in function space** $\Phi : \mathbb{R} \to \mathscr{Q}_{field}$

Example 2 (.../...)

"Usual" case	QFT case
• $\psi(x_1,...,x_N)$ is **posit. represt**. coeffs of abstract $\lvert\psi\rangle$ Diagonalization space of $(\hat{x}_j\psi)(x_1,...,x_N) = x_j\psi(x_1,...,x_N)$	$\Psi[\phi]$ is **field represt**. coeffs of abstract. $\lvert\Psi\rangle$ Diagonalization space of $(\hat{\phi}(\vec{x})\Psi)[\phi] = \phi(\vec{x})\Psi[\phi]$
• $H\psi(x_1,...,x_N) = -\sum_{j=1}^{N}\frac{\partial^2\psi}{\partial x_j^2}$	$H\Psi[\phi] = -\int_{\mathbb{R}^3}\frac{\delta^2\Psi}{\delta\phi(\vec{x})^2}d^3x$ **Functional derivative**
• Particle **configurt traj.** Q guided by $\psi(q)$ $\frac{\mathrm{d}Q_j(t)}{\mathrm{d}t} = \frac{\hbar}{m}Im\left[\psi(q)^{-1}\frac{\partial\psi}{\partial q}\right]_{\lvert q=Q(t)}$	**Field** configurt **traj.** Φ guided by $\Psi[\phi]$ $\frac{\partial\Phi(t)}{\partial t} = \frac{\hbar}{m}Im\left[\Psi[\phi]^{-1}\frac{\partial\Psi}{\partial\phi(\vec{x})}\right]_{\lvert\phi=\Phi(t)}$

Remarks

- Obvious **Bohmian-like generalizations** when **field operators emerge.**
- Quite **un-convenient for fermions** (it would be **second quantization literally**! –e.g. different possible Schrödinger wavefunctions interacting with each other –) → **Bosons-only ontology possible, so not strictly necessary!**

Remarks

- (As far as we know) **field operators are not "observable operators"**!
  What would even mean to measure/collapse? Everywhere at once?

- Experimental **verification of QFT seemingly due to particle picture**!

- **Field ontol**. can be **rigorously** done **if UV cut-off and finite Universe**! **Else, NO rigorous version** exists **yet**!
  ↳ **Main hindrance:** For **probabilistic interpretation "translation invariant" measure** on **infinite dim** vector space needed (analogue of Lebesgue measure), but **it does NOT exist**

- **Ongoing research** (perhaps unsolvable, but *for bosons my favourite ontology*).

To be Continued...

(Perhaps in following seminar)

- References for approaches in

W. Struyve, "Pilot-wave approaches to quantum field theory", in Journal of Physics: Conference Series, Vol. 306, 1 (IOP Publishing, 2011), p. 012047.
-
C. T. Sebens, "The fundamentality of fields", Synthese **200**, 380 (2022).

(ii) "Particle" as Primitive Ontology

*There are **actual particles** in the Universe **"piloted" by an ODE depending** on the **wavefunction***

- **Need a position/configuration "operator"** → best **suited for fermion QFTs** ← our interest!
  But then quite **un-convenient for bosons... (e.g., ∄ position "operator" for photons yet)**
  ↳ **Fermions-only ontology** is **possible, so not strictly necessary!**

- **Main approaches:**

(1) Take **UV cutoff & finite Universe seriously** → **Dirac Sea** with **fixed N particle Bohmian ontology:** *Deterministic trajs.* and QFT practical but **not** fundamental

(2) Take **creation/annihilation of particles wrt vacuum of QFT seriously** ← **We explain this one now!**
↳ **Bell-type QFT ontology:** *Stochastic trajectories*

(3) Take **rigorous Dirac Sea seriously** → **trajectory ontology yet open question**!

d) Bell-type QFTs

D. Dürr, S. Goldstein, R. Tumulka, and N. Zanghì, *"Bell-type quantum field theories"*,
Journal of Physics A: Mathematical and General 38, R1 (2005)

- Generalization of Bell's Bohmian-QFT (in discretized space QFT):
  Bell, J.S.: *Beables for quantum field theory.* Phys. Rep., 137, 49–54 (1986)

- A **General QFT**:

a **Hilbert** space $$\mathscr{H} = \mathcal{F}_{(anti)sym}(\mathcal{H}_1) \otimes \cdots \otimes \mathcal{F}_{(anti)sym}(\mathcal{H}_{m_{species}})$$

\+ a **Hamiltonian** $\mathbb{H} = \mathbb{H}_0 + \mathbb{H}_I$

- $\mathbb{H}_I$ **Interaction** Hamiltonian (creat. ops. etc.)
- $\mathbb{H}_0$ **Free** Hamiltonian of **each species**

$H_{1,k}$ 1 free particle of species k → $$\mathbb{H}_0 := \sum_{k}^{n_{species}} \Gamma(H_{1,k})$$

\+ **Time Evolution PDE :** $$i\hbar \frac{\mathrm{d}}{\mathrm{d}t}\psi_t = \mathbb{H}\psi_t \qquad \psi_t \in \mathscr{H}$$

- **Missing? Positions**! (Cross-sections should not be the only contact with reality!)

d) Bell-type QFTs

Why Position needed also for orthodox QFT?

- A **complete theory** should **prescribe** a way to **predict** the **probability** distribution of the **configuration** of particles that **detectors** find (which is **experimentally determinable**)

- Possibly **interesting phenomena** happen **at finite times:** that **scattering matrices** ($t \to \pm\infty$ part of the evolution) will **never show** (so **not enough with asymptotic position/momentum blending** arguments –s.th. at long times different momenta separate in position and position and momentum distributions agree)

- **Without Born rule** giving position probabilities **no way to link** a **quantum state with** actual **macroscopic configs**

- **Generalizations** of QFT to **curved space-times** possibly **different for different position operators**!

The **exploration** of **position ops.** is **beneficial independently** of **ontological preferences**!

What do we mean by position in N(t)? – Recall: the Unordered Configuration Space

- **Naïve configuration space** for **1-specie** QFT: $\tilde{Q} = \bigcup_{n=0}^{\infty} \mathbb{R}^{3n}$

  The **ordered** configuration space

  example $\tilde{q} \in \tilde{Q}$: $\tilde{q} = (\vec{x}_1, ..., \vec{x}_{321})$, $\vec{x}_j \in \mathbb{R}^3$

  ↳ **Measure** is **Lebesgue** meas. $d\tilde{q}$ $\quad d\tilde{q}(B) = \sum_{n=0}^{\infty} d^n q(B \cap \mathbb{R}^{3n})$ $\quad B \subseteq \tilde{Q}$

d) Bell-type QFTs

What do we mean by position in N(t)? – Recall: the Unordered Configuration Space

- But in $\tilde{\mathcal{Q}}$ we consider $(\vec{x}_1, \vec{x}_2) \neq (\vec{x}_2, \vec{x}_1)$ (hence potentially different probabilities) **when they should be the same configuration for 2 identical particles!** ⟶ Identify all configs. with same elements!

- **Correct* config. space** for **1-specie** QFT: The **unordered** config. space

$$\mathcal{Q} = \Gamma(\mathbb{R}^3) := \bigcup_{n=0}^{\infty} \left\{ q \subset \mathbb{R}^3 \mid \#q = n \right\}$$

example $q \in \mathcal{Q}$: $\tilde{q} = \{\vec{x}_1, \dots, \vec{x}_{321}\}$

- **Forgetful mapping:**

forget the ordering!

$$\pi : \tilde{\mathcal{Q}} \setminus \Delta \to \mathcal{Q}$$
$$(\vec{x}_1, \dots, \vec{x}_N) \mapsto \{\vec{x}_1, \dots, \vec{x}_N\}$$

$\Delta := \{(\vec{x}_1, \dots, \vec{x}_n) \in \tilde{\mathcal{Q}} \mid \vec{x}_j = \vec{x}_k \text{ for some } j, k\}$

ordered configs. with collisions (are measure 0)

↳ $\forall q \in \mathcal{Q}$ $\boldsymbol{\pi}$ **has** $\mathbf{(\#q)!}$ **pre-images** *(possible ways to order it):* $\pi^{-1}(q)$ is a set of $(\#q)!$ ordered vecs.

- **Measure** on $\mathcal{Q}$ is dq induced by Lebesgue meas. ⟶ $dq(B) := d\tilde{q}\left(\pi^{-1}(B)\right)$ $B \subseteq \mathcal{Q}$

- So, **position** op.'s **POVM** $P(B)$ should be **indexed** by the **unordered configs**. $B \subseteq \mathcal{Q}$

(we want prob. of the system to have a particular config. irrespective of particle-labelling)

*For **max generlty** must allow **multiplicities** for positions in *q* ⟶ Full def. of $\mathcal{Q}$: ***all*** $\mathbb{N} \cup \{0\}$ ***valued measures on*** $\mathbb{R}^3$

- **In general, for *m* species** *(e.g. e-, e+, tauon etc)*
  - **Ordered** configuration space $\tilde{\mathcal{Q}} = \tilde{\mathcal{Q}}_1 \times \cdots \times \tilde{\mathcal{Q}}_m$ $\quad \tilde{\mathcal{Q}}_j = \bigcup_{n=0}^{\infty} \mathbb{R}^{3n}$
  - "**Unordered**" configuration space $\mathcal{Q} = \mathcal{Q}_1 \times \cdots \times \mathcal{Q}_m$ $\quad \mathcal{Q}_j = \Gamma(\mathbb{R}^3)$

    (keep species distinguished!)
  - **Forgetful** mapping

    $\pi := \pi_1 \times \cdots \times \pi_m$

example $\tilde{q} \in \tilde{\mathcal{Q}}$

$$\tilde{q} = (\tilde{q}_1, ..., \tilde{q}_m)$$
$$= (\vec{x}_1^{(1)}, ..., \vec{x}_1^{(n_1)}, ..., \vec{x}_m^{(1)}, ..., \vec{x}_m^{(n_m)})$$

e.g., $q \in \mathcal{Q} : q = (q_1, ..., q_m) = (\{\vec{x}_1^{(1)}, ..., \vec{x}_1^{(n_1)}\}, ..., \{\vec{x}_m^{(1)}, ..., \vec{x}_m^{(n_m)}\})$

- **Measure** on $\mathcal{Q}$ $\quad dq := dq_1 \otimes \cdots \otimes dq_m$

A **position operator** for **general QFT** is hence a **POVM** $\{P(B)\}_{B \subseteq \mathcal{Q}}$ **over** $\mathcal{Q}$

- **Goal :** Get **QFT predictions** using **particles** that **exist concretely somewhere.**
- **Need :** **Positions for those particles** + **a law of motion**

Ingredient 1 : **right position POVM** $P, \mathcal{Q}$ **giving posit. probs.** $\langle\psi|P(dq)|\psi\rangle$

probability equivariant law for trajectories ***: Ingredient 2***

(i) Simple Particular Case

- If **symmetric** species with spin 0: $\tilde{\mathscr{H}} = \mathcal{F}_{sym}(\mathcal{H}_1) \otimes \cdots \otimes \mathcal{F}_{sym}(\mathcal{H}_m)$ and **if** $\mathcal{H}_j = L^2(\mathbb{R}^3, \mathbb{C})$

  ↳ $\tilde{\mathscr{H}} \subset L^2(\tilde{\mathcal{Q}}, \mathbb{C}; d\tilde{q})$ and any $\phi \in \tilde{\mathscr{H}}$ is a (class of) "function" $\tilde{\mathcal{Q}} \to \mathbb{C}$ normalizable in $d\tilde{q}$'s $L^2 - norm$

- **But** for all $q \in \mathcal{Q}$, $\phi(\tilde{q}) = \phi(\tilde{q}')$ $\forall \tilde{q}, \tilde{q}' \in \pi^{-1}(q)$ ⟶ ∃ canonical **isomorphism of** $\tilde{\mathscr{H}}$ with an

  $\mathscr{H} \subset L^2(\mathcal{Q}, \mathbb{C}; dq)$ "erasing the redundancy": **def.** $\mathscr{H}$ **identifying** $\phi \in \tilde{\mathscr{H}}$ **with** $\psi \in \mathscr{H}$ s.th.

$$\forall q \in \mathcal{Q} \qquad \psi(q) = \phi(\tilde{q}) \qquad \forall \tilde{q} \in \pi^{-1}(q)$$

- **Assuming** this **identifict.**, ∃ a **standard position** op. $\{P(B)\}_{B \subseteq \mathcal{Q}}$, $P(B) = \mathcal{M}(\chi_B)$

  gives the $|\psi(q)|^2$ **distrib.** $\mathbb{P}(q \in B \mid \psi) = \int_B |\psi(q)|^2 dq \qquad B \subseteq \mathcal{Q}$

  } **Ingredient 1 obtined!** *Now need an equivariant trajectory law*

- It's absolutely continuous measure!

  $\mathbb{P}(q \in dq|\psi) = \langle \psi | P(dq) | \psi \rangle = \rho(q) dq$ ⟶ $\rho(q) = |\psi(q)|^2$ ⟶ **"posit." probability density!**

- Then **since** $i\hbar \frac{\mathrm{d}}{\mathrm{d}t} \psi_t = \mathbb{H} \psi_t$ ⟶

$$\frac{\partial |\psi_t(q)|^2}{\partial t} = \frac{2}{\hbar} Im\{\psi_t^*(q)(\mathbb{H}\psi_t)(q)\}$$

= **Evolution equation of probability density!**

- **As Ingredient 2: Postulate a stochastic trajectory** of **exponentially distributed jumps** with **jump-rate** density $\sigma_t(q' \to q)$ and **deterministic drift** $v_t(q)$

$$\frac{\partial \rho_t(q)}{\partial t} = -div(\rho_t(q)v_t(q)) + \int_{q' \in \mathcal{Q}} \Big(\rho_t(q')\sigma_t(q' \to q) - \rho_t(q)\sigma_t(q \to q')\Big)dq'$$

- The **PDE** that QM **probability density** follows

$$\frac{\partial \rho_t(q)}{\partial t} = \underbrace{\frac{2}{\hbar} Im\{\psi_t^*(q)(\mathbb{H}_0\psi_t)(q)\}}_{\text{Current due to } \mathbb{H}_0} + \underbrace{\frac{2}{\hbar} Im\{\psi_t^*(q)(\mathbb{H}_I\psi_t)(q)\}}_{\text{due to } \mathbb{H}_I}$$

Could we equate them?

Step 1: Accommodate $\mathbb{H}_0$'s current as **deterministic**

- Recall, $\mathbb{H}_0 := \sum_k^m \mathbb{H}_{0,k}$ with $\mathbb{H}_{0,k} := \Gamma(H_{1,k})$ (**lift** of **free 1-particle Hamiltonian of k-th species**)
- Since in general $[\mathbb{H}_{0,k}, \hat{N}_j] = 0$ $\forall k, j$ **free evolut**. alone causes **no flow of probability between sectors**

↳ **If there is N particle Bohmian velocity for each species, just lift to its Fock spc., then to joint $\mathscr{H}$!**

↳ **"Trick"** to get it when $\mathbb{H}_{0,k}$ is **differential operator of order 2** or less in posit. repr.:

"the local expectation of velocity is the Bohmian velocity"

- Def $\hat{v} := \frac{i}{\hbar}[\mathbb{H}_0, q]$ then

"the local expectation of velocity is the Bohmian velocity"

Orthodox expected velocity $=$ $\langle\psi|\hat{v}|\psi\rangle = \langle\psi|\int_{\mathcal{Q}} P(dq)\hat{v}|\psi\rangle = \int_{\mathcal{Q}} Re\langle\psi|P(dq)\hat{v}|\psi\rangle =: \int_{\mathcal{Q}} v(q)\langle\psi|P(dq)|\psi\rangle = \int_{\mathcal{Q}} \underbrace{v(q)\rho(q)}dq$

PVM "completns. relat"

expect of s.a. op. is real

measure has density

Expected velocity around q

Rigorously using $\hat{q} = \mathcal{M}_q$ *in chosen* $\mathscr{H}$ *or with formal* $P(dq) =$ "$P(q)dq = |q\rangle\langle q|dq$"

$$v(q) := Re\frac{\langle\psi|P(q)\hat{v}|\psi\rangle}{\langle\psi|P(q)|\psi\rangle} = Re\frac{\psi^\dagger(q)\frac{i}{\hbar}[\mathbb{H}_0, q]\psi(q)}{\psi(q)^\dagger\psi(q)}$$

Also result of the obvious "careful" velocity measurement

$$-div(v(q)\rho_t(q)) = \frac{2}{\hbar}Im\{\psi_t^*(q)(\mathbb{H}_0\psi_t)(q)\}$$

Yields a velocity field guiding trajs. ***equivariantly wrt*** $\mathbb{H}_0$ *dynamics*

e.g.

