# Towards a Rigorous Interacting Fermion QFT: the DIRAC-HILBERT OCEAN

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# PLANNING

MOTIVATION & CONTEXT

1. A SINGLE DIRAC PARTICLE
2. THE USUAL FERMION QFT: FOCK SPACE
3. AN INSTANCE of HAAG'S THEOREM
4. DIMOCK'S DIRAC SEA
5. HAAG'S THEOREM from the SEA

METHOD ⇒ 6. LEARN TO GENERALIZE (feat. von Neumann)

RESULT ⇒ 7. THE INFINITE WEDGE PRODUCT

# 1. A single Dirac Particle

1928

- A Dirac Spinor: $\psi \in L^2(\mathbb{R}^3, \mathbb{C}^4) \rightsquigarrow \psi(x) = \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \\ \psi_3(x) \\ \psi_4(x) \end{pmatrix}$

- The Dirac Eqt.: $i\hbar \frac{d}{dt} \psi_t = H \psi_t$

  for some s.a. $H$ ← A 4x4 matrix per $x \in \mathbb{R}^3$

- The Free Hamiltonian: $H_0 := c\vec{\alpha} \cdot (-i\hbar \vec{\nabla}) + \beta m c^2$

  (where $\alpha_j := \begin{pmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{pmatrix}$ & $\beta := \begin{pmatrix} Id_{2\times 2} & 0 \\ 0 & -Id_{2\times 2} \end{pmatrix}$. )

- It is s.a. in $H^1(\mathbb{R}^3, \mathbb{C}^4) \Rightarrow \exists$ SCOPUG $e^{-\frac{i}{\hbar} H_0 t}$.

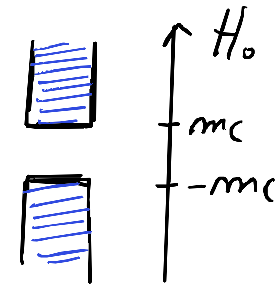
# THE FREE HAMILTONIAN



- **Energy spectrum**: $\sigma(H_0) = (-\infty, -mc] \cup [mc, +\infty)$

- **Spectral PVM**

$$\mathbb{P}^{H_0} \qquad \begin{cases} \mathbb{P}^{H_0}((0,+\infty)) =: \mathbb{P}^{H_0}_+ \\ \mathbb{P}^{H_0}((-\infty,0)) =: \mathbb{P}^{H_0}_- \end{cases}$$




- **Positive Energy subspace** $\quad \mathbb{P}^{H_0}_+(L^2(\mathbb{R}^3; \mathbb{C}^4)) =: h^{H_0}_+$

  **Negative Energy subspace** $\quad \mathbb{P}^{H_0}_-(L^2(\mathbb{R}^3; \mathbb{C}^4)) =: h^{H_0}_-$

$$L^2(\mathbb{R}^3, \mathbb{C}^4) = h^{H_0}_+ \oplus h^{H_0}_- .$$

# THE HEURISTIC FEAR

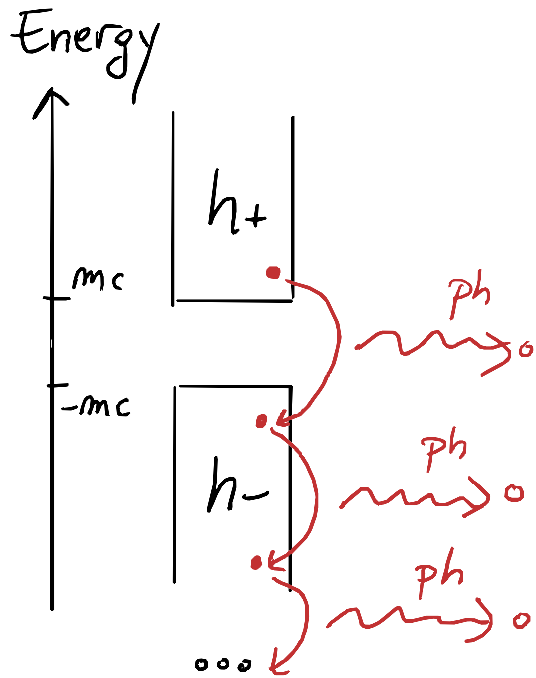
- If coupled with light

$\Downarrow$

- "Energy exchange" with EM field could decay $e^-$ to arbitrarily low energy

$\Downarrow$

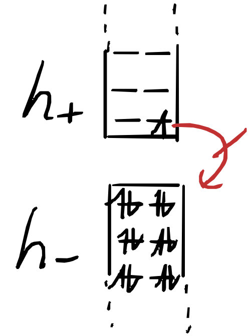
- $\infty$ "light energy"! $\rightsquigarrow$ CONTRADICTION with EXPERIMENTS!



# Dirac's Heuristic Solution

1.

- If $h_-$ "Filled" with a "Sea" of fermions } by "Pauli exclusion" No unbounded decay!




- But $\infty$ fermions! Contradictory with experiments!

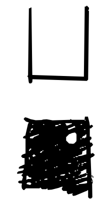
- unless... only "inhomogeneities" of this uniform background are observable to us...

- But then many new predictions would follow:

# DIRAC'S HEURISTIC SOLUTION

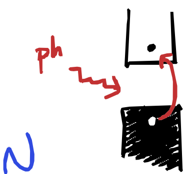
1.

⇝ If Dirac Sea was true, then :

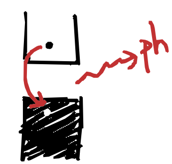


- A "hole" in Sea would behave as an opposite charge & = mass particle } ⇒ We call it an ANTI-PARTICLE

- "Inject" enough energy to apparently vacuum space ⇒ particle & hole emerge! } PAIR CREATION




- A fermion could vanish to vacuum if matched with a hole! } PAIR ANNIHILATION




↪ ALL EXPERIMENTALLY VERIFIED!

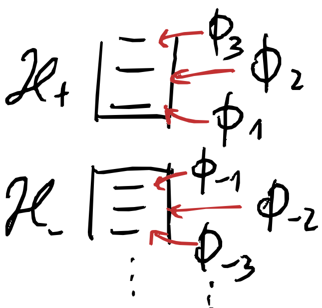
e.g., in 1932 Anderson discovers $e^+$

# Historical Account

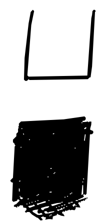


- To describe Dirac Sea, need $\infty$ particle theory...

  i.e, given ONBs $\{\phi_{-j}\}_{j\in\mathbb{N}} \subset h_-$ , $\{\phi_{+j}\}_{j\in\mathbb{N}} \subset h_+$

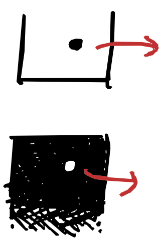



Ground state of Dirac Sea would be :



"$\phi_{-1} \wedge \phi_{-2} \wedge \phi_{-3} \wedge \dots$"

↳ VACUUM for us.



$\phi_3 \in h_+$

$\phi_2 \in h_-$

"$\underline{\underline{\phi_{+3}}} \wedge \phi_{-1} \wedge \phi_{-3} \wedge \phi_{-4} \wedge \phi_{-5} \wedge \dots$"

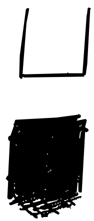
↑ Missing $\phi_{-2}$

- But they decided NOT to rigorize it !

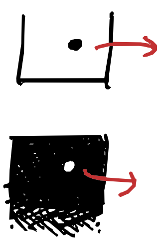
# Historical Account

They decided to take EFFECTIVE account as serious & Dirac Sea as metaphor:

Ground state of Dirac Sea would be:



$\cong$ No particle, No antiparticle $\Big\}$ $\Omega$ VACUUM



$\phi_3 \in h_+$

$\phi_{-2} \in h_-$

$\cong$ Particle in $\phi_3$, Anti-particle in "$\phi_{-2}$" $\Big\}$ $\phi_3 \wedge \phi_{-2}$

- But then, they needed an EOM for anti-particle congruent with anti-particle wavefunction.

