SCHRÖDINGER QUANTUM MECHANICS OF COUNTABLY MANY DEGREES OF FREEDOM

What could one mean by "$L^2(\mathbb{R}^\infty, d^\infty x)$" as a wavefunction space?

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BY

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What could one mean by “$L^2(\mathbb{R}^\infty,\ d^\infty x)$” *as a wavefunction space?*

The space $\mathbb{R}^\infty := \prod_{j\in\mathbb{N}} \mathbb{R}$ is the model configuration space of *countably many distinguishable degrees of freedom* and it can parameterize, among others, the expansion coefficients of physical-space fields over an orthonormal basis. In this thesis, the author presents a mathematical structure that embodies the limit $n \to |\mathbb{N}|$ of the wavefunction space $L^2(\mathbb{R}^n, d^n x)$ —a structure symbolically denoted by “$L^2(\mathbb{R}^\infty, d^\infty x)$”. This reveals a rigorous way to formulate quantum theories that employ wavefunctions over field configuration spaces, exposing a common arena for quantum field theories in “Schrödinger picture” —including in particular, the pilot-wave theories with field ontology.

To achieve this, the author uses the postulate that the state space of a composite quantum system is obtained by tensoring together the state spaces of its subsystems. In particular, von Neumann’s infinite tensor product $\otimes_{j\in\mathbb{N}} L^2(\mathbb{R}, dx)$ is considered —which has a unique construction and hence bypasses the lack of a distinguished infinite-dimensional Lebesgue measure “$d^\infty x$”. As it happens with $\otimes_{k=1}^{n} L^2(\mathbb{R}, dx)$ and the wavefunctions over $\mathbb{R}^n$, $\otimes_{j\in\mathbb{N}} L^2(\mathbb{R}, dx)$ is a space of conjugate multilinear forms with no immediate connection to wavefunctions over $\mathbb{R}^\infty$. In the former case, one solves this by noting that the “lifted” position operators $\widehat{q}_k := Id \otimes \cdots \otimes \hat{q} \otimes Id \otimes \cdots \otimes Id$ can be “jointly diagonalized” to obtain $L^2(\mathbb{R}^n, d^n\mu)$ (with some $d^n\mu$ equivalent to $d^n x$): this is a so-called “position representation”. The author considers such “lifted” position operators, now for $\otimes_{j\in\mathbb{N}} L^2(\mathbb{R}, dx)$, and then finds their joint spectral diagonalization space: namely, a “position representation” for $\otimes_{j\in\mathbb{N}} L^2(\mathbb{R}, dx)$. The resulting space is an uncountable direct sum of $L^2(\mathbb{R}^\infty, d^\infty \mu_{\mathfrak{C}})$ spaces with different (but well-behaved) spectral measures $d^\infty \mu_{\mathfrak{C}}$. All this suggests that even for a “quantized scalar field”, the rigorous “Schrödinger picture” state vector is an *uncountable* tuple of L^2-wavefunctions.

Methodologically, the author abstracts the archetype of a pilot-wave theory to guide an explicit decomposition of $\otimes_{j\in\mathbb{N}} L^2(\mathbb{R}, dx)$, which is used as an ansatz diagonal representation. After developing a joint spectral theory for arbitrarily many (strongly) commuting self-adjoint operators on non-separable Hilbert spaces, they prove that a “diagonalization space” for the “lifted” position operators exists indeed, and then they check that the ansatz is an example of such a diagonalization.

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THESIS SUPERVISOR

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EBERHARD KARLS
UNIVERSITÄT
TÜBINGEN

DAAD
Deutscher Akademischer Austauschdienst
German Academic Exchange Service

To the "Tübingen School" —our fantasy:

Ulaş, Gandeeb, Jonte, Cameron, Noah, Ruben and the whole crew (despite not all of you might be that interested): Namitha, Hannah, Simone, Tejal and Andrés.

Because the will to share discoveries with you has been my motive force for all the work that culminated in this thesis, and because this monograph is a wholehearted invitation for you to join me in the excavation of this apparently unexplored gold mine :D

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CONTENTS

PREFACE: ON THE SHOULDERS OF GIANTS

"If I have seen further it is by standing on the shoulders of Giants." (Newton, 1675)