- if $\mathbb{H}_0$ lifted Schrödinger free Hamilt. → $v(q) = \frac{\hbar}{m}Im\frac{\psi^*\nabla\psi}{\psi^*\psi}$
- If $\mathbb{H}_0$ lifted Pauli free Hamilt. → $v(q) = \frac{\hbar}{m}Im\frac{\psi^\dagger\nabla\psi}{\psi^\dagger\psi}$
- If $\mathbb{H}_0$ lifted Dirac free Hamilt. → $v(q) = c\frac{\psi^\dagger(q)\alpha_N\psi(q)}{\psi^\dagger(q)\psi(q)}$ $\quad \alpha_N = (\vec{\alpha}^{(1)}, ..., \vec{\alpha}^{(N)})$

It's the Bohmian Velocity!

- General EOM for a **stochastic trajectory** ensemble of **jump rate** density $\sigma_t(q' \to q)$ and **deterministic drift** $v_t(q)$

$$\frac{\partial \rho_t(q)}{\partial t} = -div(\rho_t(q)v_t(q)) + \int_{q'\in\mathcal{Q}} \Big(\rho_t(q')\sigma_t(q' \to q) - \rho_t(q)\sigma_t(q \to q')\Big)dq'$$

- The **PDE** that QM **probability density** follows

$$\frac{\partial \rho_t(q)}{\partial t} = \underbrace{\frac{2}{\hbar} Im\{\psi_t^*(q)(\mathbb{H}_0\psi_t)(q)\}}_{\text{Current due to } \mathbb{H}_0} + \underbrace{\frac{2}{\hbar} Im\{\psi_t^*(q)(\mathbb{H}_I\psi_t)(q)\}}_{\text{due to } \mathbb{H}_I}$$

Step 2: Accommodate $\mathbb{H}_I$'s current as **random jumps**

$$\hat{N} = \sum_{j=1}^{m} Id \otimes \cdots \otimes \hat{N}_j \otimes \cdots \otimes Id$$

- Since in general $[\mathbb{H}_I, \hat{N}] \neq 0$ **flow of probability between sectors (meaning also different species!)**
- $\mathbb{H}_I$ **polynomial in create/annihilation** ops. ⟶ has a kernel: cut-off smearing ⟶ $\mathbb{H}_I$ **is integral operator**

integral, just like jump balance!

$$(\mathbb{H}_I\psi)(q) = \int_{q'\in\mathcal{Q}} \langle q|\mathbb{H}_I|q'\rangle\psi(q')dq'$$

- General EOM for a **stochastic trajectory** ensemble of **jump rate** density $\sigma_t(q' \to q)$ and **deterministic drift** $v_t(q)$

$$\frac{\partial \rho_t(q)}{\partial t} = -div(\rho_t(q)v_t(q)) + \int_{q'\in\mathcal{Q}} \Big(\rho_t(q')\sigma_t(q' \to q) - \rho_t(q)\sigma_t(q \to q')\Big)dq'$$

- The **PDE** that QM **probability density** follows

$$\frac{\partial \rho_t(q)}{\partial t} = \underbrace{\frac{2}{\hbar}Im\{\psi_t^*(q)(\mathbb{H}_0\psi_t)(q)\}}_{\text{Current due to } \mathbb{H}_0} + \underbrace{\int_{q'\in\mathcal{Q}} \frac{2}{\hbar}Im\{\psi_t^*(q)\langle q|\mathbb{H}_{\mathbb{I}}|q'\rangle\psi_t(q')\}dq'}_{\text{due to } \mathbb{H}_I}$$

Step 2: Accommodate $\mathbb{H}_I$'s current as **random jumps**

- Since in general $[\mathbb{H}_I, \hat{N}] \neq 0$ **flow of probability between sectors (meaning also different species!)**
- $\mathbb{H}_I$ **polynomial in create/annihilation** ops. → has a kernel: cut-off smearing → $\mathbb{H}_I$ **is integral operator**

integral, just like jump balance!

$$(\mathbb{H}_I\psi)(q) = \int_{q'\in\mathcal{Q}} \langle q|\mathbb{H}_I|q'\rangle\psi(q')dq'$$

Equate in PDEs: $$\rho_t(q')\sigma_t(q' \to q) - \rho_t(q)\sigma_t(q \to q') = \frac{2}{\hbar}Im\{\psi_t^*(q)\langle q|\mathbb{H}_{\mathbb{I}}|q'\rangle\psi_t(q')\}$$

So, **for equivariance:**

$$\rho_t(q')\sigma_t(q' \to q) - \rho_t(q)\sigma_t(q \to q') = \frac{2}{\hbar} Im\{\psi_t^*(q)\langle q|\mathbb{H}_\mathbb{I}|q'\rangle\psi_t(q')\}$$

Restrictions:

- RHS **anti-symmetric** exchang. *q, q'*
(bec. kernel of s.a. op is conjug. symmetric)
- $\sigma_t(q' \to q) \geq 0$

$$\sigma_t(q' \to q)\rho(q') = \frac{2}{\hbar}\left[Im\{\psi_t^*(q)\langle q|\mathbb{H}_\mathbb{I}|q'\rangle\psi_t(q')\}\right]^+$$

$$\sigma_t(q' \to q) = \frac{2}{\hbar}\frac{\left[Im\{\psi_t^*(q)\langle q|\mathbb{H}_\mathbb{I}|q'\rangle\psi_t(q')\}\right]^+}{\psi_t^*(q')\psi_t(q')}$$

(Details of minimality in [9])

Minimal jump rates!

$q_1 \to q_2$ and $q_1 \leftarrow q_2$ **simultaneously prohibited**!

(content of $[\cdot]^+$ *is anti-sym. so if exchange q, q' get opposed sign only one order can be positive!)*

- Only **allowed** jumps **if** $\langle q|\mathbb{H}|q'\rangle \neq 0$ → if $\langle q|\mathbb{H}|q'\rangle = 0$ ∀ q=q' $\Longrightarrow$ jump = creation/destruction
- **Why not** jump in $\mathbb{H}_0$? → **Diff. operator** only $\langle q|H|q'\rangle \neq 0$ in **diagonal**, → **Minimal jump for them** is **continuous** transition!

  e.g. Schrödinger $\langle q|H|q'\rangle = \delta''(q-q') + V(q)\delta(q-q')$
- In **discretized space** minimal jump rate, yield **continuum limit** → **Bohmian EOM** (continuous+deterministic)

How to do this with Anti-symmetrized Fock spaces

- **If** $\tilde{\mathscr{H}}$ has **anti-symmetrized** Fock space, e.g., $\tilde{\mathscr{H}} = \mathscr{F}_{anti}(L^2(\mathbb{R}^3, \mathbb{C}^s))$ **not immediate identification** with **states** over **unordered** $\mathcal{Q}$, bec. for **a same** $q \in \mathcal{Q}$ $\tilde{q} \in \pi^{-1}(q)$ yield **same** $\phi(\tilde{q})$ **only up to a sign**

- Still ∃ **convenient approach** (the "*fermionic line bundle*"):

Essentially **identify** $\phi \in \tilde{\mathscr{H}}$ with functions that for **each** $q \in \mathcal{Q}$ **give** a **vector** of $(\#q)!$ **spinor entries** *(one per each order)* and **save in those slots** the **values** of ϕ on each order of q:

- Let *E* be the **vector bundle over** $\mathcal{Q}$ with **fibres** $E_q := \bigoplus_{\tilde{q} \in \pi^{-1}(q)} \mathbb{C}^{s \otimes (\#q)}$

↳ $\forall \phi \in \tilde{\mathscr{H}}$ **identify** with $\psi \in \Gamma(E)$ s.th. $\psi(q) := \bigoplus_{\tilde{q} \in \pi^{-1}(q)} \phi(\tilde{q})$

They form a 1D sub-bundle of E (the anti-symmetric wfs). Rigurous details on

S. Goldstein, J. Taylor, R. Tumulka, and N. Zanghi, "*Fermionic wave functions on unordered configurations*", arXiv preprint arXiv:1403.3705 (2014).

- **Up to** this **identification** and **understanding** by $|\psi(q)|^2$ a **summation over** the **spin** indices, almost **same steps** yield an **obvious posit.** POVM $\{P(B)\}_{B \subseteq \mathcal{Q}}$ *(ingredt. 1)* and the "same" **jump process** *(ingredt. 2)*

- **Goal :** Get **QFT predictions** using **particles** that **exist concretely.**
- **Need :** **Positions for those particles** + **a law of motion**

Ingredient 1 : = **right position POVM $P, \mathcal{Q}$ giving position* probabilities** $\langle\psi|P(dq)|\psi\rangle$

probability equivariant law for trajectories = *: Ingredient 2*

d) Bell-type QFTs – (ii) The General Case

First assume *Ingredient 1* *(POVM)* known

Ingredient 2: A Minimal Process

- A General QFT:

a **Hilbert** space $\mathscr{H} = \mathcal{F}_{(anti)sym}(\mathcal{H}_1) \otimes \cdots \otimes \mathcal{F}_{(anti)sym}(\mathcal{H}_{m_{species}})$ with IP $\langle\psi, \phi\rangle$

+ a **Unitary OPG** $\{\mathbb{U}_t\}_{t\in T}$ **with Hamiltonian** $\mathbb{H}$ → $\mathbb{U}_t = e^{-\frac{i}{\hbar}\mathbb{H}t}$ → $i\hbar\frac{\mathrm{d}}{\mathrm{d}t}\psi_t = \mathbb{H}\psi_t$

+ **We know a POVM** $P, \mathcal{Q}$ **on** $\mathscr{H}$ **s.t** → **Probability density for configuration is** $\mathbb{P}_t(dq) := \langle\psi_t|P(dq)\psi_t\rangle$

Leibniz rule on: $\frac{\mathrm{d}}{\mathrm{d}t}\langle\psi_t|P(dq)\psi_t\rangle$ *and plug eqts.* →

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{P}_t(dq) = \frac{2}{\hbar} Im\langle\psi_t|P(dq)\mathbb{H}|\psi_t\rangle$$

meaning it holds changing dq for any $B \subseteq \mathcal{Q}$

- **If $\mathbb{H}$ has a "free particle part"** $\mathbb{H}_0$, let $\mathbb{H} = \mathbb{H}_0 + \mathbb{H}_I$

(i) If $P, \mathcal{Q}$ is **PVM** and $\mathbb{H}_0$ **differential** op. **max deg 2** in *P's* diag. repres. (e.g., Schrödinger, Pauli, Dirac fields)

As seen, $\langle\psi|\hat{v}|\psi\rangle = \int_{\mathcal{Q}} Re\langle\psi|P(dq)\hat{v}|\psi\rangle =: \int_{\mathcal{Q}} v(q)\langle\psi|P(dq)|\psi\rangle$ defined via **Radon-Nikodým derivative as:**

$$v(q) := Re\frac{\langle\psi|P(dq)\frac{i}{\hbar}[\mathbb{H}_0, \hat{q}]|\psi\rangle}{\langle\psi|P(dq)|\psi\rangle}$$

→ In HS where $\hat{q} := \int_{q\in\mathcal{Q}} qP(dq)$ is multipl. op } $-div(v(q)\mathbb{P}_t(dq)) = \frac{2}{\hbar}Im\langle\psi_t|P(dq)\mathbb{H}_0|\psi_t\rangle$

(ii) If $\mathbb{H}_0 = \sum_k^{n\leq n_{species}} \Gamma(H_{1,k})$ and **equivariant 1-particle Markov process is known** for $H_{1,k}$

↓

lift 1-particle law via tensor product to N-particle → Make it N-th sector EOM

Details in [9]

- In general, (or for $\mathbb{H}_I$ if there was $\mathbb{H}_0$)

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{P}_t(dq) = \frac{2}{\hbar} Im\langle\psi_t|P(dq)\mathbb{H}|\psi_t\rangle$$

$$I = \int_{q'\in\mathcal{Q}} P(dq')$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{P}_t(dq) = \int_{q'\in\mathcal{Q}} \frac{2}{\hbar} Im\langle\psi_t|P(dq)\mathbb{H}P(dq')|\psi_t\rangle$$

RHS antisymmetric

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{P}_t(dq) = \int_{q'\in\mathcal{Q}} \left(\frac{2}{\hbar}\left[Im\langle\psi_t|P(dq)\mathbb{H}P(dq')|\psi_t\rangle\right]^+ - \frac{2}{\hbar}\left[Im\langle\psi_t|P(dq')\mathbb{H}P(dq)|\psi_t\rangle\right]^+\right)$$

Equate with

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{P}_t(dq) = \int_{q'\in\mathcal{Q}} \left(\rho_t(dq')\sigma_t(q' \to dq) - \rho_t(dq)\sigma_t(q \to dq')\right)$$

via Radon-Nikodým derivative

$$\sigma(q' \to dq) = \frac{2}{\hbar}\frac{\left[Im\langle\psi|P(dq)\mathbb{H}P(dq')|\psi\rangle\right]^+}{\langle\psi|P(dq')|\psi\rangle}$$

General Minimal Jump rates!

Precise conditions for rates in [10]

Any $\mathbb{H}$ is **approximable** by $\mathbb{H}_n$ (Hilbert-Schmidt ops.) for which **rates well-defined and equivariant**

Resulting Jump Process and Ontology

config. of species k

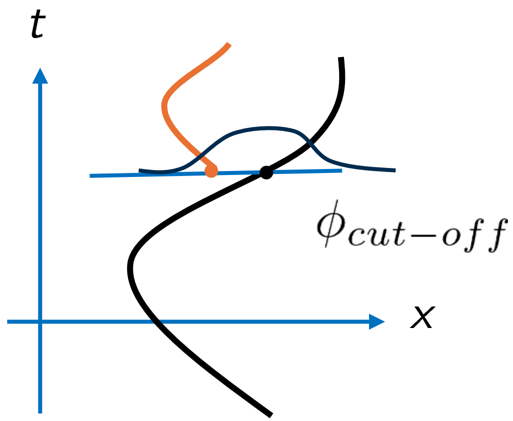
$$Q : I \subseteq \mathbb{R} \longrightarrow \mathcal{Q} = \mathcal{Q}_1 \times \cdots \times \mathcal{Q}_n \longleftarrow \mathcal{Q}_k := \Gamma(\mathcal{Q}_k^{(1)}) \equiv \bigcup_{n=0}^{\infty} \mathcal{Q}_k^{(n)} \equiv \bigcup_{n=0}^{\infty} \{q \subset Q_k^{(1)} \mid \#q = n\}$$

- **Deterministic** motion $\frac{\mathrm{d}}{\mathrm{d}t} Q_k(t) = v_k(Q(t))$ for species *k* with liftable **Bohmian** law

$+$

- Exponentially distributed **random creation/annihilation between interacting particles,** with rate density

$$\sigma(dq' \to dq)$$



Potential criticisms?

- *If UV cut-off :* Particles are **created** at close but **non-zero distance**

  ↳ Worry not! **Interior Boundary Condition technique** has been developed!

- Trajectories "**jump**" fundamentally! ⟶ at least for fermions **Dirac Sea pict.** may allow **deterministic** motion...

Ingredient 1: A Position "Operator" (= POVM)

We assumed we had a position POVM....but **at first glimpse**, many **QFT come with no distinguished posit. op**!

Need methods to find a position POVMs!