# STATE FOR ANTIPARTICLE...

- $e^-/e^+$ behaviour different only if EM present so...

- EXTERNAL EM Field HAMIL if charge is $-e$:
$$\begin{cases} H(-e) = H_0 + \underbrace{\frac{e}{c}\,\vec{\alpha}\cdot\vec{A}(x,t) - e\,V(\vec{x},t)}_{H_I(\vec{A},V)} \\ \hookrightarrow \text{it is s.a. for general } \vec{A}, V. \end{cases}$$

- $\exists$ conjugate-unitary $C : L^2(\mathbb{R}^3; \mathbb{C}^4) \to L^2(\mathbb{R}^3; \mathbb{C}^4)$ s.th.
  - $\to$ if $\psi_t$ solves Dirac eq. for $H(-e) \Rightarrow C\psi_t$ for $H(+e)$
  - $\to$ $H(-e)\psi = E\psi \Longrightarrow H(+e)\underline{C\psi} = -E\,\underline{C\psi}$
    - $\hookrightarrow$ $C\,h_-(-e) = h_+(e)$

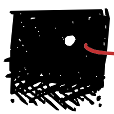
# STATE FOR ANTIPARTICLE...



- Thus $\begin{pmatrix} E>0 \text{ solutions} \\ \text{to } H(+e) \end{pmatrix} \underset{\text{via } C}{\Longleftrightarrow} \begin{pmatrix} E<0 \text{ solutions} \\ \text{to } H(-e) \end{pmatrix}$

- Hence:

 $\to \phi_3 \in h_+$ $\quad \cong \quad$ Particle in $\phi_3$

 $\to \phi_{-2} \in h_-$ $\quad$ Anti-particle in "$\cancel{\phi_{-2}}$" $\sim C\phi_{-2}$.

$\Phi_3 \wedge C\Phi_{-2}$

- Thus, if over-sea particles and holes taken seriously SPACE of N fermions & M anti-fermions is:

$$\big(\underbrace{h_+ \wedge h_+ \wedge \cdots \wedge h_+}_{N \text{ times}}\big) \otimes \big(\underbrace{Ch_- \wedge \cdots \wedge Ch_-}_{M \text{ times}}\big).$$

↳ FREE HAM'S SPLITTING!!

# II. The Usual Fermion QFT

- The space of variable (but finite) number of particles and anti-particles:

$$\mathcal{H}_{e^-/e^+} := \mathcal{F}_{\text{Anti}}\Big(\underbrace{\mathbb{1}_+^{H_0} L^2(\mathbb{R}^3,\mathbb{C}^4)}_{h_+}\Big) \otimes \mathcal{F}_{\text{Anti}}\Big(\underbrace{C\cdot \mathbb{1}_-^{H_0} L^2(\mathbb{R}^3,\mathbb{C}^4)}_{C\, h_-}\Big)$$

1 positive free energy fermion space

1 positive free energy antiparticle space

where:

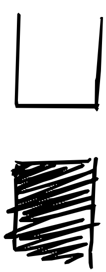
$$\mathcal{F}_{\text{Anti}}(h) := \underset{\substack{\downarrow \\ \text{vacuum} \\ \text{sector}}}{\mathbb{C}} \oplus \underset{\substack{\downarrow \\ \text{1 particle} \\ \text{sector}}}{h} \oplus \underset{\substack{\downarrow \\ \text{2 particle} \\ \text{sector}}}{h \wedge h} \oplus h \wedge h \wedge h \oplus \cdots$$

. . .

- So, each $\Psi \in \mathcal{H}_{e^-/e^+}$ is a tuple of one slot per space of N fermions & M anti-fermions.

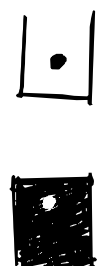
$$\Psi = (\underset{\mathbb{C}}{\underset{\downarrow}{a}},\ \underset{\mathcal{H}_+}{\underset{\downarrow}{\psi_{1e^-}}},\ \underset{C\mathcal{H}_-}{\underset{\downarrow}{\psi_{1e^+}}},\ \underset{\mathcal{H}_+\otimes(C\mathcal{H}_-)}{\underset{\downarrow}{\psi_{1e^-1e^+}}},\ \underset{(\mathcal{H}_+\wedge\mathcal{H}_+)\otimes(C\mathcal{H}_-)}{\underset{\downarrow}{\psi_{2e^-1e^+}}},\ \ldots)$$

- Examples:



$\cong \Omega = (1, 0, 0, \ldots)$

"VACUUM" state



$\cong (0, 0, 0, \phi_3 \wedge C\phi_{-2}, 0, \ldots)$

# "CREATION & ANNIHILATION" OPs.

Given $\Psi_{n,m} := (f_1 \wedge \cdots \wedge f_n) \otimes (Cg_1 \wedge \cdots \wedge Cg_m)$

- CREATE / ANNIH. FERMION in state $f \in h_+$:

$$\begin{cases} a^+(f)\Psi_{n,m} := (f \wedge f_1 \wedge \cdots \wedge f_n) \otimes (Cg_1 \wedge \cdots \wedge Cg_m) \\ a(f)\Psi_{n,m} := \left( \sum_{k=1}^{n} (-1)^k \langle f, f_k \rangle f_1 \wedge \cdots \wedge \hat{f}_k \wedge \cdots \wedge f_n \right) \otimes (Cg_1 \wedge \cdots \wedge Cg_m) \end{cases}$$

- CREATE / ANNIH. ANTI-FERMION in state $Cg \in h_+$:

$$\begin{cases} b^+(g)\Psi_{n,m} := (f_1 \wedge \cdots \wedge f_n) \otimes (Cg \wedge Cg_1 \wedge \cdots \wedge Cg_m) \\ b(g)\Psi_{n,m} := (f_1 \wedge \cdots \wedge f_n) \otimes \left( \sum_{k=1}^{m} (-1)^{n+k} \langle Cg, Cg_k \rangle Cg_1 \wedge \cdots \wedge \widehat{Cg_k} \wedge \cdots \wedge Cg_m \right) \end{cases}$$

All bounded ops. & $a(f)\Omega = 0 = b(g)\Omega$.

# THE FIELD OPERATORS

$$\hat{\Psi}(f) := a(P_+^{H_0} f) + b^\dagger(P_-^{H_0} f)$$

for $f \in L^2(\mathbb{R}^3; \mathbb{C}^4)$.

Very UNMOTIVATED (unless Dirac sea seriously taken)

- Can recover: $a(f) = \hat{\Psi}(f)$ if $f \in h_+$ ; $b(g) = \hat{\Psi}^\dagger(g)$ if $g \in h_-$
- They are the standard irred. representation of the CANONICAL ANTI COMMUTATION RELATIONS.

$$\{\hat{\Psi}(f), \hat{\Psi}^\dagger(g)\} = \langle f, g \rangle \, \mathrm{Id} \qquad \forall f, g \in L^2(\mathbb{R}^3; \mathbb{C}^4)$$

$$\{\hat{\Psi}(f), \hat{\Psi}(g)\} = 0 = \{\hat{\Psi}(f)^\dagger, \hat{\Psi}^\dagger(g)\}$$

# IMPLEMENTABILITY of 1-PARTICLE DYNAMICS in the FOCK QFT

- 1-particle unitary $\mathcal{U}: \mathcal{H} \to \mathcal{H}$ tells also how to "move" $n$ particles $f_1 \wedge \cdots \wedge f_n \in \mathcal{H} \wedge \cdots \wedge \mathcal{H}$ by:

  Factorwise applict.: $\mathcal{U} f_1 \wedge \cdots \wedge \mathcal{U} f_n$.

- But in $\mathcal{H}_{e^-/e^+}$, to lift $\mathcal{U}: L^2(\mathbb{R}^3; \mathbb{C}^4) \longrightarrow L^2(\mathbb{R}^3; \mathbb{C}^4)$ this way: $\mathcal{U} f_1 \wedge \cdots \wedge \mathcal{U} f_n \otimes \mathcal{C}\mathcal{U} g_1 \wedge \cdots \wedge \mathcal{C}\mathcal{U} g_m$ only makes sense if $\mathcal{U}$ preserves $h_+, h_-$

  (Else mix $E>0$ & $E<0$ .... $\rightsquigarrow$ pair creation)

$\Rightarrow$ Hard to say how to lift it .... except:

- Action on field op. obvious (Heisenb.-p.) 2. [redacted]

$$\hat{\Psi}(f) \longmapsto \hat{\Psi}(\mathcal{U}f)$$ creat/annih. evolved state.