Method A : Lift Sector-1 Position "Operator"

If 1-particle (sectors) $\mathscr{H}^{(0,..,1,...0)} = \mathcal{H}_{1,k}$ **had position "operators"**: **lift them** to $\mathscr{H}$!

- **Corollary** 7 in [10]:

Given **two Hilbert spaces** $\mathcal{H}_a, \mathcal{H}_b$, two standard Borel measure spaces $\mathcal{Q}_a, \mathcal{Q}_b$ and **POVMs,** E_a **on** $\mathcal{Q}_a$ **acting on** $\mathcal{H}_a$ **and** E_b **on** $\mathcal{Q}_b$ **acting on** $\mathcal{H}_b$**,** then, there **exists a unique POVM** E **on** $\mathcal{Q}_a \times \mathcal{Q}_b$ **acting on** $\mathcal{H}_a \otimes \mathcal{H}_b$ satifying

$$E(B_a \times B_b) = E_a(B_a) \otimes E_b(B_b)$$

for all $B_a \subseteq \mathcal{Q}_a$ and $B_b \subseteq \mathcal{Q}_b$. E is called the tensor product POVM, denoted by $E_a \otimes E_b$.

If E_a and E_b are PVMs, then E is a PVM.

Method A : Lift Sector-1 Position "Operator"

- **How to lift POVM** P_k acting on **k-th species** **1-particle configuration** and **Hilbert** space, $\mathcal{Q}_k^{(1)}, \mathcal{H}_k$ to its "second quantized" config.- and Fock space $\mathcal{Q}_k, \mathscr{F}_{(anti)sym}(\mathcal{H}_k)$?

Recall, second quantized config. space is unordered one $$\mathcal{Q}_k := \Gamma(\mathcal{Q}_k^{(1)}) \equiv \bigcup_{n=0}^{\infty} \mathcal{Q}_k^{(n)} \equiv \bigcup_{n=0}^{\infty} \{q \subset Q_k^{(1)} \mid \#q = n\}$$

1) Take **ordered** $(\mathcal{Q}_k^{(1)})^{\times n}$ **apply iteratively corollary** ⟶ $P_k^{\otimes n}$ **unique POVM** on $(\mathcal{Q}_k^{(1)})^{\times n}, \mathcal{H}_k^{\otimes n}$ s.th

$$U_j \subseteq \mathcal{Q}_j^{(1)} \qquad (P_k)^{\otimes n}(U_1 \times \cdots \times U_n) = P_k(U_1) \otimes \cdots \otimes P_k(U_n)$$

ordered conf!

2) Define **action on unordered configs.**: $$P_k^{(n)}(B) := P_k^{\otimes n}\left(\pi^{-1}(B)\right); \qquad B \subseteq \mathcal{Q}_k^{(n)}$$ *(unordered!)*

Recall forgetful mapping $$\pi^{-1}(B_n) := \left\{(q_1, ..., q_n) \in (\mathcal{Q}^{(1)})^n | \{q_1, ..., q_n\} \in B_n\right\}$$

3) $P_k^{(n)}(\cdot)$ is invariant under permutats., so it maps **(anti)sym.** ⟶ **(anti)sym** elements of $\mathcal{H}_k^{\otimes n}$

↳ **By restriction, gives a POVM on** $(Anti)Symm(\mathcal{H}_k^{\otimes n})$

Method A : Lift Sector-1 Position "Operator"

4) Assemble POVM $P_k^{(n)}(\cdot)$ **of different. sectors** $\mathcal{Q}_k^{(n)}$**,** $(Anti)Symm(\mathcal{H}_k^{\otimes n})$ **in single POVM on** $\mathcal{Q}_k, \mathscr{F}_{(anti)sym}(\mathcal{H}_k)$ **by direct sum:**

$$\Gamma\Big(P_k\Big)(B) := \bigoplus_{n=0}^{\infty} P_k^{(n)}(B \cap Q_k^{(n)}) \qquad B \subseteq \mathcal{Q}_k$$

Called **"second quantizated"/lift of POVM** P_k

(acting on the **"second quantization"** of config. $\mathcal{Q}_k^{(1)}$ and Hilbert space $\mathcal{H}_k$ $\longrightarrow$ $\Gamma(\mathcal{Q}_k^{(1)}) := \mathcal{Q}_k$, $\mathscr{F}_{(anti)sym}(\mathcal{H}_k)$)

5) But there are $m_{species}$ species! So, use again corollary to make **position POVM for all species**

- **joint config. space** (*preserving order btw. –distinguishbl.– species*)

$$\mathcal{Q} := \mathcal{Q}_1 \times \cdots \times \mathcal{Q}_{m_{species}}$$

- **joint Hilbert space**

$$\mathscr{H} = \mathcal{F}_{(anti)sym}(\mathcal{H}_1) \otimes \cdots \otimes \mathcal{F}_{(anti)sym}(\mathcal{H}_{m_{species}})$$

- **Position op. on them**

$$P_{QFT} := \Gamma\Big(P_1\Big) \otimes \cdots \otimes \Gamma\Big(P_{m_{species}}\Big)$$

Example 1 : Standard Position "Operator" for Dirac-particles (it is a PVM!)

- **1-particle** HS $L^2(\mathbb{R}^3, \mathbb{C}^4)$, P_{std} **PVM of std. position** op. $\hat{x}\psi^{(1)}(x) = x\psi^{(1)}(x)$ for $B \subseteq \mathbb{R}^3$

  $P_{std}(B)\psi^{(1)}(x) = \chi_B(x)\psi^{(1)}(x)$ associated position prob. is a.c. $\langle\psi^{(1)}|P_{std}(dx)|\psi^{(1)}\rangle = |\psi^{(1)}(x)|^2 d^3x$

- **Lift** to $\mathscr{F}_{Dirac} := \mathscr{F}_{anti}(L^2(\mathbb{R}^3, \mathbb{C}^4)),\ \mathcal{Q} := \Gamma(\mathbb{R}^3)$ is the **PVM** $(\Gamma(P_{std})(B)\psi)^{(n)} = \chi_{\pi^{-1}(B\cap^n\mathbb{R}^3)}\ \psi^{(n)}$

  s.th. the position prob. is

$$\mathbb{P}(q \in B \mid \psi) = \langle\psi|\Gamma(P_{std})(B)|\psi\rangle = \sum_{n=0}^{\infty}\int_{\pi^{-1}(B\cap^N\mathbb{R}^3)} |\psi(q)|^2 d^{3n}q \qquad B \subseteq \mathcal{Q}$$

Example 2 : Obvious Position "Operator" for Positrons (it is a POVM!)

–forget C for a moment–

- **1-particle** HS $\mathcal{H}_- \subset L^2(\mathbb{R}^3, \mathbb{C}^4) = \mathcal{H}_+ \oplus \mathcal{H}_-$ let proj. (to E<0), $P_- : L^2(\mathbb{R}^3, \mathbb{C}^4) \to \mathcal{H}_-$

  Obvious position POVM was $P_{e+,obv}(\cdot) = P_- P_{std}(\cdot) incl$ with inclusion $incl : \mathcal{H}_- \to L^2(\mathbb{R}^3, \mathbb{C}^4)$

- **Lift** to $\mathscr{F}_{e+} := \mathscr{F}_{anti}(\mathcal{H}_-),\ \mathcal{Q} := \Gamma(\mathbb{R}^3)$ is the **POVM** $(\Gamma(P_{e+,obv})(B)\psi)^{(n)} = (P_-)^{\otimes n}\chi_{\pi^{-1}(B\cap^n\mathbb{R}^3)}\ \psi^{(n)}$

  s.th. the position prob. is

$$\mathbb{P}(q \in B \mid \psi) = \langle\psi|\Gamma(P_{e+,obv})(B)|\psi\rangle = \sum_{n=0}^{\infty}\int_{\pi^{-1}(B\cap^N\mathbb{R}^3)} |\psi(q)|^2 d^{3n}q \qquad B \subseteq \mathcal{Q}$$

Example 3 : Obvious (and not-that-obvious) Position "Operator" for Electron/Positron QFT

$$\mathscr{H} = \mathscr{F}_{e-} \otimes \mathscr{F}_{e+} = \bigoplus_{n=0}^{\infty} Anti((\mathcal{H}_+)^{\otimes n}) \otimes \bigoplus_{\tilde{n}=0}^{\infty} Anti((\mathcal{H}_-)^{\otimes \tilde{n}})$$

Using algorithm: **Obvious POVM** → **A Bell jump proc. for e- & e+**

$$\mathcal{Q}_{e\pm} = \Gamma(\mathbb{R}^3) \times \Gamma(\mathbb{R}^3) \quad , \quad P_{obv} = P_{e-,obv} \otimes P_{e+,obv} = \Gamma(P_+ P_{std} I) \otimes \Gamma(P_- P_{std} I)$$

subspace of

$$\mathscr{F}_{Dirac} \otimes \mathscr{F}_{Dirac} = \bigoplus_{N=0}^{\infty} Anti((L^2(\mathbb{R}^3, \mathbb{C}^4))^{\otimes N}) \otimes \bigoplus_{N=0}^{\infty} Anti((L^2(\mathbb{R}^3, \mathbb{C}^4))^{\otimes N})$$

By Projection to subspace $\mathcal{H}$ **we recover**

$$\mathcal{Q}_{2D} = \Gamma(\mathbb{R}^3) \times \Gamma(\mathbb{R}^3) \qquad P = \Gamma(P_{std}) \otimes \Gamma(P_{std})$$

It's a **PVM**!

So even after secd. qtzt. they're relatd by a projection!

we found it is unitarily **identified with**

$$\bigoplus_{N=0}^{\infty} Anti((\mathcal{H}_+ \oplus \mathcal{H}_-)^{\otimes N}) = \bigoplus_{N=0}^{\infty} Anti((L^2(\mathbb{R}^3, \mathbb{C}^4))^{\otimes N}) = \mathscr{F}_{Dirac}$$

$$\mathcal{Q}_D = \Gamma(\mathbb{R}^3) , \qquad P = \Gamma(P_{std})$$

Not-that-obvious PVM (!) for $\mathscr{H}$

It would yield trajectories that take e+/e- as same particle!

Which is the Correct Position Operator for e-/e+?

- **Only one** of the possibilities **can yield the correct position representation**

  How to find? **Experimental** verification and/**or relativistic mathematical arguments**

Desiderata \ Posit. op.	$P_{NW}(\cdot)$	$P_{obv}(\cdot)$	Wanted! ?
Propagation locality*	✗	"✓"	✓
Interaction locality**	"✗"	✗	✓

Obtainable by lift from 1-particle theory (Method A) → **In fact, we found no more "nice" 1-particle posit. ops!** → **Then must rely on N(t)-particle theory?** → **Method B!**

** = No faster than light wavefunction propagation*

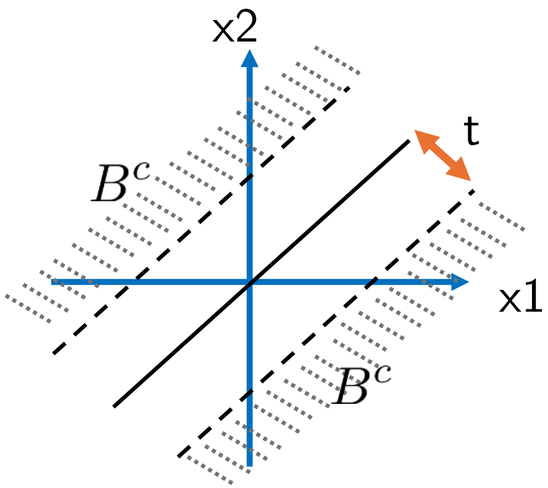
*** = Absence of interact terms betw spacelike sep regions in unitary evolution/Hamiltonian*

Revisiting the Difficulty with $P_{obv}(\cdot)$ (now in N(t)) : Violation of Interaction Locality

Following [11]

Reason 2: Else conservation of particle number violated

- Consider **2e- interacting** and $\psi_t := \psi^{(2,0)}_{s_1,s_2}(x_1, x_2, t)$
- Let **free** time **evolution part** be $e^{-i\mathbb{H}_0 t}$



- **If interaction locality** ⟶ $\psi_t|_{B^c} = e^{-i\mathbb{H}_0 t}\psi_0|_{B^c}$

$$B := \text{"}t\ neightood\ of\ diagonal:\ \{(x_1, x_2)|x_1 = x_2\}\text{"}$$

- If **both** ψ_0 and ψ_t **in** $\mathcal{H}_+ \otimes \mathcal{H}_+$ (i.e., (2,0) sector) ⟶ $\psi_t - e^{-i\mathbb{H}_0 t}\psi_0$ **too (vect. sp)**

But $\psi_t - e^{-i\mathbb{H}_0 t}\psi_0 = 0 \quad in\ B^c$ can **only be if** $\psi_t - e^{-i\mathbb{H}_0 t}\psi_0 = 0$ **everywhere** ⟶ $\psi_t = e^{-i\mathbb{H}_0 t}\psi_0$

Contradicts interaction!

Conclusion:
- **<u>Either</u>** $\psi_t \neq 0$ in **other sectors immediately after *t* = 0** ⟶ **no particle number conservation**
- **<u>Or</u> local interaction** for $P_{obv}(\cdot)$ **impossible**

- A **NOVM** is a pair $W, \{\hat{N}(B)\}_{B\in W}$
  - W a measurable space, e.g., $W=\mathbb{R}^3$ (with Borel sigma algebra)
  - $\hat{N}(B):\mathscr{H}\to\mathscr{H}$ **un**-bounded linear operators, s.th.
    - **commuting ops.** $[\hat{N}(B),\hat{N}(\tilde{B})]=0 \quad \forall B,\tilde{B}\subseteq W$
    - $spectrum(N(B))\subseteq\{0\}\cup\mathbb{N}$
    - **s.a. ops.** $\hat{N}(B)^*=\hat{N}(B)$
    - **strong σ-addtvty** for $\{B_k\}_{k\in\mathbb{N}}$ disjoint in *W*

$$N\left(\bigsqcup_{k\in\mathbb{N}} B_k\right)=\sum_{k=0}^{\infty}N(B_k)$$

(strong sense)

- Yields a measure on *W* $\quad Num(\cdot):=\langle\psi,\hat{N}(\cdot)\psi\rangle$

An unormalized POVM, like a PVM but spectrum is in $\{0,1,2,\ldots\}$ *rather than* $\{0,1\}$

- $\{\hat{N}(B)\}_{B\in W}$ **commuting family of s.a. ops** ⟶ **Joint spectrum** are the $\mathbb{N}\cup\{0\}$**-valued measures:**

Any **joint eigenvalue** is a **map** $n.$ taking **subsets of *W*** and giving **a natural number** or 0 (*infty not allowed*), **σ-addtvely,** so in 1-1-correspdce with **an unordered configuration** *(up to multpltcties),* i.e. for all eigenvalue $n.$

$$\exists q\in\Gamma(W):=\{q\subseteq W \mid \#q<+\infty\} \text{ s.th } n.=n.(q) \quad \text{with} \quad n_B(q)=\#(q\cap B) \qquad \forall B\subseteq\mathbb{R}^3$$

- By **spectral theorem one-to-one correspondence**

$\{\hat{N}(B)\}_{B\in W}$ **NOVM** on W	⟷	$\{P(A)\}_{A\in\Gamma(W)}$ **PVM** on $\Gamma(W)$

via

$$\hat{N}(B)=\int_{q\in\Gamma(W)}n_B(q)P(dq) \qquad B\subseteq W \ (e.g., \mathbb{R}^3)$$

- One-to-one relation between a NOVM $\hat{N}(\cdot)$ *on* $\mathbb{R}^3$ and a PVM $P(\cdot)$ *on* $\Gamma(\mathbb{R}^3)$ i.e., position operator for QFT:

$\{\hat{N}(D)\}_{D\subseteq\mathbb{R}^3}$ NOVM on $\mathbb{R}^3$ → **Joint spectrum indexed by unordered configts of $\Gamma(\mathbb{R}^3)$** maps giving numb. in $\mathbb{N}\cup\{0\}$ to each subset of $\mathbb{R}^3$

$$n_D(q) = \#(q\cap D) \qquad \forall q\in\Gamma(\mathbb{R}^3) := \{q\subseteq\mathbb{R}^3 \mid \#q < +\infty\}$$

(up to multipl.)

Joint PVM $\{P(A)\}_{A\in\Gamma(\mathbb{R}^3)}$

A position operator!

$$\hat{N}(D) = \int_{q\in\Gamma(\mathbb{R}^3)} n_D(q)P(dq)$$

- **So spectral Hilbert space where all $\{\hat{N}(D)\}_{D\subseteq\mathbb{R}^3}$ are diagonal is a position representation!**

d.ii.2) Method B : Use the Field Operators! (They were actually "for this"!)

- We said "**at first glimpse**, many QFT come with no...distinguished configuration"

It turns out that **many "secretly" do,** insofar they define **field operators** $\Psi(x)$!

– Don't lift, take PVM of Fock space operators!

- Usually $\Psi(x)$ introduced "merely" to define Hamiltonians with good spacetime properties...

Dürr et al. say more : **field ops.** are **exactly the structure to properly define position ops.**!

via Number operators that yield a NOVM

How to obtain a NOVM over $\mathbb{R}^3$?

- Any **single specie** QFT $\mathscr{H} = \mathscr{F}_{(anti)sym}(\mathcal{H}_1)$ has a **create/annih op.** $\Psi(\cdot)$ and hence a **number operator**

$$\hat{N} = \sum_{j\in\mathbb{N}} \Psi^\dagger(f_j)\Psi(f_j) \quad \text{for } \{f_j\}_{j\in\mathbb{N}} \subseteq \mathcal{H}_1 \text{ arbitry ONB} \longleftarrow \textbf{formally} \quad \hat{N} = \text{“}\int_{\mathbb{R}^3} \Psi^\dagger(\vec{x})\Psi(\vec{x})d^3x\text{”}$$

- Now, **formally**, one can **define** the **number operator for general measurable subsets** $D \subseteq \mathbb{R}^3$

$$\hat{N}(D) = \text{“}\int_D \Psi^\dagger(\vec{x})\Psi(\vec{x})d^3x\text{”}$$

they are a family of ops. $\{\hat{N}(D)\}_{D\subseteq\mathbb{R}^3}$ that **formally satisfy**

- **commute** $[\hat{N}(D), \hat{N}(\tilde{D})] = 0 \quad \forall D, \tilde{D} \subseteq \mathbb{R}^3$
- **s.a.** $\hat{N}(D)^* = \hat{N}(D)$
- **eigenvals. in $\mathbb{N} \cup \{0\}$**
- **σ-addtvty** $N\left(\bigsqcup_{k\in\mathbb{N}} B_k\right) = \sum_{k=0}^{\infty} N(B_k)$

- So **formally**, $\{\hat{N}(D)\}_{D\subseteq\mathbb{R}^3}$ is a **NOVM**
- By **spectral theorem ∃ simultans. diagonlzt. PVM**, $P(\cdot)$
- **Joint spectrum eigenvalues** n. **are indexable by** $q \in \Gamma(\mathbb{R}^3)$ $\quad n_q(D) := \#(q \cap D)$

$$\hat{N}(D) = \int_{q\in\Gamma(\mathbb{R}^3)} n_q(D)\ P(dq)$$

- So formally, **any QFT** would have a **natural position op.** (it would even be a **PVM**!) $\quad \hat{X} := \int_{q\in\mathcal{Q}} q\ P(dq)$

  Natural because *it is the one position op.* ***encoded by the fields of QFT***!

How to obtain a NOVM over $\mathbb{R}^3$?

- However, this is **true only** if we can **"enrigorize"** $\hat{N}(D) = \text{“}\int_D \Psi^\dagger(\vec{x})\Psi(\vec{x})d^3x\text{”}$

- **If** $\mathcal{H}_1 = L^2(\mathbb{R}^3, \mathbb{C}^s)$ (e.g., the Schröding. field s=1 or the Dirac-particle field s=4 –not caring if E>0,E<0 etc.)

"rigorifiable" via: $\{\varphi_j\}_{j\in\mathbb{N}} \subset L^2(D, \mathbb{C}^s)$ **1-particle ONB of** $L^2(D, \mathbb{C}^s)$ ⟶ $\hat{N}(D) = \sum_{j\in\mathbb{N}} \Psi^\dagger(\varphi_j)\Psi(\varphi_j)$

↳ This yields the **lift of the standard posit. op.** $\Gamma(P_{std})$ *(use def. of $\hat{N}$ as multipl. op in $\mathscr{F}_{(anti)sym}(\mathcal{H}_1)$)*

- **BUT**, when $\mathcal{H}_1$ is a **closed subset** of $L^2(\mathbb{R}^3, \mathbb{C}^s)$ where **no compactly supported wavefunction exists**

this is **not possible**: **not obvious if** $\exists\{\varphi_j\}_{j\in\mathbb{N}} \subset \mathcal{H}_1$ **ONB** that when **restricted to D is also ONB of** $L^2(D, \mathbb{C}^s)$

- For example: **No clear NOVM for e-/e+,** since $\mathcal{H}_1 = \mathcal{H}_+, \mathcal{H}_- \subset L^2(\mathbb{R}^3, \mathbb{C}^4)$ and **we found** by [1, Cor. 1.7.]

$f \in \mathcal{H}_\pm$ **vanishing** in **open** set ⟶ **vanishes everywhere** ⟶ $\nexists$ compact supp $f \in \mathcal{H}_\pm$

↳ **"Rigorification" of** $\hat{N}_{e^-}(D) = \text{“}\int_D a^\dagger(\vec{x})a(\vec{x})d^3x\text{”}$ & $\hat{N}_{e^+}(D) = \text{“}\int_D b^\dagger(\vec{x})b(\vec{x})d^3x\text{”}$ is an ***Open Problem***

e) On the Natural e-/e+ Position Operator

But why do we want e-/e+ NOVM derived from field ops. (hence the natural posit. op.) to exist?

Because as **proven by Tumulka** [11] *(we will see)*, if it did, the resulting e-/e+ posit. op. $P_{nat}(\cdot)$ would be the **relativistic salvation**!

Desiderata \ Posit. op.	$P_{NW}(\cdot)$	$P_{obv}(\cdot)$	$P_{nat}(\cdot)$
Propagation locality*	✗	"✓"	✓
Interaction locality**	"✗"	✗	✓

Obtainable from 1-particle theory ($P_{NW}(\cdot)$, $P_{obv}(\cdot)$)

Only Obtainable from N(t)-particle theory! ($P_{nat}(\cdot)$)

** = No faster than light wavefunction propagation*

*** = Absence of interact terms betw spacelike sep regions in unitary evolution/Hamiltonian*

e) On the Natural e-/e+ Position Operator

(i) Towards a Sufficient Condition for an e-/e+ NOVM

- Recall, given **e-/e+ number ops.** $\hat{N}_{e^-} = \sum_{j\in\mathbb{N}} a^\dagger(h_j)a(h_j)$ & $\hat{N}_{e^+} = \sum_{j\in\mathbb{N}} b^\dagger(g_j)b(g_j)$ $\left\{ \begin{array}{l} \{h_j\}_{j\in\mathbb{N}} \subset \mathcal{H}_+ \\ \{g_j\}_{j\in\mathbb{N}} \subset \mathcal{H}_- \\ \text{ONBs} \end{array} \right.$

we **found two** distinguished **Number Operators for e-/e+ QFT** $\mathscr{H} = \mathscr{F}_{e-} \otimes \mathscr{F}_{e+}$

$\hat{N}_{e^-e^+} = \hat{N}_{e^-} + \hat{N}_{e^+}$ ⟵ **e- & e+ number operator**

$\hat{Q} = q(\hat{N}_{e^-} - \hat{N}_{e^+})$ ⟵ **Charge operator = excess e- operator**

- **IF** there was a **way to "enrigorize"** $\{\hat{N}_{e^-}(D)\}_{D\subseteq\mathbb{R}^3}, \{\hat{N}_{e^+}(D)\}_{D\subseteq\mathbb{R}^3}$ then, by **spectral theorem** there **would exist**

$\{\hat{N}_{e^-}(D)\}_{D\subseteq\mathbb{R}^3}$ ⟶
- $\exists$ **simultans. diagonlzt.** PVM $P_{e^-}(\cdot)$ on $\Gamma(\mathbb{R}^3)$ $\mathscr{F}_{e+}$
- **Joint spectrum :** $n_{e^-,D}(q_{e^-}) := \#(q_{e^-} \cap D)$ $\quad q_{e^-} \in \Gamma(\mathbb{R}^3)$

$\Big\}$ $\hat{N}_{e^-}(D) = \int_{q\in\Gamma(\mathbb{R}^3)} n_{e^-,D}(q)\; P_{e^-}(dq)$

$\{\hat{N}_{e^+}(D)\}_{D\subseteq\mathbb{R}^3}$ ⟶
- $\exists$ **simultans. diagonlzt.** PVM $P_{e^+}(\cdot)$ on $\Gamma(\mathbb{R}^3), \mathscr{F}_{e-}$
- **Joint spectrum :** $n_{e^-,D}(q_{e^-}) := \#(q_{e^-} \cap D)$ $\quad q_{e^-} \in \Gamma(\mathbb{R}^3)$

$\Big\}$ $\hat{N}_{e^+}(D) = \int_{q\in\Gamma(\mathbb{R}^3)} n_{e^+,D}(q)\; P_{e^+}(dq)$

- **By** the **corollary**, $P_{nat} := P_{e^-} \otimes P_{e^+}$ would a **PVM** on $\mathcal{Q} := \Gamma(\mathbb{R}^3) \times \Gamma(\mathbb{R}^3)$ acting on $\mathscr{H} = \mathscr{F}_{e-} \otimes \mathscr{F}_{e+}$

(Note it is not $P_{obv}(D)$ **since that was only a POVM!)**

(i) Towards a Sufficient Condition for an e-/e+ NOVM

(IF...)

- But then $P_{nat}(\cdot)$ would be **the PVM diagonalizing** $\hat{N}_{e^-e^+}(D) := \hat{N}_{e^-}(D) + \hat{N}_{e^+}(D)$ since:

$$\hat{N}_{e^-e^+}(D) = N_{e^-}(D) \otimes Id_{\mathscr{F}_{e^+}} + Id_{\mathscr{F}_{e^-}} \otimes N_{e^+}(D)$$

$$= \int_{q\in\Gamma(\mathbb{R}^3)} n_{e^-,D}(q)\ P_{e^-}(dq) \otimes Id + Id \otimes \int_{q\in\Gamma(\mathbb{R}^3)} n_{e^+,D}(q)\ P_{e^+}(dq)$$

$$= \iint_{q',q\in\Gamma(\mathbb{R}^3)} n_{e^-,D}(q)\ P_{e^-}(dq) \otimes P_{e^+}(dq') + \iint_{q',q\in\Gamma(\mathbb{R}^3)} n_{e^+,D}(q)\ P_{e^-}(dq') \otimes P_{e^+}(dq)$$

Last corollary →

$$= \int_{(q',q)\in\mathcal{Q}} n_{e^-,D}(q)\ P_{e^+} \otimes P_{e^-}(dq' \times dq) + \int_{(q',q)\in\mathcal{Q}} n_{e^+,D}(q)\ P_{e^+} \otimes P_{e^-}(dq \times dq')$$

$$= \int_{(q',q)\in\mathcal{Q}} \Big(n_{e^-,D}(q) + n_{e^+,D}(q')\Big)\ P_{nat}(dq' \times dq) = \int_{q\in\mathcal{Q}} N_D(q)\ P_{nat}(dq)$$

Total number of e-&e+ in *D*

$$N_D((q,q')) := n_{e^-,D}(q) + n_{e^+,D}(q')$$

$$= \#(q \cap D) + \#(q' \cap D) \qquad \forall (q,q') \in \mathcal{Q} \equiv \Gamma(\mathbb{R}^3) \times \Gamma(\mathbb{R}^3)$$

(i) Towards a Sufficient Condition for an e-/e+ NOVM

(IF...)

- And **importantly** $P_{nat}(\cdot)$would **also be the PVM diagonalizing** $\hat{Q}(D)$ since by the same steps:

$$\hat{Q}(D) = -e\Big(N_{e^-}(D) \otimes Id_{\mathscr{F}_{e+}} - Id_{\mathscr{F}_{e-}} \otimes N_{e^+}(D)\Big)$$

$$= \int_{(q',q)\in\mathcal{Q}} -e\Big(n_{e^-,D}(q) - n_{e^+,D}(q')\Big)\; P_{nat}(dq' \times dq)$$

$$= e\int_{q\in\mathcal{Q}} \lambda_D(q)\; P_{nat}(dq)$$

we assume e=1 (natural units)

$$\lambda_D((q,q')) := n_{e^+,D}(q) - n_{e^-,D}(q')$$

$$\forall (q,q') \in \mathcal{Q} \equiv \Gamma(\mathbb{R}^3) \times \Gamma(\mathbb{R}^3)$$

Total "charge" in *D*
=
number of e- until sea filled
=
- excess of e-s in sea

- This has an interesting **consequence**! **If instead of looking for a rigorization** of $\hat{N}_{e^-}(D), \hat{N}_{e^+}(D)$ we **look for** $\hat{N}_{e^-e^+}(D)$ or $\hat{Q}(D)$ we will have **enough by the uniqueness of spectral PVM** if their **"eigenvalues" are "writable"** as $N_D((q,q')) := n_{e^-,D}(q) + n_{e^+,D}(q')$ or $\lambda_D((q,q')) := n_{e^+,D}(q) - n_{e^-,D}(q')$

→ *"just would need to reverse-engineer previous steps"*

Key point: is that $\hat{Q}(D)$ is **indeed "rigorifiable"!** So "**done" if** it is **true th.** $\lambda_D((q,q')) := n_{e^+,D}(q) - n_{e^-,D}(q')$

(i) Towards a Sufficient Condition for an e-/e+ NOVM

- Recall, there was **no simple shape** of $\hat{N}_{e^- e^+}$ **in terms of full** theory **field ops.** $\Psi(\cdot) = a(\cdot) + b^\dagger(\cdot)$

Yet, we found

$$\boxed{\hat{Q}} = q \sum_{j=1}^{\infty} \Big(a^\dagger(P_+ f_j) a(P_+ f_j) \; - \; b^\dagger(P_- f_j) b(P_- f_j) \Big) =$$

$$= q \sum_{j=1}^{\infty} \left(\Psi^\dagger(f_j)\Psi(f_j) - ||P_- f_j||^2 \right) \qquad \boxed{\equiv q \sum_{j=1}^{\infty} : \Psi^\dagger(f_j)\Psi(f_j) :}$$

Where (a priori) the **obvious rigorization would be appliable**

$\{\varphi_j\}_{j\in\mathbb{N}} \subset L^2(D, \mathbb{C}^4)$ 1-pariclе ONB of $L^2(D, \mathbb{C}^4)$ $\longrightarrow$ $\hat{Q}(D) = q \sum_{j=1}^{\infty} : \Psi^\dagger(\varphi_j)\Psi(\varphi_j) :$

It is commuting family of s.a. ops. so jointly diagonalizable: just need $\lambda_D((q,q')) = n_{e^+,D}(q) - n_{e^-,D}(q')$

↳ **Enter Tumulka's Conjecture!** ⟵ He gives a **more refined, slightly more conservative condition that already allows Bell-type QFT** *(at least in finite space)*

R. Tumulka, "***Positron position operators. I. A natural option***", Annals of Physics **443**, 168988 (2022)

e) On the Natural e-/e+ Position Operator

R. Tumulka, "***Positron position operators. I. A natural option***", Annals of Physics **443**, 168988 (2022)

(ii) Tumulka's Conjecture

Recall: $P_{nat}(\cdot)$ is the PVM on the joint spectrum of all $\hat{Q}(A);\ A \subseteq \mathbb{R}^3$ (simultaneous diagonalization of all $\hat{Q}(A)$)