- If $\exists$ corresponding $\Gamma(\mathcal{U})$ evolut. in $\mathcal{H}_{e^-/e^+}$ $\Rightarrow$ $\left\{ \text{that is the LIFT!} \right.$

DEF.: A 1-particle unitary $\mathcal{U} : L^2(\mathbb{R}^3, \mathbb{C}^4) \longrightarrow L^2(\mathbb{R}^3, \mathbb{C}^4)$ is IMPLMTBLE in $\mathcal{H}_{e^-/e^+}$ iff $\exists \Gamma(\mathcal{U}) : \mathcal{H}_{e^-/e^+} \to \mathcal{H}_{e^-/e^+}$ unitary s.th.

$$\boxed{\hat{\Psi}(\mathcal{U}f) = \Gamma(\mathcal{U})\, \hat{\Psi}(f)\, \Gamma(\mathcal{U})^*} \quad \forall f \in L^2(\mathbb{R}^3, \mathbb{C}^4).$$

Note: if $\mathcal{U}$ preserves $h_+, h_-$ $\Rightarrow$ it acts on sector $(m e^-, m e^+)$ as:

$$f_1 \wedge \cdots \wedge f_m \otimes g_1 \wedge \cdots \wedge g_m \longmapsto \mathcal{U}f_1 \wedge \cdots \wedge \mathcal{U}f_m \otimes C\mathcal{U}g_1 \wedge \cdots \wedge C\mathcal{U}g_m$$

$\hookrightarrow$ It's the case for FREE TIME EVOLUTION!

- If $\mathcal{U}_t$ is implementable, Heisenb.-pic. 2. [redacted] ops. $\hat{\Phi}_t(\cdot) := \hat{\Phi}(\mathcal{U}_t \cdot)$ are again a FIELD OP., i.e., an irrep of the CAR on $\mathcal{H}_{e^-/e^+}$, defining

→ new creat. & annih. ops: $\begin{cases} c_t(f) := \hat{\Phi}_t(f) & \text{if } f \in h_+ \\ d_t(g) := \hat{\Phi}_t(g)^* & \text{if } g \in h_- \end{cases}$

⇒ with corresponding NEW VACUUM $\tilde{\Omega}_t$ !

- For free time evolution $\tilde{\Omega}_t = \Omega$

- But in general, $\Omega \rightsquigarrow \tilde{\Omega}_t \neq \Omega$ taken to state with some $e^-/e^+$ (e.g. when strong $\vec{E}$)

POLARIZED VACUUM

⇝ $e^-/e^+$ PAIR CREATION !

**PROP (Lazarovici 2011)**: A unitary $\mathcal{U}$ is implemtble on $\mathcal{H}_{ee^+}$ $\Longleftrightarrow$ Heisenb. pict. op. $\hat{\Psi}(\mathcal{U}\cdot)$ has "VACUUM" in the same space, i.e., $\exists\, \tilde{\Omega} \in \mathcal{H}_{e^-e^+}$:

$$\begin{cases} c(f)\tilde{\Omega} := \hat{\Psi}(\mathcal{U}f)\tilde{\Omega} = 0, & \forall f \in h_+ \\ d(f)\tilde{\Omega} := \hat{\Psi}(\mathcal{U}f)^*\tilde{\Omega} = 0, & \forall g \in h_- . \end{cases}$$

- So a unitary NOT being implementable in hints at needing a NEW VACUUM! (A new Dirac Sea)

- Note: Even if NOT implmtble in $\mathcal{H}_{e^-e^+}$ it can be accomodated in a different BUT INEQUIVALENT CAR rep.!

# 3. An Instance of Haag's Theorem

- "Haag's theorem" is one of the main issues to be faced when rigorizing interacting particle QFTs

- It tells that generically we need different state spaces (say, inequivalent irreps. of CAR) to represent free & interacting QFTs.

- In a sense, "the interacting vacuum lives outside the space representing the free vacuum".

- Let $H_t := H_0 + H_I(V_t, \vec{A}_t)$ govern 1-particle in an external EM field $(V_t, \vec{A}_t) \in \mathcal{C}_c^\infty$.

It is s.a. op. $\forall t$ & $\exists$ propagator $\mathcal{U}^{(V,\vec{A})}(t, t_0)$.

THM. (RUIJSENAARS 1976):

$$\left( \mathcal{U}^{(V,\vec{A})}(t,t_0) \text{ is implementable in } \mathcal{H}_{e^-/e^+} \right) \Leftrightarrow \left( \vec{A}(t) \equiv \vec{0} \right).$$

- So ANY magnetic field requires an INEQUIVALENT representation space than the $e^-/e^+$ Fock space!

# MEANING in the e-/e+ PICTURE

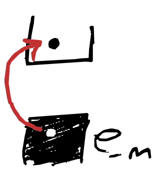
**Prop** (Thaller 1992): Given a unitary $\mathcal{U} : L^2(\mathbb{R}^3, \mathbb{C}^4) \to L^2(\mathbb{R}^3, \mathbb{C}^4)$

$$\left( \mathcal{U} \text{ is implementable in } \mathcal{H}_{e^-/e^+} \right) \Longleftrightarrow \left( \begin{array}{l} \mathcal{U}_{+-} := P_+^{H_0} \mathcal{U} P_-^{H_0} \\ \mathcal{U}_{-+} := P_-^{H_0} \mathcal{U} P_+^{H_0} \end{array} \right\} \text{ are Hilbert-Schmidt ops.} \right)$$

which is called the SHALE-STINESPRING condition.

- Given ONB $(e_m)_{m \in \mathbb{Z}\setminus\{0\}}$ with $(e_m)_{m\in\mathbb{N}} \subset h_+$, $(e_{-m})_{m\in\mathbb{N}} \subset h_-$

$$\|\mathcal{U}_{+-}\|^2_{\text{Hilb.Sch.}} := \sum_{m\in\mathbb{Z}\setminus\{0\}} \|\mathcal{U}_{+-} e_m\|^2 = \sum_{m\in\mathbb{N}} \underbrace{\|P_+^{H_0} \mathcal{U} e_{-m}\|^2}$$




Probab of an initial $e_{-m} \in h_-$ to transition to $h_+$

i.e., generate a pair from vacuum!

- Thus, $\mathcal{U}$ NOT implementable when $\sum_{m\in\mathbb{N}} \|P_+ \mathcal{U} e_{-m}\|^2 = +\infty \Rightarrow \mathcal{U}$ would cause an unbounded creation of $e^-/e^+$ from the vacuum / Dirac Sea. $\Rightarrow$ "VACUUM POLARIZATION"

  ↳ External EM field strong enough $\rightsquigarrow$ creation of $e^-/e^+$ pairs!

- But of course, in $\mathcal{H}_{e^-/e^+}$ there is NO $\infty$ particle number sector $\Rightarrow$ cannot represent time evolution there!

# INTERESTING CONSEQUENCE 3.

- Let $W(\Lambda): L^2(\mathbb{R}^3; \mathbb{C}^4) \longrightarrow L^2(\mathbb{R}^3; \mathbb{C}^4)$ be the unitary representat. of the Lorentz transf. $\Lambda$
- A Lorentz transformation may

$$(V_t, \vec{A}_t) \xrightarrow{\Lambda} (\tilde{V}_t, \tilde{\vec{A}}_t)$$

$$\vec{A}_t = 0 \longmapsto \vec{A}_t \neq 0$$

- Thus, Lorentz transf. $W(\Lambda)$ may not be implementable in $\mathcal{H}_{e^-/e^+}$. $\Rightarrow$ The observer frame $\Lambda$ sees many $e^-/e^+$ where there was "vaccum"!