- Point in joint spectrum **=** provides eigenvalue λ_B per each $B \subseteq \mathbb{R}^3$ **=** a map $\lambda_\cdot : B \subseteq \mathbb{R}^3 \mapsto \lambda_B \in \mathbb{Z}$
- $\hat{Q}(\cdot)$ are σ-additive: $\hat{Q}(B_1 \cup B_2 \cup \cdots) = \hat{Q}(B_1) + \hat{Q}(B_2) + \cdots \quad if\ A_i \cap A_j = \emptyset\ \forall i \neq j$ → $Q_\cdot$ inherit **additivity**

Tumulka's Conjecture:

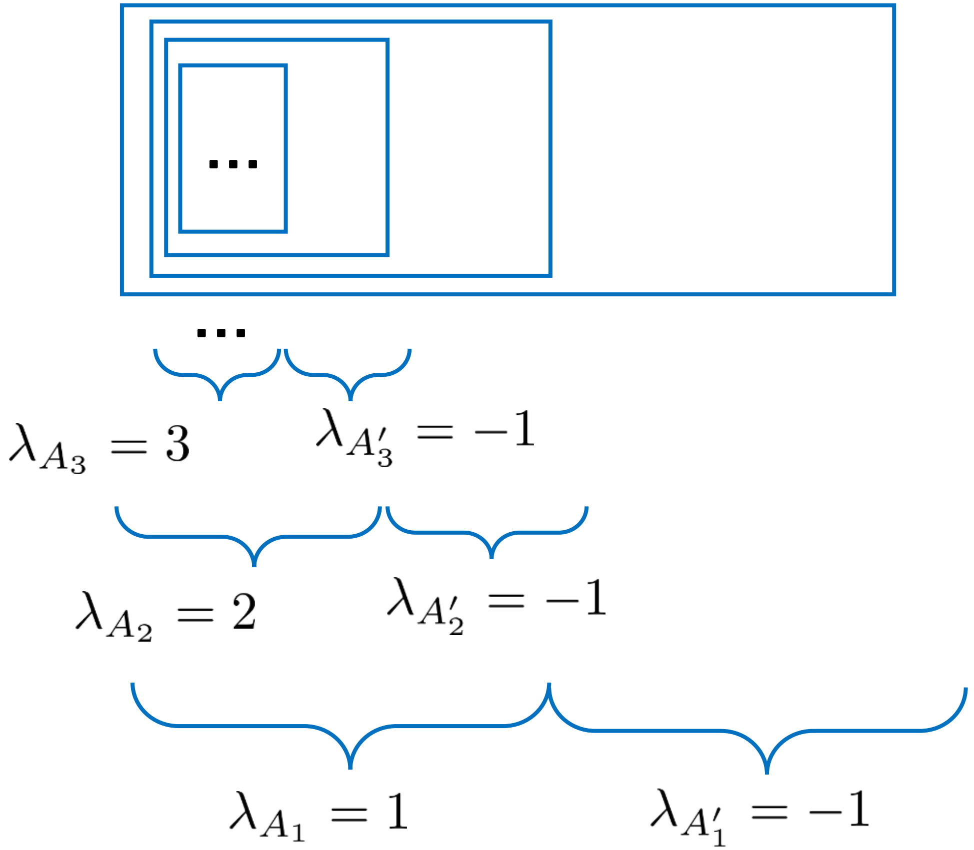
> The joint spectrum of $\{\hat{Q}(D)\}_{D \subseteq \mathbb{R}^3}$ consists (up to P_{nat}-null sets) of **locally bounded functions** $\lambda_\cdot : \mathcal{P}(\mathbb{R}^3) \mapsto \mathbb{Z}$.

$\lambda_\cdot : B \subseteq \mathbb{R}^3 \mapsto \lambda_B \in \mathbb{Z}$ locally bounded **=** $\forall x \in \mathbb{R}^3\ \exists r > 0$ and (finite) $C > 0$ s.th. $\forall A \subseteq B_r(x),\ |\lambda_A| < C$

= **$\nexists$ point x where arbitrarily small sets *A* have arbitrarily big λ_A**

= **always $\exists$ a radius from which on, λ_A's are bounded**

Example of
NON-locally-bounded
$\mathbb{Z}$ valued function



It is a **box** with **finite charge**
but unbounded charge
sub-boxes

Tumulka's Conjecture says this is NOT a possible joint eigenvalue

A **general locally-bounded additive** set function

$\lambda_\cdot : B \subseteq \mathbb{R}^3 \mapsto \lambda_B \in \mathbb{Z}$

$\exists q_\lambda \subset \mathbb{R}^3$ s.th. $\lambda_B = \sum_{x \in B \cap q_\lambda} \lambda_x$

locally finite
(any $x \in \mathbb{R}^3$ has neigh. with finite cardnlty. intersection with S)

λ_x = charge at x

("usually" $\lambda_x = \pm 1$)

(up to multiplicities)

$\exists q_+, q_-$ locally finite s.th. $\lambda_B = \#(B \cap q_+) - \#(B \cap q_-)$ for $B \subset \mathbb{R}^3$ bounded

Positions of positive and negative charges

so, **charge eigenvalues indexable by** e-/e+ **configuration**!

$$\lambda_B((q_+, q_-)) := \#(B \cap q_+) - \#(B \cap q_-)$$

as heuristically anticipated!

Specify a (generalized) configuration = **Specify q_+, q_-**

(up to "unlikely" multiplicities)

P_{nat} is a PVM on

$$\mathcal{Q}_{loc.fin} = \Gamma_{loc.fin}(\mathbb{R}^3) \times \Gamma_{loc.fin}(\mathbb{R}^3)$$

$$\Gamma_{loc.fin}(\mathbb{R}^3) := \left\{ q \subset \mathbb{R}^3 \mid \#(q \cap B_r(x)) < \infty \; \forall r > 0 \; \forall x \in \mathbb{R}^3 \right\}$$

Note conjecture is a bit more conservative than Bell-type QFT config. space

e) On the Natural e-/e+ Position Operator

(iii) Example : $P_{nat}(\cdot)$ in Finite Volume

- If **Universe finite** 3-**volume** → **locally finite** = **finite**

  by Conjecture → $P_{nat}(\cdot)$ **concentrated** on **finite configurations**

- $[0, L)^3$ *periodic boundary* $= \mathbb{T}^3$ $\quad \mathcal{H}_1 = L^2(\mathbb{T}^3, \mathbb{C})$ $\quad \mathscr{H} = \mathscr{H}_+ \oplus \mathscr{H}_-$ etc.

  ↳ $\mathcal{Q}_{loc.fin} = \Gamma_{loc.fin}(\mathbb{T}^3) \times \Gamma_{loc.fin}(\mathbb{T}^3) = \Gamma(\mathbb{T}^3) \times \Gamma(\mathbb{T}^3) = \mathcal{Q}$ { *same config. space as Bell-type QFTs*

- **Ontology proposal:**

  Variable but **finite** number of **particles** of **2 species** { **e-**, **e+**

- Configuration **distribution** (*Born Rule*):

$$\mathbb{P}(B) = \langle \psi | P_{nat}(B) \psi \rangle \quad \forall B \subseteq \mathbb{T}^3 \times \mathbb{T}^3$$

Bohmian Trajectories via Bell-type QFT recipe

We **have all** the **ingerdients**

- A **Hilbert** Space $\mathscr{H}_{\mathbb{T}^3}$ $\quad H = -i\hbar c\vec{\alpha}\cdot\vec{\nabla} + \beta mc^2$
- A **Hamiltonian** on it $\mathbb{H}_0 := \Gamma(H)$ with **unitaries** $\mathbb{U}_t = e^{-\frac{i}{\hbar}\mathbb{H}t}$
- A **PVM** $P_{nat}(\cdot)$ on $\mathcal{Q} = \Gamma(\mathbb{T}^3)\times\Gamma(\mathbb{T}^3)$ **configuration** space

we get:

$$Q_t = (Q_{e^-}(t), Q_{e^+}(t))$$

stochastic process in $\mathcal{Q}$

$$Q_{e^-}(t) \equiv X_1(t), ..., X_n(t)$$

- **Deterministic Bohm-Dirac-like motion in fixed sector**

$$\mathcal{Q}^{(n,\bar{n})} := \{(q_{e^-}, q_{e^+}) \in \mathcal{Q} \mid \#q_{e^-} = n \;\&\; \#q_{e^+} = \bar{n}\}$$

- **Stochastic jumps**

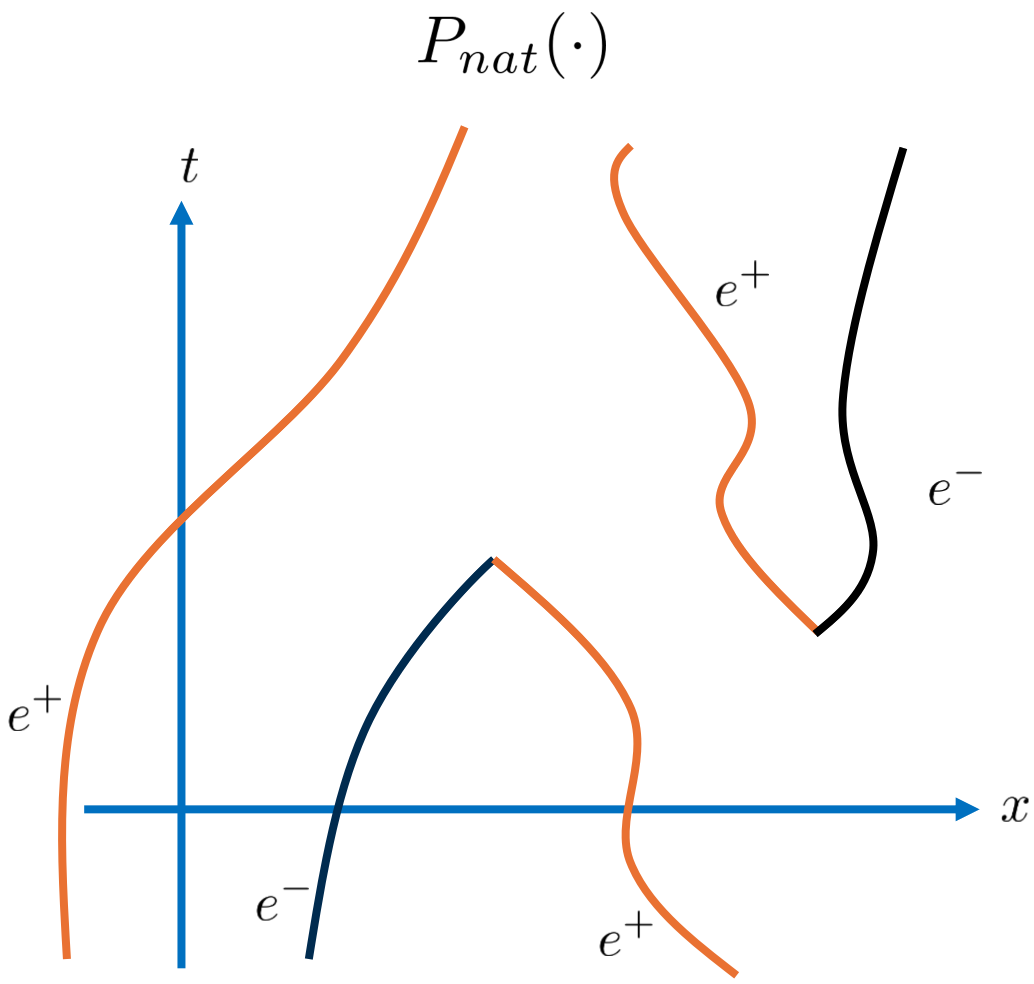
$$\sigma = 2Im^+ \frac{\langle\psi|P_{nat}(q_{e^-}\cup x,\ q_{e^+}\cup x)\mathbb{H}P_{nat}(q_{e^-}, q_{e^+})|\psi\rangle}{\langle\psi|P_{nat}(q_{e^-}, q_{e^+})\psi\rangle}$$

only **allowed transitions** $(q_{e^-}, q_{e^+}) \longleftrightarrow (q_{e^-}\cup x, q_{e^+}\cup x)$

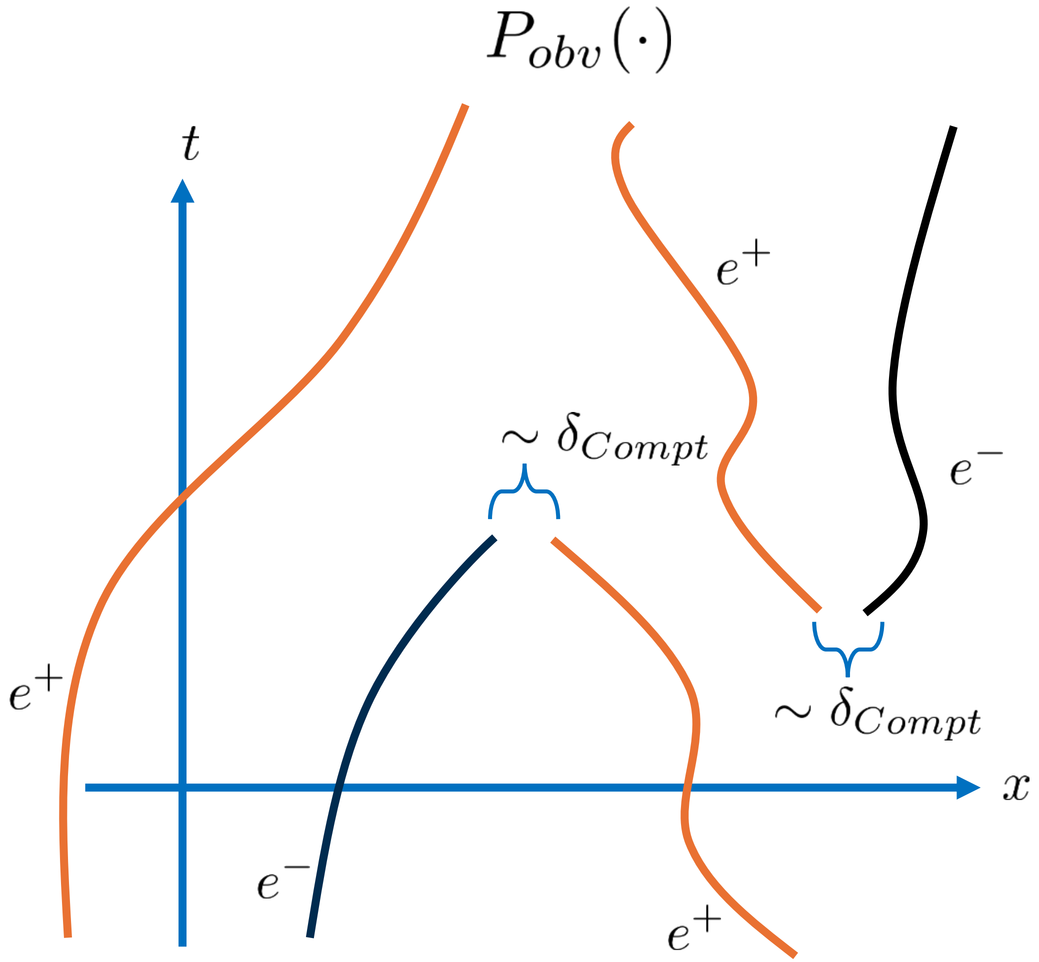
- which is **equivariant!**

$$if\ Q_0 \in \mathcal{Q} \sim \langle\psi_0|P_{nat}(dq)\psi_0\rangle \longrightarrow Q_t \in \mathcal{Q} \sim \langle\psi_t|P_{nat}(dq)\psi_t\rangle \quad \forall t$$

Bohmian Trajectories via Bell-type QFT recipe



e-/e+ create/destr. in situ



e-/e+ create/destr. at dist $\sim width(P_+\delta^3(x)) \sim \delta_{Compton}$

- So far so good. But, unlike $P_{obv}(\cdot)$, **vacuum subspace of** $P_{nat}(\cdot)$: $range\ P_{nat}(\emptyset)$ is $+\infty$ **dimensional!** (Instead of 1)…. **Why?** ⟶ **related to the Dirac sea picture**! *Lets **dive** till the bottom…*

(iv) $P_{nat}(\cdot)$ from Discretized Space

$$\mathcal{L} := \text{“}N^3 \text{ site square lattice”} = ([0,L) \cap \tfrac{L}{N}\mathbb{Z})^3$$

- To **rigorously** talk about the **PVM of each** $\hat{N}(\vec{x}, s) := \Psi_s^\dagger(\vec{x})\Psi_s(\vec{x})$ **we need to discretize space!** **Then take continuum limit and (hope) intuitions will still hold…**

$$\mathcal{H}_1 = L^2(\mathcal{L}, \mathbb{C}^4)$$

General state

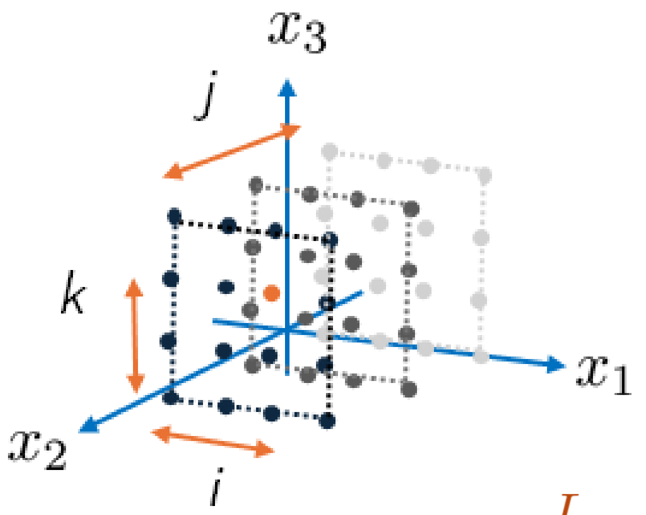
$$\psi(x) = \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \\ \psi_3(x) \\ \psi_4(x) \end{pmatrix}$$

$$x = (i,j,k)\frac{L}{N}$$

An ONB: 4 possible position deltas per site $x_0 \in \mathcal{L}$ ↓ $4N^3$ **dims**

$$\delta_{x_0,1} = \begin{pmatrix} \delta_{x_0} \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \delta_{x_0,2} = \begin{pmatrix} 0 \\ \delta_{x_0} \\ 0 \\ 0 \end{pmatrix}$$

$$\delta_{x_0,3} = \begin{pmatrix} 0 \\ 0 \\ \delta_{x_0} \\ 0 \end{pmatrix} \quad \delta_{x_0,4} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \delta_{x_0} \end{pmatrix}$$



$$x_0 = (i,j,k)\frac{L}{N}$$

Deltas now are “legal”! $\delta_{x_0}(x) := \begin{cases} 1 & if\ x = x_0 \\ 0 & else \end{cases}$

$$\mathcal{H}_1 = L^2(\mathcal{L}, \mathbb{C}^4) \longrightarrow \mathscr{H} = \mathscr{F}_{anti}(\mathcal{H}_1) = \bigotimes_{x\in\mathcal{L}} \mathscr{H}_x$$

At each site a Fock space of $\mathbb{C}^4$

$$dim(\mathscr{H}) = 16^{N^3}$$

At each site "a $\mathbb{C}^4$"

"0 particle" sector, "1 particle" sector, "2 particle" sector, "3 particle" sector, "4 particle" sector

$$\mathscr{H}_x := \mathscr{F}_{anti}(\mathbb{C}^4) = \mathbb{C} \oplus \mathbb{C}^4 \oplus anti(\mathbb{C}^4 \otimes \mathbb{C}^4) \oplus anti(\mathbb{C}^4 \otimes \mathbb{C}^4 \otimes \mathbb{C}^4) \oplus anti(\mathbb{C}^4 \otimes \mathbb{C}^4 \otimes \mathbb{C}^4 \otimes \mathbb{C}^4)$$

Has dim 16 = 1 + 4 + 6 + 4 + 1

Why?

- Canonical basis

$$e_1, e_2, e_3, e_4 \in \mathbb{C}^4$$

- Basis of $\mathbb{C}^4 \otimes \mathbb{C}^4 \otimes \mathbb{C}^4$

$$e_i \otimes e_j \otimes e_k$$

$$i, j, k \in \{1, 2, 3, 4\}$$

4^3 element

Recall...

$$Anti(e_i \otimes e_j \otimes e_k) = e_i \wedge e_j \wedge e_k$$

rules:

$$e_j \wedge e_j \wedge e_k = 0$$

Repetition → 0

$$e_i \wedge e_k \wedge e_k \propto e_k \wedge e_j \wedge e_i$$

Reordering **not LI**

Basis of $anti(\mathbb{C}^4 \otimes \mathbb{C}^4 \otimes \mathbb{C}^4)$**:**

$$e_i \wedge e_j \wedge e_k$$

With no repetition and order doesn't matter

$$\frac{4 \cdot 3 \cdot 2}{3!} = 4$$

elements!

Note that in $\mathscr{H} = \mathscr{F}_{anti}(L^2(\mathcal{L}, \mathbb{C}^4))$ there is no e+/e- splitting yet!

- $\Psi(\delta_{\vec{x}_0,s}), \Psi^\dagger(\delta_{\vec{x}_0,s}) =: \Psi_s(\vec{x}_0), \Psi_s^\dagger(\vec{x}_0)$ are now **"legal" creation/annihilation** operators

Joint spectral PVM of all $\hat{N}(\vec{x}, s) := \Psi_s^\dagger(\vec{x})\Psi_s(\vec{x})$ is P_{std}, the PVM on $\mathcal{Q}$ acting on $\mathscr{H}$ as $P_{sea}(B) = \mathcal{M}(\chi_B)$

- $H = -i\vec{\alpha} \cdot \nabla + \beta m$ on $\mathcal{H}_1 = L^2(\mathcal{L}, \mathbb{C}^4)$ with $\nabla_j \psi(\vec{x}) := \dfrac{\psi(\vec{x} + \frac{L}{N}\vec{u}_j) - \psi(\vec{x})}{L/N}$ → $\mathbb{H} = \Gamma(H)$

  Again $\mathcal{H}_1 = \mathcal{H}_+ \oplus \mathcal{H}_-$ but now $dim(\mathcal{H}_\pm) = 2N^3$ **finite!** → **(again) Dirac Sea rigorously takeable!**

  $|\Omega_{sea}\rangle = g_1 \wedge \cdots \wedge g_{2N^3}$ s.th. $\{g_j\}_{j=1}^{N^3} \subset \mathcal{H}_-$ any ONB

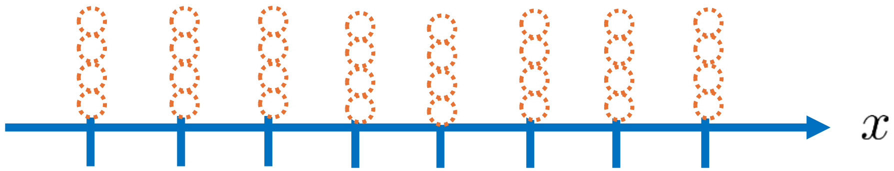
- **Configuration** space $\mathcal{Q} = \{0,1,2,3,4\}^{\mathcal{L}}$ → $q \in \mathcal{Q}$ gives **occupation number per site** (Bell's QFT!)

- **Max occupation** number **per site 4** (by fermionic antisym. + dim 4 of $\mathbb{C}^4$) – *(recall Pauli exclus. in diagonals!)*

In Dirac Sea approach: occupation in site *x* = **"number of electrons in site *x*"**

In Tumulka's approach: occupation in site x = **"pre-particle"** = **encodes number of e- and e+**

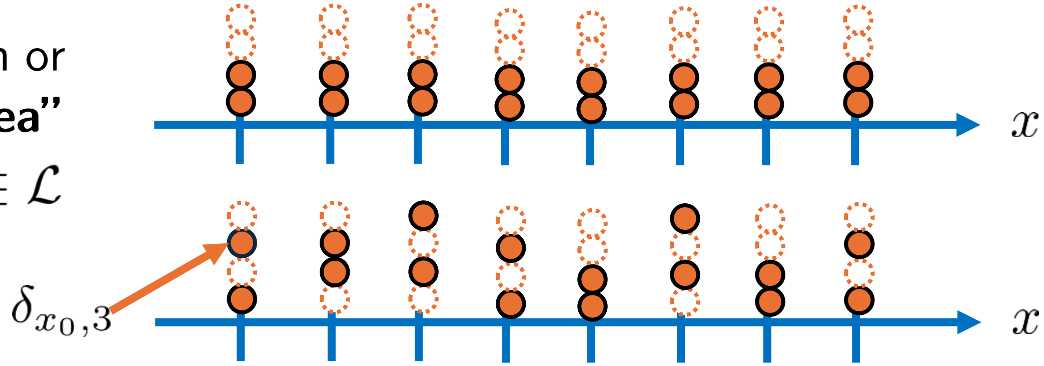
- **Bottom** configuration
  $q_B(x) := 0 \ \forall x \in \mathcal{L}$



the Bottom state $|B\rangle$ → In all sectors 0 "pre"-particles → **True of Dirac Sea picture!**

$$P_{std}(\{q_B\}) = |B\rangle\langle B|$$

- Level configuration or **"position-Dirac sea"**
  $q_L(x) := 2 \ \forall x \in \mathcal{L}$



Level states/"position-Dirac seas"

$$|L_j\rangle \in range\ P_{std}(\{q_L\}) =$$

$$= \bigotimes_{x\in\mathcal{L}} anti(\mathbb{C}^4 \otimes \mathbb{C}^4) \qquad 6^{N^3} \text{dims}$$

range $P_{std}(\{q_L\})$ **=** eigenfcs of eigv 1 of $P_{std}(\{q_L\})$ **=** superpos. of ψ that are **0 in all sectors except** sector $\mathbf{2N^3}$, where they are **superpositions of 2-deltas-per-site-states**

- **Sea state** $|\Omega_{sea}\rangle = g_1 \wedge \cdots \wedge g_{2N^3}$ → **not occupation number eigenstate!** → $|\Omega_{sea}\rangle \notin Level\ state!$

or **"energy-Dirac sea"**

→ eigenstate of? $\Psi^\dagger(g_k)\Psi(g_k)$ ✔ $\Psi_s^\dagger(\vec{x})\Psi_s(\vec{x})$ ✘

- $P_{std}(q)\psi$ → Computes $n := \sum_{\vec{x}\in\mathcal{L}} q(\vec{x})$ & for all sectors $m \neq n$ $(P(q)\psi)^{(m)} \equiv 0$
  in sector n leaves "superps of q(x)-deltas-per-site-states"

- **Let's re-interpret the occupation per site: Net Balance wrt level**

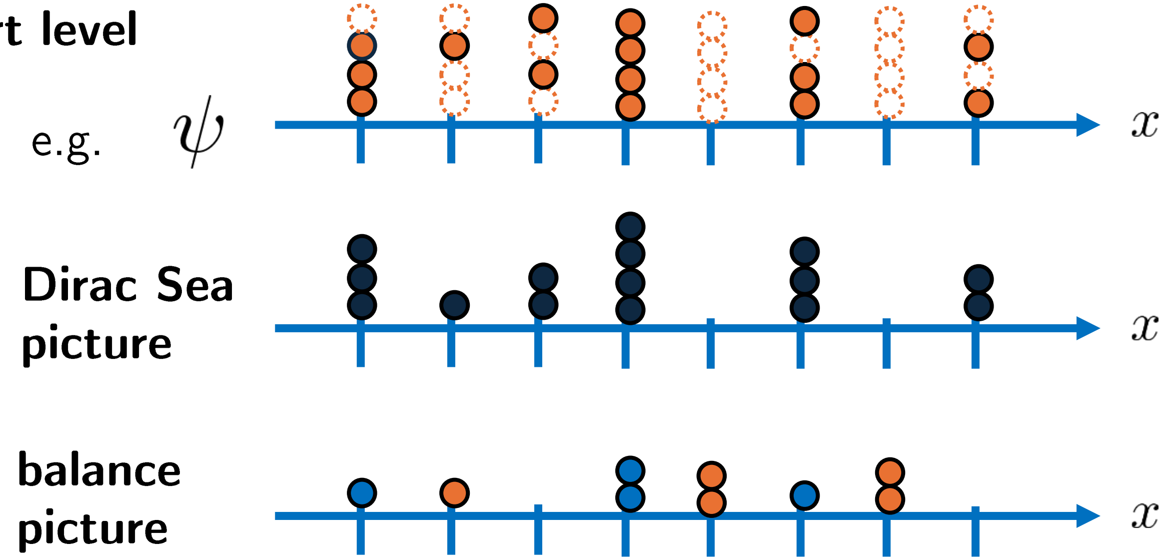
4 Dirac-particles in x (**level+2**) ⟷ **0 e+ — 2 e-**

3 Dirac-particles in x (**level+1**) ⟷ **0 e+ — 1 e-**

2 Dirac-particles in x (**level**) ⟷ **0 e+ — 0 e-**

1 Dirac-particles in x (**level-1**) ⟷ **1 e+ — 0 e-**

0 Dirac-particles in x (**level-2**) ⟷ **2 e+ — 0 e-**



- **This defines main candidate to** $P_{nat}(\cdot)$**'s discretization:** $\boldsymbol{P_{nat,N}}(\cdot)$, by **re-def of spectral sets of** $\boldsymbol{P_{sea}}(\cdot)$ **wrt level**

$$P_{nat,N}(q) := P_{std}(q_L - q)$$

⟶ **still same PVM so still diagonalizes all** $\hat{N}(x,s) := \Psi_s^\dagger(x)\Psi_s(x)$

$$q \in \{-2,-1,0,1,2\}^{\mathcal{L}}$$

Possible charges per site if level-occupation is 0 charge

As we saw, in **continuum** case, $P_{nat}(\cdot)$ arises from Dirac **sea pict. by latter's number op**. $\hat{N} = \int_{\mathbb{R}^3} \Psi^\dagger(x)\Psi(x)d^3x$ via "**subtracting**" **sea e-s**, i.e., counting wrt "sea level". Hence **discrete** vers. must be the **PVM of** Dirac sea pict. **numb**. op $\hat{N}$ **"normalized" to "sea level"** (–in particular ***position*** *sea-level* bec. we talk bt. posit. space NOVM $\hat{N}(D)$). This directly gives **"normal ordering" as limit**, bec. through the **"net balance"** we **prevent contribut of** ∞ **(position)-sea e-s!**

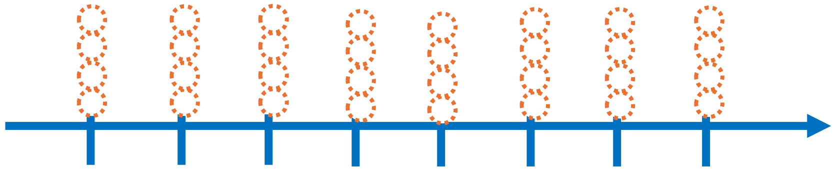
in Sea picture

in Balance wrt level picture

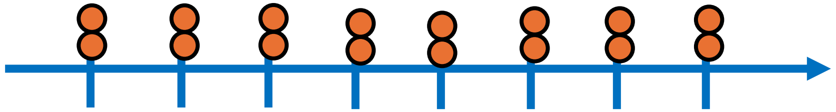
- **Bottom** configuration

  $q_B(x) := 0 \ \forall x \in \mathcal{L}$

Dirac Sea vacuum



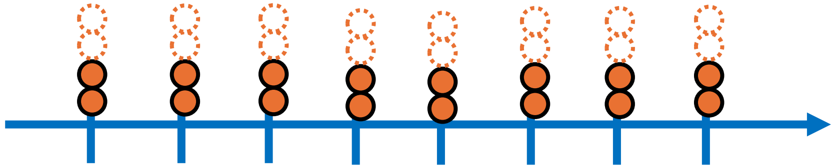
is max positive-charge/"hole" configrt.