↳ Related to UNRUH EFFECT.

↳ effective $e^-/e^+$ picture "over the sea" is INCONSISTENT!

# Naive Solution: 3. [redacted]

## Adapted Time-Dependent State Spce

- Given $H_t := H_0 + H_I(V_t, \vec{A}_t)$ splitting $\sqcup$ $h_+(t)$

depends on time $\begin{cases} h_+(t) := P_+^{H_t} L^2(\mathbb{R}^3;\mathbb{C}^4) \\ h_-(t) := P_-^{H_t} L^2(\mathbb{R}^3;\mathbb{C}^4) \end{cases}$ $h_-(t)$

- Build: $\mathcal{H}(t) := \mathcal{F}_{anti}(h_+(t)) \otimes \mathcal{F}_{anti}(C h_-(t))$

s.th. at each $t$ we have different

↳ vacuum $\Omega_t \in \mathcal{H}(t)$

↳ field ops. $a_t(\cdot), a_t(\cdot)$ on $\mathcal{H}(t)$

3.

- But then, state $\psi(t) \in \mathcal{H}(t)$ would live in inequivalent representation spaces as time evolves $\Rightarrow$ unclear how to talk about continuity, differentiability etc.

$\Rightarrow$ $e^-/e^+$ interpretation would depend on time & on our choice of $(V, \vec{A})$ or on the observer's Lorentz frame!

$\Rightarrow$ No consistent particle - antiparticle picture! $\Rightarrow$ We should realize back that $e^-/e^+$ is an effective description!

# 4. Dimock's (2010) Dirac Sea Representation

- Let $h$ a Hilbert space (e.g., $L^2(\mathbb{R}^3, \mathbb{C}^4)$) and $V \subset h$ $\infty$ dim & codim. closed subspace (e.g., $V = h_+$)

- $(e_n)_{n \in \mathbb{Z} \setminus \{0\}}$ ONB s.th.
  $\{e_1, e_2, \dots\} \subset V^\perp$ ONB $\qquad \boxed{\begin{matrix} \vdots \\ e_2 \\ e_1 \end{matrix}}\ V^\perp$
  $\{e_{-1}, e_{-2}, \dots\} \subset V$ ONB $\qquad \boxed{\begin{matrix} e_{-1} \\ e_{-2} \\ \vdots \end{matrix}}\ V$

- We will build an <u>abstract rigorization</u> of the <u>Dirac Sea</u> with vacuum $e_{-1} \wedge e_{-2} \wedge e_{-3} \wedge \dots$ that is an <u>equivalent CAR irrep.</u> to $\underline{\mathcal{H}_{e\text{-}let}}$.

**DEF:** (The Sea states and $e_{-1}\wedge e_{-2}\wedge\cdots$) 4.

(1) Define abstract-symbol vector space:

$$W := \operatorname{span}\left\{ e_{j_1}\wedge\cdots\wedge e_{j_K}\wedge \underline{e_{-m}\wedge e_{-m-1}\wedge e_{-m-2}\wedge\cdots} : \text{for some } K, m\in\mathbb{N} \text{ \& no repeated } j\text{'s} \right\}$$

(2) Define inner product s.th.: $\langle\cdot,\cdot\rangle : W\times W \to \mathbb{C}$

$$\left\langle \sum c_j\, e_{j_1}\wedge e_{j_2}\wedge\cdots, \sum d_j\, e_{k_1}\wedge e_{k_2}\wedge\cdots \right\rangle := \sum \overline{c_j}\, d_j\, \delta_{(j_1,j_2,\dots),(k_1,k_2,\dots)}$$

(3) The Cauchy completion of $(W, \langle\cdot,\cdot\rangle)$ is the Hilbert space $\left(\bigwedge_{j\geq 1}^{\infty} h, (e_m)_{m\in\mathbb{Z}\setminus\{0\}}\right)$ that we call the $\infty$ WEDGE SPACE GENERATED by $(e_m)_m$.

# DEF.: Obvious def for Field Ops.

$$\hat{\psi}_{Sea}(e_\ell)\,(e_{j_1}\wedge e_{j_2}\wedge\cdots) := \begin{cases} 0 & \text{if } \ell \neq \text{all } j_k \\ (-1)^{s+1}\cdot e_{j_1}\wedge\cdots\wedge \cancel{e_\ell}\wedge\cdots & \text{if } \ell = \text{some } j_k \end{cases}$$

$$\hat{\psi}^{+}_{Sea}(e_\ell)\,(e_{j_1}\wedge e_{j_2}\wedge\cdots) := \begin{cases} (-1)^{s}\cdot e_{j_1}\wedge\cdots\wedge e_\ell\wedge\cdots & \text{if missing } \ell \text{ in } j_1, j_2, \ldots \\ 0 & \text{if } \ell = \text{some } j_k \end{cases}$$

- Then, for $f \in h$

$$\hat{\psi}_{Sea}(f) := \sum_{j\in\mathbb{Z}\setminus\{0\}} \langle e_j, f\rangle\, \hat{\psi}_{Sea}(e_j), \qquad \hat{\psi}^{+}_{Sea}(f) := \sum_{j\in\mathbb{Z}\setminus\{0\}} \overline{\langle e_j, f\rangle}\, \hat{\psi}^{+}_{Sea}(e_j)$$

naturally satisfy CAR!

## Now Vacuum is LITERALLY Dirac Sea

- If $\Omega_{sea} := e_{-1} \wedge e_{-2} \wedge e_{-3} \wedge \dots$

naturally:

$$\Psi_{sea}(h)\,\Omega_{sea} = 0, \quad \forall h \in V^{\perp}$$

"No overseas fermion in $\Omega$"

$$\Psi_{sea}(g)^{+}\,\Omega_{sea} = 0, \quad \forall g \in V$$

"Pauli Exclusion"

**DEF** (Just for comparison):

- $a_{sea}(h) := \Psi_{sea}(h)$, $\quad a^{+}_{sea}(h) := \Psi_{sea}(h)$ $\quad$ if $h \in V^{\perp}$

- $b^{+}_{sea}(g) := \Psi_{sea}(g)$, $\quad b_{sea}(g) := \Psi^{+}(g)$ $\quad$ if $g \in V$.

annihilation of a ~~state~~ $g$ in sea

is creation of a hole!

# EQUIVALENCE

**THM.** The CAR reps. given in $\mathcal{H}_{e^-/e^+}$ and in $\left(\bigwedge_{j=1}^{\infty} L^2(\mathbb{R}^3, \mathbb{C}^4), (e_n)_{n\in\mathbb{Z}}\right)$ with $V = h_-$ are unitarily equivalent, i.e., $\exists$ unitary $\mathcal{U} : \left(\bigwedge^{\infty} L^2(\mathbb{R}^3, \mathbb{C}^4), (e_n)\right) \to \mathcal{H}_{e^-e^+}$

- $\mathcal{U}\Omega_{sea} = \Omega_{Fock}$
- $\mathcal{U}\,\hat{\Psi}_{sea}(f)\,\mathcal{U}^{-1} = \hat{\Psi}_{Fock}(f)$
- $\mathcal{U}\,a_{sea}(f)\,\mathcal{U}^{-1} = a_{Fock}(f)$
- $\mathcal{U}\,b_{sea}(f)\,\mathcal{U}^{-1} = b_{Fock}(f)$

Hence, whatever is done in Fock space QFT has a rigurous counterpart in Dirac Sea terms!