- **Level** configuration

  $q_L(x) := 2 \ \forall x \in \mathcal{L}$

Position-Dirac sea



Tumulka's vacuum

is vacuum "charge" configuration

So Level/"position-Dirac-sea" is the "Natural" Vacuum!

- $range\ P_{nat,N}(\{\emptyset\}) = range\ P_{sea}(q_L)$

  6^{N^3}dims

$\neq$ **the Dirac Sea pict. vacuum** $rangeP_{std}(\{\emptyset\}) = |B\rangle\langle B|$

$\neq$ **the (energy-)Dirac-sea or e-/e+ "obvious" vacuum** $|\Omega\rangle\langle\Omega|$

Apparent Continuum limit

- L *fixed*
- $N \to +\infty$
- consider **states that differ from** $|\Omega\rangle$ **in finitely many** sea **particles**

$\mathcal{L} \longrightarrow \mathbb{T}^3$ ⟶ $P_{nat,N} \longrightarrow P_{nat}$

by Tumlka's Conjecture

$N \to +\infty$

$$\langle\Omega|P_{nat,N}(\{\emptyset\})|\Omega\rangle \longrightarrow \langle\Omega|P_{nat}(\{\emptyset\})|\Omega\rangle \in (0,1)$$

and

$L \to +\infty$

$$\langle\Omega|P_{nat}(\{\emptyset\})|\Omega\rangle \longrightarrow 0$$

Probab. that **no charge in** $|\Omega\rangle$ according to P_{nat}

$|\Omega\rangle$ is **not orthog. to** $range P_{nat}(\{\emptyset\})$ **nor** is **inside** it

Even if locally finite, **prob. no charge** in $|\Omega\rangle$ accdg. to P_{nat}, **vanishes**

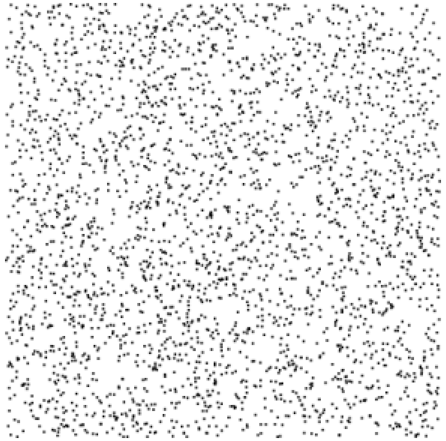
(v) A typical photo of vacuum

vacuum of $P_{nat}(\cdot)$:

$\psi \in range\ P_{nat}(\{\emptyset\})$

vacuum of $P_{obv}(\cdot)$:

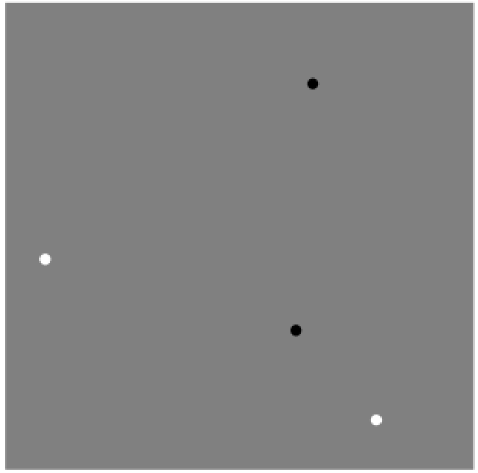
$|\Omega\rangle$ (energy-)sea state



a priori imaginable typical configuration

if conjecture true →

$|\Omega\rangle$ (energy-)sea state



only "finitely away" from level config

"position-sea"/level state

Max uniform distribut. a typical configuration

According to **Dirac Sea picture**

$N \to \infty$ already taken

But if reality is this:

Maybe more reasonable to consider only "white and black dots" → **Tumulka's picture**

(v) A typical photo of vacuum

vacuum of $P_{obv}(\cdot)$:

$|\Omega\rangle$ (energy-)sea state

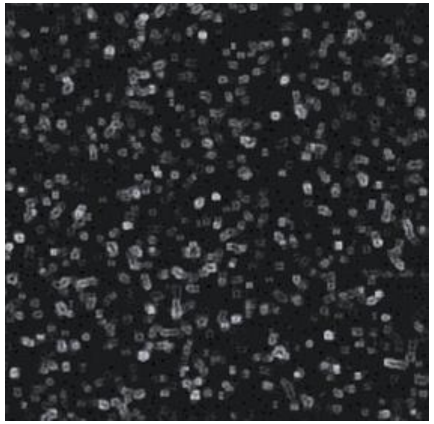
$|\Omega\rangle$ (energy-)sea state

vacuum of $P_{nat}(\cdot)$:

$\psi \in range\ P_{nat}(\{\emptyset\})$

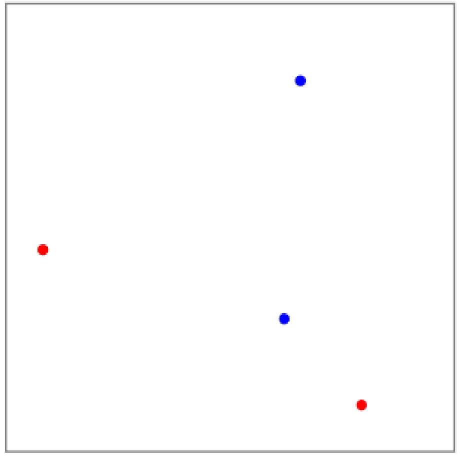
"position-sea"/level state

According to **Tumulka's picture**

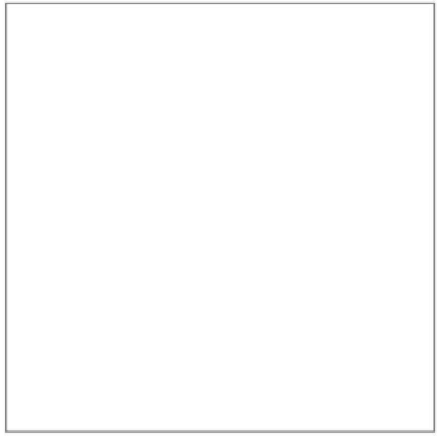


a priori imaginable typical configuration

if conjecture true →



only "finitely away" from level config



Max uniform distribut. a typical configuration

$N \to \infty$ already taken

Then his proposal is the "continuum limit of the Dirac sea except for a reinterpretation"

(vi) The Locality Properties of $P_{nat}(\cdot)$

A. Propagation Locality (PL)

"same" argument

Reason 1.1:

In lattice approx. PL holds for Dirac sea picture, because **we are considering free Dirac particles** → $P_{nat}(\cdot)$ **ontology** is just a **re-interpretation of "excess and lack of" Dirac particles/site** → **PL still holds**

Reason 1.2:

In Dimock-Deckert-like **rigorous Dirac Sea picture** $P_{nat}(\cdot)$ **extendible to** $\mathscr{H}_\infty$ by extend config. space **allowing** ∞ **particles** → again, **ontology** merely **reinterpretation of (now infinite) free Dirac particles** in **continuum** → **PL holds**

Reason 2:

- **Bohmian trajs move no faster than light**

  +

- Bohmian trajs **are** $\langle\psi_t|P_{nat}(\cdot)|\psi_t\rangle$**-equivariant** and $P_{nat}(\cdot)$ **is PVM**

→ **complex magnitude of** ψ_t **in** $P_{nat}(\cdot)$**–"position representation" propagates v<c** → **PL holds**

(vi) The Locality Properties of $P_{nat}(\cdot)$

B. Interaction Locality (IL)

"same" argument

Reason 1.1:

In lattice approx. IL holds for Dirac sea picture, because **we are considering free Dirac particles** → $P_{nat}(\cdot)$ **ontology** is just a **re-interpretation of "excess and lack of" Dirac particles/site** → **IL still holds**

Reason 1.2:

In Dimock-Deckert-like **rigorous Dirac Sea picture** $P_{nat}(\cdot)$ **extendible to** $\mathscr{H}_\infty$ by extend config. space **allowing** ∞ **particles** → again, **ontology** merely **reinterpretation of (now infinite) free Dirac particles** in **continuum** → **IL holds**

Reason 2: Spacelike separated "Local Algebras" commute

- $\mathscr{A}(A)$ = The **(spatial)** ***algebra of local observables*** **(in space** $\mathbb{T}^3$**) on the open subset** $A \subseteq \mathbb{T}^3$

  ⊃ **all spectral projections of** $Q(A')$ **for all** $A' \subset A$

- $\mathcal{Q}_A := \Gamma(A) \times \Gamma(A) \longrightarrow \mathcal{Q} = \mathcal{Q}_A \times \mathcal{Q}_{A^c}$ because
  $$\forall A \cap B = \emptyset \longrightarrow \Gamma(A \cup B) = \Gamma(A) \times \Gamma(B)$$
  if identify $q \subset A \cup B = (q \cap A, q \cap B)$

(vi) The Locality Properties of $P_{nat}(\cdot)$

B. Interaction Locality (IL)

Reason 2: Spacelike separated "Local Algebras" commute

$\mathcal{B}_A$ = subsets of $\mathcal{Q}$ of the form $B \times \mathcal{Q}_{A^c}$, $B \subseteq \mathcal{Q}_A$ = (σ-algebra of) events only depending on A not on A^c

Then,

$$P_{nat}(\tilde{B}) \in \mathscr{A}(A) \quad \forall \tilde{B} = B \times \mathcal{Q}_{A^c} \in \mathcal{B}_A$$

i.e., projectors to ↑ are local in A operators

- By def. if spacetime regs. $\mathcal{O}_1, \mathcal{O}_2$ **spacelike separated** ⟶ local **algebras commute:**

$$[\mathscr{A}(\mathcal{O}_1), \mathscr{A}(\mathcal{O}_1)] = 0$$

- $\forall A \subset \mathbb{T}^3, t \in \mathbb{R},\ s.th.\ \{t\} \times A \subset \mathcal{O}_1,$

$$e^{iHt} P_{nat}(\tilde{B}) e^{-iHt} \in \mathscr{A}(\mathcal{O}_1) \qquad \forall \tilde{B} \in \mathcal{B}_A$$

↳ all **local ops in** $\mathcal{O}_2$ **commute with** $P_{nat}(\tilde{B})$ at time t

↳ **No superliminal signalling betw** $\mathcal{O}_1, \mathcal{O}_2$

IV – OUTLOOK

The two reasons leading us to the N(t) description are still “unhappy”…

a) Ideal N(t)-Particle Treatment: Multi-time wavefunctions

Recall, **consistency conditions** make interaction **potentials unimplementable** in multi-time eqts.: instead, interactions could only be introduced by (BCs or) **particle creation/annihilation terms**....*well, here we are!*

- In **seen QFT formalization**, **only 1 time** variable **per sector** $\psi^{(n)}(t, \vec{x}_1, ..., \vec{x}_n)$

  ↳ **cannot** be **fully Lorentz covariant**, since a Lorentz transf. of each $(t, \vec{x}_j)$ will yield a different time t'_j

- Also, proper **connection** to **physicist's QFT was** *(for pairwise distinct configs.)*

$$\psi^{(n)}(x_1, ..., x_n, t) \quad = \langle \Omega | \hat{\Psi}(x_n, t) \cdots \hat{\Psi}(x_1, t) | \psi \rangle$$

*if **1 species.***

Schrödinger pict. **Heisenberg pict.**

- or say, if there are ***3 species***, with creation operators $a^\dagger(\cdot), b^\dagger(\cdot), c^\dagger(\cdot)$ e.g., **e-, e+** and **scalar meson**

$$\psi^{(n,\bar{n},m)}(x_1, ..., x_n, y_1, ..., y_{\bar{n}}, z_1, ..., z_m, t) \quad = \langle \Omega | \hat{c}(z_m, t) \cdots \hat{c}(z_1, t) \hat{b}(y_n, t) \cdots \hat{b}(y_1, t) \hat{a}(x_n, t) \cdots \hat{a}(x_1, t) | \psi \rangle$$

where in the **lhs** the **(anti)symmetry** properties come **from the definition of the Hilbert space** (tensor product of (anti)symmetrized Fock spaces) & in the **rhs from postulated (anti)commutation relations** of the create/annih ops.

a) Ideal N(t)-Particle Treatment: Multi-time wavefunctions

- **But what** about the **physicist's expression** $\langle \Omega | \hat{\Psi}(x_n, t_n) \cdots \hat{\Psi}(x_1, t_1) | \psi \rangle$ **?**

  That's **precisely** to consider **a multi-time wavefunction per sector!**

$$\psi^{(n)}(x_1, t_1, ..., x_n, t_n) = \langle \Omega | \hat{\Psi}(x_n, t_n) \cdots \hat{\Psi}(x_1, t_1) | \psi \rangle$$

(recover the single-time expressions by setting all times to be the same)

- In multi-time QFT, **naively** a **wavefunction** would be on $\Gamma(\mathbb{R}^4) = \bigcup_{N=0}^{\infty} (\mathbb{R}^4)^N$ (i.e., a **variable** number of **space-time variables**) but can be proven that **like** for **usual multi-time wfs.**, **only on space-like** configs.

$$\mathcal{S} := \bigcup_{N=0}^{\infty} \{(x_1, ..., x_N) \in (\mathbb{R}^4)^N \mid x_i, x_j \text{ spacelike sep}\}$$

  are general multi-time equations **consistent** (with particle creation and annihilation).

- Time evolution would be as many multi-time equations as time variables per sector

$$i\hbar \frac{\partial}{\partial t_j} \psi^{(n)}(\vec{x}_1, t_1, ..., \vec{x}_n, t_n) = (\mathbb{H}_j \psi)^{(n)}(\vec{x}_1, t_1, ..., \vec{x}_n, t_n)$$

a) Ideal N(t)-Particle Treatment: Multi-time wavefunctions

- For each **space-like hypersurface** $\Sigma \subset \mathbb{R}^4$ **, multi-time** ψ yields ϕ_Σ in Fock space $\mathscr{H}_\Sigma$ and **config. space** $\Gamma(\Sigma) \subset \mathcal{S}$

$$(\phi_\Sigma)^{(n)}(x_1, ..., x_n) := \psi^{(n)}(x_1, ..., x_n) \ \forall x_j \in \Sigma$$

which is **compatible** with **Tomonaga-Schwinger's surface wavefunction** QFT (another fomralization).

- Main **physical relevance** of doing **multi-time description**?
  - experiments where **detectors placed in space-like surfaces** *(hypersurface Born rule)*
  - **equations** can be given in a more **frame independent** way
  - a **generalization of QFT to curved spacetime posible!**

S. Petrat and R. Tumulka, *"Multi-time wave functions for quantum field theory"*, Annals of Physics **345**, 17 (2014).

To be Continued...
(Perhaps in following seminar)

b) Inconsistencies of e-/e+ Dirac QFT

(i) External EM fields re-signify e-/e+

- 1 **free** Dirac particle → 1 Dirac particle in an **external EM field**

$$H = c\vec{\alpha}\cdot(-i\hbar\vec{\nabla}) + \beta mc^2$$

$$H(t) = c\vec{\alpha}\cdot\left(-i\hbar\vec{\nabla} - \frac{e}{c}\vec{A}(t,\vec{x})\right) + \beta mc^2 + e\phi_{el}(ct,\vec{x})$$

- (Under reasonable conditions [1, Thm. 4.9]) $\exists U(t,s) = \text{“}e^{-i\int_s^t H(\tau)d\tau}\text{”}$ **unitary** propagator in $\mathcal{H}_1 = L^2(\mathbb{R}^3,\mathbb{C}^4)$
- **BUT: E>0, E<0 spectral subspace splitting of** $\mathcal{H}_1 = L^2(\mathbb{R}^3,\mathbb{C}^4)$ **depends on time!**

How to face it?