# 5. Haag's Theorem seen from the Dirac Sea

**Prop.** (Deckert et al. 2010): The single particle unitary $\mathcal{U}$ on $L^2(\mathbb{R}^3;\mathbb{C}^4)$ is implementable in $\left(\bigwedge_{n=1}^{\infty} L^2(\mathbb{R}^3;\mathbb{C}^4), (e_m)_m, h_-\right)$

$\Longleftrightarrow$ $P(h_-) - P(\mathcal{U}h_-)$ is Hilbert-Schmidt.

orthogonal projector

But: Given $V, \tilde{V} \subset h$

$P_V - P_{\tilde{V}}$ Hilb. Sch $\Longleftrightarrow$ $(P_V - P_{\tilde{V}})^*(P_V - P_{\tilde{V}})$ is TRACE class $\Longleftrightarrow$ $\operatorname{tr}(P_V - P_W P_V + P_W - P_V P_W)$ is FINITE

$\Downarrow$

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(i) $\operatorname{tr}(P_V) = \dim(V)$

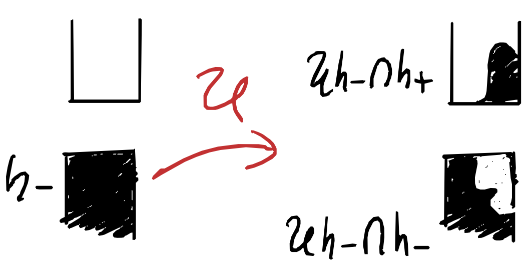
(ii) If $P_V P_{\tilde{V}} = P_{\tilde{V}} P_V \Rightarrow P_V P_{\tilde{V}} = P_{V \cap \tilde{V}}$

(iii) $\operatorname{tr}(P_V - P_{\tilde{V}} P_V) = \dim$ in $V \perp$ to $V \cap \tilde{V}$

(iv) $\operatorname{tr}(P_V - P_{\tilde{V}} P_V + P_{\tilde{V}} - P_V P_{\tilde{V}})$ is essentially the difference of dim of the part of $V$ $\perp$ to $V \cap \tilde{V}$ & part of $\tilde{V}$ $\perp$ to $V \cap \tilde{V}$.

5.

- Hence, implementable when:




$$\dim\big(h_- / (\mathcal{U}h_- \cap h_-)\big) - \dim\big(\mathcal{U}h_- / (\mathcal{U}h_- \cap h_-)\big)$$

is FINITE & well defined

- i.e., #holes in sea caused by $\mathcal{U}$ − #electrons over sea promoted by $\mathcal{U}$

  is <u>FINITE</u> & well-defined.

- So indeed, the issue is an "unbounded generation of e-/e+" ⇒ too many sea particles get promoted to $h_+$ out of $h_-$.

- In Dimock's construction, only states built by changing finitely many factors in $\underline{e}_{-1}\wedge\underline{e}_{-2}\wedge\cdots$ exist in the same space $\Rightarrow$ if some $\mathcal{U}$ makes $(\mathcal{U}e_{-1})\wedge(\mathcal{U}e_{-2})\wedge\cdots$ have infinitely many changes, say sending $\mathcal{U}\underline{e}_{-m}\mapsto\mathcal{U}e_{m}$, then we need to employ the space generated by a different "vacuum"/sea!
- In order to be able to describe dynamics in a SINGLE space, we need a complete definition of INFINITE WEDGE PRODUCT, where all Dimock spaces (relative to the possible Dirac seas) are naturally embedded as subspaces.

# 7. The Infinite Wedge-Product of Hilbert Spaces

**Objective:** Given a Hilbert space $\mathcal{H}$, build a rigorous: $\bigwedge\limits_{k=1}^{\infty} \mathcal{H} := \mathcal{H} \wedge \mathcal{H} \wedge \dots := \mathrm{Anti}(\mathcal{H} \otimes \mathcal{H} \otimes \dots)$

naturally containing all spaces generated by finite perturbations of any "Sea" $f_1 \wedge f_2 \wedge \dots$ with

$(f_j)_{j \in \mathbb{N}} \subset \mathcal{H}$ linearly indep. & $\|f_j\| = 1$.

- Then, the "Dirac-Hilbert Ocean" would be

$\bigwedge\limits_{k=1}^{\infty} L^2(\mathbb{R}^3; \mathbb{C}^4)$ $\Big\{$ containing all possible states of the Dirac Sea!

# 6. Learning How to Generalize (Feat. von Neumann)



A def. of $\mathcal{H}\otimes\cdots\otimes\mathcal{H} =: \bigotimes_{j=1}^{m}\mathcal{H}$ in abstract Cantor way:

(i) Def. abstract vector space: $W := \operatorname{span}\{f_1\otimes\cdots\otimes f_m \mid f_j\in\mathcal{H}\setminus\vec{0}\}$

(ii) Def. inner product as sesquilinear form:

$$\langle f_1\otimes\cdots\otimes f_m, g_1\otimes\cdots\otimes g_m\rangle := \prod_{j=1}^{m}\langle f_j, g_j\rangle .$$

(iii) $\bigotimes_{j=1}^{m}\mathcal{H}$ is the Cauchy completion of $(W,\langle\cdot,\cdot\rangle)$

i.e., the set of equivalence classes of Cauchy sequences in $W$ with extended $\langle\cdot,\cdot\rangle$.

- Simple, but leaves a structure whose elements are abstract symbols $\Rightarrow$ hard to proof stuff!

6.

- The more convenient definition is in terms of conjugate multilinear forms $\widetilde{\mathcal{L}}(\underbrace{\mathcal{H}\times\cdots\times\mathcal{H}}_{m}, \mathbb{C})$.

(1). Def. $f_1\otimes\cdots\otimes f_m$, $f_j\in\mathcal{H}$ as the conj. multilin. form:

$$f_1\otimes\cdots\otimes f_m : \mathcal{H}\times\cdots\times\mathcal{H} \longrightarrow \mathbb{C}$$
$$(g_1,\cdots,g_m) \longmapsto \prod_{j=1}^{m} \langle g_j, f_j\rangle$$

Hence, $f_j$ is the conj. lin. form $\langle\cdot, f_j\rangle$

(2). Def. $W := \mathrm{span}\{f_1\otimes\cdots\otimes f_m \mid f_j\in\mathcal{H}\}$: it is a DENSE vector subspace of $\widetilde{\mathcal{L}}(\mathcal{H}\times\cdots\times\mathcal{H}, \mathbb{C})$.

(3.) Def. $\bigotimes_{j=1}^{m}\mathcal{H}$ as the closure (hence, $\widetilde{\mathcal{L}}(\mathcal{H}\times\cdots\times\mathcal{H}, \mathbb{C})$)

Equip it with the unique conjugate linear extension of:

$$\langle g_1\otimes\cdots\otimes g_m, f_1\otimes\cdots\otimes f_m\rangle := f_1\otimes\cdots\otimes f_m(g_1,\cdots,g_m).$$

$$\langle g_1\otimes\cdots\otimes g_m, \psi\rangle := \psi(g_1,\ldots,g_m). \quad \psi\in\bigotimes_{j=1}^{m}\mathcal{H}$$

# How von Neumann 1939 built $\bigotimes_{k=1}^{\infty} \mathcal{H}$

6.

- First he noted some DESIDERATA that any generalization of $\bigotimes_{j=1}^{m} \mathcal{H}$ is expected to satisfy, e.g.:

(i) $f_1 \otimes f_2 \otimes \cdots$ expected to have norm $\prod_{j=1}^{\infty} \|f_j\|$

$\Rightarrow$ imposes restriction: only $f_1 \otimes f_2 \otimes \cdots$ with $\prod_{j=1}^{\infty} \|f_j\| < \infty$ (he denotes C-sequences) need to be considered. & if $\prod_{j=1}^{\infty} \|f_j\| = 0 \Rightarrow f_1 \otimes f_2 \otimes \cdots = \vec{0}$.

(ii) $\langle f_1 \otimes f_2 \otimes \cdots, g_1 \otimes g_2 \otimes \cdots \rangle$ expected $= \prod_{j=1}^{\infty} \langle f_j, g_j \rangle$. just as in finite m. $\Rightarrow$ Imposes restriction:

Need to find notion of $\prod_{j=1}^{\infty}$ allowing this!

⇛ Even if $(f_j)_j, (g_j)_j \in \mathcal{C}$-sequences 6.

it may happen $\prod_{j=1}^{\infty} \langle f_j, g_j \rangle$ NON-CONVERGENT

(when $\sum_{j=1}^{\infty} \arg(\langle f_j, g_j \rangle)$ NON-CONVERGENT. !)

↪ He redefines convergence to be 0 if so

(nicely justifiable via generalized Banach limits)

---

• Then he chooses the multilinear map definit. of $\bigotimes_{j=1}^{\infty} \mathcal{H}$ to generalize because simpler proofs.

& starts defining:

$$f_1 \otimes f_2 \otimes \cdots : \quad \mathcal{C}\text{-sequences} \xrightarrow{\ \mathcal{H}\times\mathcal{H}\times\cdots\ } \mathbb{C}$$

$$(g_1, g_2, \ldots) \longmapsto \prod_{j=1}^{\infty} \langle g_j, f_j \rangle$$

etc.

6. [redacted]

## IMPORTANT!

- When cooking a generalization $\exists$ different ways. Ideally can prove all are equivalent:

**THM.** (von Neumann 1939): Given any other generalizat. $\mathcal{K}$, if it satisfies the obvious desiderata:

(i) $\forall (f_j)_{j\in\mathbb{N}} \in$ C-seq $\exists$ element identifiable with the symbol $f_1 \otimes f_2 \otimes \cdots$ in $\mathcal{K}$.

(ii) $\langle f_1 \otimes f_2 \otimes \cdots, g_1 \otimes g_2 \otimes \cdots \rangle = \prod_{j=1}^{\infty} \langle f_j, g_j \rangle$

(iii) $\operatorname{span}\{f_1 \otimes f_2 \otimes \cdots\}$ is dense in $\mathcal{K}$

Then $\exists!$ identification with his $\bigotimes_{j=1}^{\infty} \mathcal{H}$.

# STEPS TO GENERALIZE to $\infty$ 6. [redacted]

(1) Study finite case defs. & formulate the desiderata for generalization

(2) Extend math operations employed in finite case (possibly ad-hoc for desiderata)

(3) Build generalization for def. with structures having most known results (easier proofs later)

⚠ Be ready to abandon some desiderata!

(5) Proof alterntve generalizts. would be equivalent!

# 7. The Infinite Wedge-Product of Hilbert Spaces

**Objective:** Given a Hilbert space $\mathcal{H}$, build a rigorous: $\bigwedge_{k=1}^{\infty} \mathcal{H} := \mathcal{H} \wedge \mathcal{H} \wedge \dots := \mathrm{Anti}(\mathcal{H} \otimes \mathcal{H} \otimes \dots)$

naturally containing all spaces generated by finite perturbations of any "Sea" $f_1 \wedge f_2 \wedge \dots$ with $(f_j)_{j \in \mathbb{N}} \subset \mathcal{H}$ linearly indep. & $\|f_j\| = 1$.

- Then, the "Dirac-Hilbert Ocean" would be

$\bigwedge_{k=1}^{\infty} L^2(\mathbb{R}^3; \mathbb{C}^4)$ $\Big\{$ containing all possible states of the Dirac Sea!

## THE STRUCTURE to be GENERLIZED 7.

DEF. as SUBSPACE of $\overset{m}{\otimes}\mathcal{H}$: $\mathcal{H}\wedge\cdots\wedge\mathcal{H}$ is the closed vector subspace of $\mathcal{H}\otimes\cdots\otimes\mathcal{H}$ given by closure of the span of:

$$f_1\wedge\cdots\wedge f_m := \frac{1}{\sqrt{m!}}\sum_{\sigma\in S_m}(-1)^{|\sigma|} f_{\sigma(1)}\otimes\cdots\otimes f_{\sigma(m)}$$

s.th. $\langle f_1\wedge\cdots\wedge f_m, g_1\wedge\cdots\wedge g_m\rangle_{\otimes\mathcal{H}} = \det\left(\left(\langle f_i, g_j\rangle\right)_{ij=1}^{m}\right)$

ISSUE: Not directly generalizable because

$$\frac{1}{\sqrt{m!}} \xrightarrow[m\to\infty]{} 0$$

DEF. as CANTOR:

(i) Choose ONB $\{\phi_j\}_{j\in\mathbb{N}} \subset \mathcal{H}$ & def. abstract vector sp.:

$$V := \mathrm{span}\{\phi_{j_1}\wedge\cdots\wedge\phi_{j_m} \mid j_1 \le j_2 \le \cdots \le j_m\}$$

(ii) Def. sesquil. form s.th.:

$$\langle \phi_{j_1}\wedge\cdots\wedge\phi_{j_m}, \phi_{k_1}\wedge\cdots\wedge\phi_{k_m}\rangle := \delta_{(j_1,\dots,j_m),(k_1,\dots,k_m)}.$$

(iii) $\mathcal{H}\wedge\cdots\wedge\mathcal{H}$ is the Cauchy COMPLETION of $(V, \langle\cdot,\cdot\rangle)$

NOTE: This is the def. generalized by Dimock, But we saw it had multiple inequivalent generalizations!

# DEF. as ALTERNATING FORMS:

7.

(i) Def. $f_1 \wedge \cdots \wedge f_m$, as the alternating conjugate-linear form:

$$f_1 \wedge \cdots \wedge f_m : \mathcal{H} \times \cdots \times \mathcal{H} \longrightarrow \mathbb{C}$$
$$(g_1, \cdots, g_m) \longmapsto \det\left( (\langle g_j, f_k \rangle)_{j,k \in \{1, \dots, m\}} \right)$$

(ii) Def. $W := \operatorname{span}\{ f_1 \wedge \cdots \wedge f_m \mid f_j \in \mathcal{H} \}$: it is a vector subspace of $\tilde{\mathcal{L}}(\mathcal{H} \times \cdots \times \mathcal{H}, \mathbb{C})$.

(iii) Def. $\bigwedge_{j=1}^{m} \mathcal{H}$ as the closure (all alternating forms) equipped with the unique conjugate linear extension of:

$$\langle g_1 \wedge \cdots \wedge g_m, f_1 \wedge \cdots \wedge f_m \rangle := f_1 \wedge \cdots \wedge f_m \, (g_1, \cdots, g_m).$$



$$\langle g_1 \wedge \cdots \wedge g_m, \psi \rangle := \psi(g_1, \cdots, g_m). \quad \psi \in \bigwedge_{j=1}^{m} \mathcal{H}$$

# DESIDERATA for $\bigwedge_{j=1}^{\infty} \mathcal{H}$



(a) We want the norm & inner product to be given by a generalized notion of determinant

$$\left\{ \begin{array}{l} \text{if } f_1 \wedge f_2 \wedge \cdots, g_1 \wedge g_2 \wedge \cdots \in \bigwedge_{j=1}^{\infty} \mathcal{H} \text{ then} \\ \langle f_1 \wedge f_2 \wedge \cdots, g_1 \wedge g_2 \wedge \cdots \rangle = \text{"} \det\left( \langle f_i, g_j \rangle_{i,j \in \mathbb{N}} \right) \text{"} . \end{array} \right.$$

- One could take the Fredholm det, but it was designed for operators and $(\langle f_i, f_j \rangle)_{i,j \in \mathbb{N}}$ needn't be!

$\hookrightarrow$ $\left\{ \begin{array}{l} \text{We want a generalization} \\ \text{of Fredholm det.} \end{array} \right.$

(b) We need that if

$$\exists \text{``}\|f_1 \wedge f_2 \wedge \cdots\|^2 = \det(\langle f_i, f_j \rangle_{i,j\in\mathbb{N}})\text{''} < \infty$$

and also for $g_1 \wedge g_2 \wedge \cdots \Rightarrow \exists \text{``}\det(\langle f_i, g_j \rangle_{i,j\in\mathbb{N}})\text{''}$

$\Rightarrow$ maybe need to adequate det (as von Neum.)

(use generalized Banach limits...)

---

(c) Expect that: $\operatorname{span}\left\{ \bigwedge_{j=1}^{\infty} \phi_j \in \bigwedge_{j=1}^{\infty} \mathcal{H} \right\} = \bigwedge_{j=1}^{\infty} \mathcal{H}$

(d) From def. 1: $\bigwedge_{j=1}^{\infty} \mathcal{H} \subset \bigotimes_{j=1}^{\infty} \mathcal{H}$ (von Neum.'s)

Easiest way to satisfy this is via def. 3

(e) From def. 2:

7.