Idea 1: Fock spaces obeying *H(t)* positive/negative energy splitting

- Given **positive** and **negative spectral projectors** $P_+(t), P_-(t)$ of $H(t)$

$$\mathcal{H}_+(t) := P_+(t)\mathcal{H}_1$$
$$\mathcal{H}_-(t) := P_-(t)\mathcal{H}_1$$

$\downarrow \forall t$

$$\mathcal{H}_1 = \mathcal{H}_+(t)\oplus\mathcal{H}_-(t)$$

$$\mathscr{H}_t := \mathscr{F}_{anti}(\mathcal{H}_+(t))\otimes\mathscr{F}_{anti}(C\mathcal{H}_-(t))$$

at each *t* a different

- **vacuum** $|\Omega_t\rangle \in \mathscr{H}_t$
- **field ops.** $a_t(\cdot), b_t(\cdot)$ **acting on** $\mathscr{H}_t$

$$a_t(P_+(t)f)|\Omega_t\rangle = 0 = b_t(P_-(t)f)|\Omega_t\rangle$$

- $\psi(t)$ is in **a different Fock space per *t***
  - → **Unclear** how to talk about time evolving state: **continuity**, **differentiability** etc.
- **e-/e+ interpret. depends on time?!**
- **Worse**: since **we can choose** the **external field**, we **could choose what a particle and antiparticle is!**

Idea 2: Force fixed splitting (if you can...)

- If **particles** are **asymptotically free**, it may be **reasonable** to **commit to their splitting**

$$\mathcal{H}_1 = \mathcal{H}_+^{(free)} \oplus \mathcal{H}_-^{(free)}$$

and their **Fock space**

$$\mathscr{H}_{free} := \mathscr{F}_{anti}(\mathcal{H}_+) \oplus \mathscr{F}_{anti}(\mathcal{H}_-)$$

whose **native field ops** $a_0(\cdot), b_0(\cdot)$

satisfy anticomt. relts.

$$\begin{cases} \{a_0(h_1), a_0^\dagger(h_2)\} = \langle h_1, h_2\rangle Id \\ h_j \in \mathcal{H}_+^{(free)} \text{ etc.} \end{cases}$$

- For **positive/negative spectral projs.** $P_+(t), P_-(t)$ of $H(t)$

$$\left| \begin{array}{l} \mathcal{H}_+(t) := P_+(t)\mathcal{H}_1 \\ \mathcal{H}_-(t) := P_-(t)\mathcal{H}_1 \end{array} \right. \longrightarrow \mathcal{H}_1 = \mathcal{H}_+(t) \oplus \mathcal{H}_-(t)$$

$$\mathscr{H}_t := \mathscr{F}_{anti}(\mathcal{H}_+(t)) \otimes \mathscr{F}_{anti}(\mathcal{H}_-(t))$$

with its **interacting particle anticomt. relts**

- **To represent this "interacting field algebra" in** $\mathscr{H}_{free}$ we **look for ops**. $a_t(\cdot), b_t(\cdot)$ **acting on** $\mathscr{H}_{free}$ **but**

$$\begin{cases} \{a_t(h_1), a_t^\dagger(h_2)\} = \langle h_1, h_2\rangle Id \\ \forall h_j \in \mathcal{H}_+(t) \quad \text{etc.} \end{cases}$$

BUT this is **only possible if 1-particle time evol.** $U(t,s) =$ "$e^{-i\int_s^t H(\tau)d\tau}$" is **liftable** to a unitary $\mathbb{U}(t,s)$ on $\mathscr{H}_{free}$

Idea 2: Force fixed splitting (if you can…)

Shale-Stinespring [1, Thm. 10.7]

1-particle time evolution $U(t,s)$ is **liftable to a unitary** $\mathbb{U}(t,s)$ **on** $\mathscr{H}_{free}$ ***iff*** ⟷ $P_+^{(free)}U(t,s)P_-^{(free)}$, $P_-^{(free)}U(t,s)P_+^{(free)}$ **are Hilbert-Schmidt ops.**

A Hilbert-Schmidt iff $\sum_{j\in\mathbb{N}\ ONB}\langle f_j|A^2 f_j\rangle =: tr(A^2) < +\infty$

Intuition:

Def. $U_{\pm,\pm}(t,s) := P_\pm^{(free)}U(t,s)P_\pm^{(free)}$ ⟶ $U(t,s) = \begin{pmatrix} U_{++}(t,s) & U_{+-}(t,s) \\ U_{-+}(t,s) & U_{--}(t,s) \end{pmatrix}$

- **Diagonal terms** are **the part of evolution** that **leaves e-/e+** subspaces **invariant**
- **Non-diagonal terms** describe **flow of probability from e- to e+ subsp.: pair creation and annihilation**

$$||U_{+-}(t,s)||^2_{Hilbert-Schmidt} \underbrace{\simeq}_{\text{leading order}} \sum_{n\in\mathbb{N}} ||U_{+-}(t,s)g_n||^2_{\mathcal{H}_1} = \sum_{n\in\mathbb{N}} ||P_+U(t,s)g_n||^2$$

$\{g_n\}_{n\in\mathbb{N}} \subset \mathcal{H}_-^{(free)}$ *ONB*; *prob. n-th sea e- gets E>0 part*

= **sum** of **probs. to generate** a **pair from the Sea** *(i.e., negative energy e- converts to positive energy)*

So, **Shale-Stinespring cond. ensures pair-creation probs. well-defined!**

Idea 2: Force fixed splitting (if you can...)

- **If purely electric time (in)depdt. field** → $P_+^{(free)} U(t,s) P_-^{(free)}$ $P_-^{(free)} U(t,s) P_+^{(free)}$ are **Hilbert-Schmidt** ops. ✓ → **Fixed free particle Fock sp.** $\mathscr{H}_{free}$ **possible!**

Consequence 1: Vacuum Polarization

- **Interacting** creation/annihilation op's **vacuum** ⟷ (*corresponds to* / *when represented in* $\mathscr{H}_{free}$) a **charged** $\mathscr{H}_{free}$ **state**
- ***Heuristically*: if lowest electron** bound **state reaches E<0 subspace** (which happens for *strong enough* external *electric* fields) → **transitions from e- to e+ states**
- ***Physical effect:*** the **vacuum in** the **neighbhood of strong field** becomes **charged, causing a partial screening of the field**

Consequence 2: e-/e+ creation if strong enough **electric field!**

Idea 2: Force fixed splitting (if you can...)

- **However...** (by *Ruijsenaars* [14])

$P_{+}^{(free)} U(t,s) P_{-}^{(free)}$, $P_{-}^{(free)} U(t,s) P_{+}^{(free)}$ **are Hilbert-Schmidt ops.** ⟷ ***iff*** $\vec{A} \equiv 0$

Worse: **gauge transformations** adding **non-zero spatial components** to $\vec{A}$ **cannot be lifted**!

So, **if (even static) magnetic** fields **present**, **NO lift of** $U(t,s)$ **exists** to $\mathscr{H}_{free}$ **!!!**

- **Intuition:**

$$||U_{+-}(t,s)||^2_{\substack{Hilbert-\\Schmidt}} \underbrace{\simeq}_{\substack{\text{leading}\\\text{order}}} \sum_{n\in\mathbb{N}} ||P_+ U(t,s) g_n||^2$$ **explode!** ⟷ **Unbounded e-/e+ pair creation (promotion of Dirac sea e-)!**

Impossible to deal with EM interaction in $\mathscr{H}_{free}$

- **Why Scattering Matrices still work...**

When **field off, e-s turn back** to **free** Dirac Sea → **ingoing & outgoing** particles are **in same Fock space**

↳ **for 1-particle scattering matrix** with **asymptotic. free states** S^U, $P_{\pm}^{(free)} S^U P_{\pm}^{(free)}$ is **Hilbert-Schmidt**

↳ **scat. mat.** S^U is **liftable** to $\mathscr{H}_{free}$ **even with** $\vec{A}$

So particle interpretation doomed to merely asymptotic meaning?

Idea 3: Accept e+/e- picture has to change – Let a unitary choose the evolution of the splitting

At *t=0*

Given **any splitting** (aka ***polarization***)

$$\mathcal{H}_1 = \mathcal{H}_+ \oplus \mathcal{H}_-$$

of **projectors** P_+, P_-

- **build** $\mathscr{H}_{t=0} := \mathscr{F}(\mathcal{H}_+) \otimes \mathscr{F}(C\mathcal{H}_-)$
- **vacuum** $|\Omega\rangle$
- **field ops.** $a(P_+ f)\Omega = b(P_- f) = 0$

For example the **splitting of the free** Dirac **particles**! (Because maybe all particles are free at some *t=0*)

At time *t*

Evolve splitting by some 1-particle unitary

$$\mathcal{H}_1 = U(t,0)\mathcal{H}_+ \oplus U(t,0)\mathcal{H}_-$$

of **projectors** $P_\pm(t) = U(t,0)P_\pm U(0,t)$

- $\exists$ vacuum $|\Omega_t\rangle \in \mathscr{H}_{t=0}$ s.th.

t=0 create/annh. ops!

$$a\Big(P_+(t)U(t,0)f\Big)|\Omega_t\rangle = 0 = b\Big(P_-(t)U(t,0)f\Big)|\Omega_t\rangle$$

- **so, build**

$$\mathscr{H}_t := \left(\textbf{closure of linear combts. of} \prod_{j,k} a^\dagger(U(t,0)h_j)b^\dagger(U(t,0)g_k)|\Omega_t\rangle \right)$$

$$\{h_j\}_{j\in\mathbb{N}} \subset \mathcal{H}_+, \ \{g_k\}_{k\in\mathbb{N}} \subset \mathcal{H}_- \ \ \textit{ONB}$$

Then field operators would be "the same" at all *t*! *Great!*

Idea 3: Accept e+/e- picture has to change – Let a unitary choose the evolution of the splitting

- **Escapable objection:**

"e-/e+ interpret. depends on time" → still **reasonable if** we think **Dirac Sea/vacuum is readjusted to account for** the **unbounded negative energy e- movements** (i.e., think of **e-/e+** as **effective** but not fundmntl.)

- **BUT:**

the **only non-arbitrary choice of initial splitting,** $\mathcal{H}_1 = \mathcal{H}_+^{(free)} \oplus \mathcal{H}_-^{(free)}$ **(assuming fields off at t=0),** does **not allow** to **describe pair creation by the field! –physically observed–** (using idea 3). Why?

↳ Given **any 1-particle** state **in** $\mathcal{H}_-$ **at** time **t=0**, it **will remain in** the **sea**, $U(t,s)\mathcal{H}_-$, **forever!**

Moreover, using obvious $U(t,0) =$ "$e^{-i\int_0^t H(\tau)d\tau}$" **choice of polarization** at each t **depends on the whole history of previous EM fields!**

Blended with idea 1 perhaps?

- Splitting **by** E>0 and E<0 **spectral subspaces** of $H(t)$ **(idea 1)** and we **fix** the **problem of time varying Fock** spaces by **linking them with the necessary unitaries** for this to happen**?**

↳ **pair creation may be possible...** **But** the **splitting** is **NOT Lorentz invariant!**

b) Inconsistencies of e-/e+ Dirac QFT

(ii) Lorentz Transformations re-signify e-/e+

- **Lorentz trasnfmed**. **Hamiltonian** $H(t)$ has a **different potential**!

  ↳ **Lorentz** transfs. **don't preserve splitting** $\mathcal{H}_1 = \mathcal{H}_+(t) \oplus \mathcal{H}_-(t)$ **when external field** present!

  ↳ Lorentz **transfmed**. **theory** has **different Fock space and e-/e+ interpretation!**

Related to the Unruh Effect *Diff. observers "not agreeing on presence of particles"*

- **Free** e-/e+ **splitting** $\mathcal{H}_1 = \mathcal{H}_+^{(free)} \oplus \mathcal{H}_-^{(free)}$ **is preserved** by **Poincaré** trf. and are **liftable to** $\mathscr{H}_{free}$

  ↳ **Free** particle **Fock** space $\mathscr{H}_{free}$ **is Poincaré invariant**

  But as we saw, **interacting time evolution not implementable** in $\mathscr{H}_{free}$ if **magnetic field....**

- **Conclusions**:
  - A **fixed particle/antiparticle** picture **cannot** be **based** on **spectral considerations**
  - **Consistent particle interpr. in e-/e+** picture **doomed to merely asymptotic meaning!**

But what if...we **seriously take the Dirac Sea movement metaphor?**

b) Inconsistencies of e-/e+ Dirac QFT

(iii) A solution: make Dirac Sea picture Primitive

- **Time varying Fock space** and **time varying vacuum** are a **"hell gale" in e-/e+ picture** (in "sea surface"),
  but in Dirac sea perspective, **it is merely an update of what the Sea's state is...**

(unbounded pair creation means "in the Dirac Sea's depth (Hilbert's Hotel's guts), things get revolved non-neglectably")

- **Def**. a closed **subspace** $\mathcal{H}_{sea} \subset \mathcal{H}_1$ with **infinite dim** and **co-dim** as ***polarization*** (**represents a filled Dirac Sea**)

  ↪ yields a splitting $\mathcal{H}_1 = \mathcal{H}_{sea} \oplus \mathcal{H}_{sky}$

- **Pb. of Idea 3 (only promising idea)** was **choice of polarizat.** is **arbitrary**, so consider instd. **classes** of **polarizats.**

$\mathcal{H}_{sea} \sim \mathcal{H}'_{sea}$ ***iff*** $P_- - P'_-$ *is Hilbert-Schmidt*

Why this relation?

- By intuition above, **transition amplitudes still well-defined** ("*they differ by a superficial change in the sea*")
- If **EM fields off at t_0 equivalence class time evolution** $C(t) = [U(t,s)\mathcal{H}_-]_\sim \; t > t_0$

  only depends on **potentials** at ***t*** **not on the history** of fields!

(iii) A solution: make Dirac Sea picture Primitive

- Then, a specific **choice of polarization in each class along *t*** is a **choice of *"reference frame"*** wrt which the **Dirac QFT time evolution** is **represented** (a **choice** of **Fock space** at **each *t*)**

- **Dirac time-evolution** can be **naturally implemented between time-varying Fock spaces**

  Just need to keep track of the "**polarization charge current"** as the "*frame independent quantity*"

  - To do this **construction**, the **natural** thing is to use the **infinite wedge product spaces!!!**

    - And a **very nicely working** theory **allowing EM field implementat. results**!

D.-A. Deckert, D Dürr, F Merkl, and M Schottenloher, "*Time-evolution of the external field problem in quantum electrodynamics*", Journal of Mathematical Physics **51** (2010).

To be Continued...

(Perhaps in following seminar)

So: Anti-particle is Quasi-particle or not?

(Fermionic) [annotation under "Anti-particle"]

Why Dirac Sea primitive (hence anti-matter effective)?

- Would **explain** why **(fermionic) quantum vacuum is not empty** (providing a possible explanation of **dark energy/matter…**)
- Would **explain fermionic renormalization in QFT** (of H, of Q etc.)
- Would **explain why ∄ proper position operator for anti-mat. yet**
- **Provides promising "machinery"** to do **interacting QFT**
- Could provide an **explanation** to the **matter-antimatter asymmetry**
- **Potentially** allows **deterministic trajectory ontology**

So: Anti-particle is Quasi-particle or not?

(Fermionic)

Why anti-particle as proper particle?

- Hard to **see a e+'s trace** in a **cloud chamber and deny** its actuality *(although what we really see are molecule drops...)*
- A **simpler mental picture: less omnipresent** yet **"inaccessible"** stuff *(although apparently external field and observer dependent...)*
- A **trajectory-based ontology exists** *(although stochastic...)*
- A more **symmetric theory** *(although fundamental particles zoo ↑...)*
- **Mainstream physicist consensus** (yeah....we love orthodox physics...)

CONCLUSIONS

- **Don't** buy **cheap QFT** explanations! **There exist/are currently being found** way **more coherent versions** in all senses:

mathematically*, *physically*, "*intuitionally*" and *philosophically

- **Still** many **open questions in all four** subjects, **so** consider **joining the quest!!!**

The clarification of QFT is in your hands!

V – BIBLIOGRAPHY

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- Proof that, unitary of 1 Dirac-particle implementable in free e-/e+ space iff no magnetic field present.

QUESTIONS