$\forall$ ONB $\{\phi_j\}_j \subset \mathcal{H}$ we want:

$$\psi_1 \wedge \cdots \wedge \psi_\ell \wedge \underbrace{\phi_{j_{\ell+1}} \wedge \phi_{j_{\ell+2}} \wedge \cdots}_{\text{fixed tail / "depth"}} \in \bigwedge_{j=1}^{\infty} \mathcal{H}$$

(for all $\psi_j \in \mathcal{H}$, $\ell \in \mathbb{N}$ & $(j_1, j_2, \ldots) \in \mathbb{N}^{|\mathbb{N}|}$.)

$\Rightarrow$ This way all possible Dimock seas, i.e., all releveled Dirac Fock spaces are subspaces of $\bigwedge_{j=1}^{\infty} \mathcal{H}$.

(f) The obvious $\{\hat{\Psi}(f)\}_{f \in \mathcal{H}}$ are a rep. of CAR.

# THE DETERMINANT 7.

- Let $I$ be any ordered set (e.g. $\mathbb{N}$) and let

$$\mathcal{P}_0(I) := \text{All finite subsets of } I$$

| DIRECTED by INCLUSION.

**DEF. (QA.):** We say $(a_{ij})_{i,j\in I} \subset \mathbb{C}$ has a VON-NEUMANN DET. $\alpha \in \mathbb{C}$ when, as a NET:

$$\det\left((a_{ij})_{i,j\in J}\right)_{J\in\mathcal{P}_0(I)} \longrightarrow \alpha$$

i.e., $\forall \varepsilon > 0 \;\; \exists I^{\varepsilon} \in \mathcal{P}_0(I)$ s.th. $\forall J \in \mathcal{P}_0(I),\; I^{\varepsilon} \subseteq J$

$$\left| \det\left((a_{ij})_{i,j\in J}\right) - \alpha \right| \leq \varepsilon.$$

+ General! Even for $I = \mathbb{R}$!

**LEM. (QA.):** If it exists it is UNIQUE.

# SIMPLER ALTERTVE in $\mathbb{N}$ ?



**DEF. (Q.A.):** $(a_{ij})_{ij\in\mathbb{N}} \subset \mathbb{C}$ has a **ω-VON-NEUMANN DETERMINANT** $\alpha \in \mathbb{C}$ when

$\forall$ nested exhaustions of $\mathbb{N}$ by finite sets, i.e., $(J_m)_{m\in\mathbb{N}} \subset P_0(\mathbb{N})$ with $J_m \subseteq J_{m+k}$, $\bigcup_{m\in\mathbb{N}} J_m = \mathbb{N}$

we have: $\det\left((a_{ij})_{ij\in J_m}\right) \xrightarrow[m\to\infty]{} \alpha$.

**PROP. (Q.A.)**

$$\left(\begin{array}{l}(a_{ij})_{ij\in\mathbb{N}} \subset \mathbb{C} \text{ has} \\ \text{a } \underline{\text{von Neu.}} \text{ det } \alpha\end{array}\right) \Longleftrightarrow \left(\begin{array}{l}(a_{ij})_{ij\in\mathbb{N}} \subset \mathbb{C} \text{ has} \\ \text{a } \underline{\omega\text{-von Neu.}} \text{ det } \alpha\end{array}\right)$$

# WHY GOOD GENERALZT.? 7.

**Well-Studied DEF.** (Simon 1977): Given $A \in \mathcal{L}(\mathcal{H})$

s.th. $A = \mathrm{Id} + T$ with $T \in \mathcal{J}_1(\mathcal{H})$ TRACE CLASS

(i.e., $T$ compact & $\mathrm{tr}(\sqrt{T^*T}) < +\infty$)

its FREDHOLM DETERMINANT is:

$$\det_F(A) := \prod_{n \in \mathbb{N}} (1 + \lambda_n(T)) = \sum_{k=0}^{\infty} \mathrm{tr}(\Lambda^k T)$$

Eigval nth {Lidskii} ↳ Lift of $T$ to $\bigwedge_{j=1}^{k} \mathcal{H}$

- So, well-defined only when $(a_{ij})_{ij \in \mathbb{N}} = (\delta_{ij} + t_{ij})_{ij}$, for some trace-class $(t_{ij})_{ij \in \mathbb{N}}$ bounded op. on $\ell^2(\mathbb{N})$. $\Rightarrow$ We'll see this is big req.!

**PROP.:** (Q.A.)

$\Big($ If $(a_{ij})_{ij\in\mathbb{N}} \subset \mathbb{C}$ has Fredholm det $\alpha$ – hence, is a trace class perturb. of Id. $\Big)$ $\Rightarrow$ $\Big( (a_{ij})_{ij\in\mathbb{N}} \subset \mathbb{C}$ has a von Neum. det. $\alpha \Big)$

• The converse ($\Leftarrow$) is FALSE! Counterexample:

$$\begin{pmatrix} 1 & 2 & 3 & 4 & \cdots \\ 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & & \ddots \end{pmatrix}$$

has vN det 1 but is not even a bdd op

$\Rightarrow$ has no Fredholm det.

# QUEST for RELEVANT EXCEPTION 7.

- Want a seq. $(f_\alpha)_{\alpha\in\mathbb{N}} \subset \mathcal{H}$ that $(\langle f_i, f_j\rangle)_{i,j\in\mathbb{N}}$ has a von Neum. det BUT no Fredholm, e.g., because it is NOT even a bounded op. → So need to charactz. bdness in terms of $f_j$

THM. (QA.): Given $(f_j)_{j\in\mathbb{N}} \subset \mathcal{H}$

$$\Big( (\langle f_i, f_j\rangle)_{i,j\in\mathbb{N}} \text{ is a BOUNDED OP. in } \ell^2(\mathbb{N}) \Big)$$

$$\Updownarrow$$

$\Big( \{f_j\}_{j\in\mathbb{N}}$ is a BESSEL SEQUENCE, i.e.,
$\exists\, c>0$ s.th. $\sum^{\infty} |\langle f_k, g\rangle|^2 \leq c\cdot\|g\|^2 \quad \forall g\in\mathcal{H} \Big)$

7.

**COROL.:** let $(g_k)_{k\in\mathbb{N}} \subset \mathcal{H}$ ONB

def. $\quad f_1 := g_1 \; ; \; f_m := \frac{1}{\sqrt{m}}\, g_1 + g_m$

$\Rightarrow$

- $\|f_m\| \leq 3/2 \quad \forall m$: UNIFMLY BDED
- $\det_{vN}(\langle f_i, f_j\rangle) = 1$

A very unpatholg. seqce deserving $f_1 \wedge f_2 \wedge \dots$

- But $(\langle f_i, f_j\rangle)_{i,j\in\mathbb{N}}$ has NO FREDH. det !

- So in def. of $\bigwedge_{k=1}^{\infty} \mathcal{H}$ via $\det_{vN}$, state $f_1 \wedge f_2 \wedge \dots$ is in unit sphere, while not even in space if we used Fredholm det.

# DESIDERATA to the Rescue! 7. [redacted]

• Ideally $\bigwedge_{k=1}^{\infty}\mathcal{H} \subset \bigotimes_{k=1}^{\infty}\mathcal{H}$ (in vN sense), but for that $f_1\wedge f_2\wedge\cdots$ must be conjugate-multilin. forms over $\ell$-sequences! s.th.

$$f_1\wedge f_2\wedge\cdots : \ell\text{-seq} \longrightarrow \mathbb{C}$$

$$(g_1,g_2,\ldots)\longmapsto \text{“}\det\left(\langle g_j, f_k\rangle_{j,k\in\mathbb{N}}\right)\text{”}$$

and inner product acts: $\quad \|$

$$\langle g_1\wedge g_2\wedge\cdots, f_1\wedge f_2\wedge\cdots\rangle_{\wedge\mathcal{H}} := f_1\wedge f_2\wedge\cdots(g_1,g_2,\ldots)$$

$\Rightarrow$ $(g_j)_j, (f_j)_j$ must be $\ell$-seqces too!

• So the desiderata roughly tells us that only e-sequences matter. But then...

THM. (QA.): Given $(f_j)_{j\in\mathbb{N}} \subset \mathcal{H}$ linear indep. seq. if e-seq $\left(\prod_{j=1}^{\infty}\|f_j\| < \infty\right)$, then:

(i) $(\langle f_i, f_j\rangle)_{i,j\in\mathbb{N}}$ has von Neumann det $\alpha \in \mathbb{R}^+_0$

(it makes sense to define $f_1 \wedge f_2 \wedge \dots$)

• if moreover, $\alpha \neq 0$ (hence $f_1 \wedge f_2 \wedge \dots \neq \vec{0}$):

(ii) $(f_j)_j$ is a Bessel seq. & $(\langle f_i, f_j\rangle)_{i,j\in\mathbb{N}} \in \mathcal{L}(\ell^2(\mathbb{N}))$

(iii) $\exists$ decompos. $f_j = \eta_j + \chi_j$

for some $\begin{cases} (\eta_j)_j \subset \mathcal{H} \text{ a Riesz Basis of } \operatorname{span}\{f_j\} \\ (\chi_j)_j \subset \mathcal{H} \text{ Bessel s.th } (\|\chi_j\|)_j \in \ell^2(\mathbb{N}) \end{cases}$

any ONB $\mathcal{H}$

(iv) $\sum_{j=1}^{\infty} |f_j\rangle\langle e_j|$ is a Hilbert-Schmidt perturbat. of an injective closed range op.

(v) $(\langle f_i, f_j\rangle)_{ij\in\mathbb{N}}$ is a TRACE-CLASS perturbation of the IDENTITY $\Rightarrow$ has a FREDHOLM DET!

• Thus, restricting to $\mathcal{C}$-seqces we have enough with Fredholm det. For norms of vectors: $\| f_1 \wedge f_2 \wedge \cdots \|^2 = \det(\langle f_i, f_j\rangle_{ij})$

• Still unclear if also enough for crossed:

$$\langle f_1 \wedge f_2 \wedge \cdots, g_1 \wedge g_2 \wedge \cdots \rangle = \det(\langle f_i, g_j\rangle_{ij})$$

COR.: If $(f_j)_j, (g_j)_j \in \mathcal{C}$-seqces & non-zero "norm" then $(\langle f_i, g_j\rangle)_{i,j\in\mathbb{N}}$ is a Hilbert-Schmidt perturbation of a closed range operator.

(to be continued...)

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• Sabemos que $(f_n) \in \mathrm{Bess}(\mathcal{H})$ ~~then~~ bien formado?

$$\left(\langle f_i, f_j\rangle = \delta_{ij} + t_{ij},\ t \text{ trace class}\right) \overset{\text{if } \det_F \neq 0}{\Longrightarrow} (\Longleftarrow) \left(\mathrm{Op}_E((f_n)) = U + T,\ \hookrightarrow f_k = \underbrace{\phi_k}_{\text{ONB}} + \underbrace{t_k}_{\text{something}}\right)$$

Can we use this to prove that any one with $\exists \det_F$ must actually be a $\mathcal{C}$ sequence? (Sería brutal!).

$$\prod_{k=1}^{\infty} \|f_k\|^2 = \prod_{k=1}^{\infty} \|\phi_k + t_k\|^2 = \prod_{k=1}^{\infty} \|\phi_k\|^2 + \|t_k\|^2 + 2\,\mathrm{Re}\langle \phi_k, t_k\rangle$$

$$= \underbrace{\prod^{\infty} \|\phi_k\|^2}_{1} + \prod^{\infty} \|t_k\|^2 + 2$$

$$\mathrm{Op}_E((f_n)) = \sum_{n=1}^{\infty} \left( |\phi_n\rangle\langle e_n| + |t_n\rangle\langle e_n| \right)$$

# The Construction

$\boxed{1}$ For any $(f_j)_j \subset \mathcal{H}$, $\mathcal{C}$-sequence define $f_1 \wedge f_2 \wedge \cdots$ as the alternating conjugate multil. form:

$$f_1 \wedge f_2 \wedge \cdots : \mathcal{C}\text{-seq} \longrightarrow \mathbb{C}$$
$$(g_1, g_2, \cdots) \longmapsto \det_{\forall N}\left(\langle g_j, f_k\rangle_{j,k\in\mathbb{N}}\right)$$

To do: check it is in $\bigotimes_{k=1}^{\infty} \mathcal{H}$!

$\boxed{2}$ Def. vector subspace:

$$W := \operatorname{span}\{f_1 \wedge f_2 \wedge \cdots \mid (f_j)_j \in \mathcal{C}\text{-seq}\}$$

$\boxed{3}$ Def. sesquilinear form s.th:

$$\langle \cdot , \cdot \rangle : W \times W \longrightarrow \mathbb{C}$$

$$\langle g_1 \wedge g_2 \wedge \cdots , f_1 \wedge f_2 \wedge \cdots \rangle := (f_1 \wedge f_2 \wedge \cdots)(g_1, g_2, \cdots)$$

$\boxed{4}$ Def. $\hat{\bigwedge}_{j=1}^{\infty} \mathcal{H}$ as subsp. of altntg. conjg. forms $\Psi$ that are strong limits of $W$ element-$\langle \cdot , \cdot \rangle$-Cauchy seqs., i.e., s.th. $\exists (\Phi_j)_{j \in \mathbb{N}} \subset W$ with:

(i) $\lim_{\ell, k \to \infty} \| \Phi_\ell - \Phi_k \|_{\langle \cdot , \cdot \rangle} = 0$

(ii) $\forall (g_j)_j \in \mathcal{C}$, $\lim_{\ell \to \infty} \Phi_\ell((g_j)_j) = \Psi((g_j)_j)$

$\boxed{5}$ Extend $\langle\cdot,\cdot\rangle$ from $W$ to whole $\hat{\bigwedge}_{j=1}^{\infty}\mathcal{H}$ via Cauchy seqces.

$\Rightarrow$ to do: check that indeed, $\left(\hat{\bigwedge}_{j=1}^{\infty}\mathcal{H}, \langle\cdot,\cdot\rangle\right)$ is a Hilbert-space.

$\boxed{6}$ Prove a well-generalized theorem

$\boxed{7}$ Define Sea generated by $(f_j)_{j\in\mathbb{N}}$ as

$$\hat{\bigwedge}_{j=1}^{\infty}{}^{[f_j]}\mathcal{H} := \overline{\operatorname{span}\{\phi_1\wedge\phi_2\wedge\dots\wedge\phi_\ell\wedge\underbrace{f_m\wedge f_{m+1}\wedge\cdots}_{\text{fixed tail}}\}}.$$

- Find equivalence relation $\Gamma$ of pure prods. characterizing belonging to ⊜ Dirac Sea. !

7. [redacted]

$\boxed{8}$ Check $\bigwedge\limits_{j=1}^{\infty} \mathcal{H} = \bigoplus\limits_{\varepsilon \in \Gamma} \left( \bigwedge\limits_{j=1}^{\infty}{}^{\varepsilon} \mathcal{H} \right)$

This is : Dirac-Hilbert Ocean structure!

$\boxed{9}$ Dimensionality study:

Expect $\bigwedge\limits_{j=1}^{\infty} \mathcal{H}$ NON-SEPARABLE & $\bigwedge\limits_{j=1}^{\infty}{}^{\varepsilon} \mathcal{H}$ SEP.

$\boxed{10}$ other equivalence relations characterizing implementability of unitaries,

lifts of operator algebras etc.

"Solve" External field QED?

Thank you

for your

Attention!

- Is $\hat{\wedge}\mathcal{H} < \tilde{\otimes}\mathcal{H}$ a closed subsp.?

  ↳ proj. onto the ∃

- What if symmetric part

$$\underline{\mathcal{U}f_1} \wedge \cdots \wedge \underline{\mathcal{U}f_n} \otimes C\mathcal{U}g_1 \otimes \cdots \otimes C\mathcal{U}g_n$$