It is no mystery that science is an inextricably social endeavor. Newton's Giants are not individual thinkers, but they are titanic monuments, sculpted by the intellect and creative labor of hundreds if not thousands of scientists from all over the world. No single researcher — not even a reduced group of them— could ever develop their disciplines as they do without the results (positive and negative) of all the fellows that precede (or coexist with) them; namely, without standing on the shoulders of Giants. This dependency often goes unnoticed when the contributions of an individual personality are outlined. The pinnacles they built over the gigantic monuments are zoomed in, detached from the rest of the edifice beneath them. There certainly exist geniuses, magicians, who can erect a second temple over the previous one in a single lifetime, using very few charts left around by fellow "shoulder wanderers". But the fact is that often, if not usually, the blueprints for the pinnacles that successively become the pillars and charts of the further levels of the cathedral, were already around even the greatest geniuses. As such, the usual practice is to pick them up and complete them. Consciously or not, a scientist employs their cunning to find the pieces of these blueprints, they ravel the messily wired loose ends out, and they harness their own creativity to connect them back, filling the remaining gaps of the puzzle with hard work and a clear vision of the desired assemblage. In a sense, it is this dance over the shoulders of Giants that makes up science: looking first up to project ones vision, then down and around to find the tail end and the instructions, and finally inwards to educe the vacant junctions.

This work constitutes a crystalline example of such a practice. As the reader that goes through the provided mathematical proofs will very early realize, the whole set of results we found is just a weaving of the work and blueprints provided by von Neumann (1939), Schmüdgen (2012), Arai (2018), Folland (1999), Willard (2012) and Tao (2011) (among others) —all of which (except von Neumann's) refer to text-books, meaning that they are themselves intricate depictions of the landscapes in certain shoulders.

OUTLINE OF EACH CHAPTER

In **Chapter 1**, we motivate the need of an ontology in a physical theory and in particular, in a quantum field theory (QFT). We showcase the adequacy of a *field* ontology for such theories by using non-relativistic quantum electrodynamics as a case study. In particular, we abstract the structure of a "pilot-wave theory" and apply it to the case study to understand the "photon" as a phenomenon emergent from a well-defined underlying field —following D. Bohm. Most importantly, we find that the case study will only be made rigorous after making precise sense of "$\lim_{n\to\infty} L^2(\mathbb{R}^n, d^n x)$" or "$L^2(\mathbb{R}^\infty, d^\infty x)$", symbols that have (yet) no universally accepted meaning.

In **Chapter 2**, we interpret "$\lim_{n\to\infty} L^2(\mathbb{R}^n, d^n x)$" as a limit taken on the measure. We start by proving that *there is no well-behaved Lebesgue measure on* $\mathbb{R}^\infty$*:* neither as a product measure nor as a translation invariant measure. We then prove that the Lebesgue measure $d^n x$ is actually replaceable for QM over $\mathbb{R}^n$: any mutually absolutely continuous (mut. a.c) measure gives the same probabilistic predictions and pilot-wave trajectories. This opens the door to use other measures, which despite being mut. a.c to $d^n x$ do have well-behaved infinite products —namely, certain *probability* measures. Interestingly, we find that two different sequences of probability measures $(d\mu_j)_{j=1}^{\infty}$, $(d\nu_j)_{j=1}^{\infty}$ need not yield mut. a.c infinite product measures $d^\infty \mu$, $d^\infty \nu$, even when all their finite partial products $d^n \mu$, $d^n \nu$ are mut. a.c to $d^n x$ —and hence to each other. That is, although each such $d^\infty \mu$ yields a well-behaved $L^2(\mathbb{R}^\infty, d^\infty \mu)$ that "cuts-off" to the usual $L^2(\mathbb{R}^n, d^n x)$ for every $n \in \mathbb{N}$, the $L^2(\mathbb{R}^\infty, d^\infty \mu)$ spaces can be inequivalent for different choices of $(d\mu_j)_{j=1}^{\infty}$. This leaves an "arbitrary choice problem".

In **Chapter 3**, after noting that $L^2(\mathbb{R}^n, d^n x)$ is canonically identified with $\bigotimes_{k=1}^{n} L^2(\mathbb{R}, dx)$, we interpret "$\lim_{n\to\infty} L^2(\mathbb{R}^n, d^n x)$" as a limit on the tensor product —instead of the measure. Unlike with the product measure, von Neumann proved there is essentially a unique way to build an infinite tensor product (ITP) $\bigotimes_{k\in\mathbb{N}} L^2(\mathbb{R}, dx)$. We recall the definition and main properties of this space including a decomposition into (uncountably many) mutually orthogonal subspaces which we call the *"layers of the ITP"*. In order to make this decomposition practical, we develop *the theory of arbitrary direct sums of arbitrary Hilbert spaces* —which we could not find in the literature. Unfortunately, at the end of the chapter, we find that $\bigotimes_{k\in\mathbb{N}} L^2(\mathbb{R}, dx)$ does not provide an obvious way to build a pilot-wave theory on $\mathbb{R}^\infty$ because there is no clear "Born rule".

In **Chapter 4**, we find a way to reconcile the issues with the two previous chapters by merging them together. First, we develop *practical result about $L^2(\mathbb{R}^\infty, d^\infty \mu)$ spaces* —which we could not find in the literature. Next, we prove that there is an obvious unitary identification between some special layers of $\bigotimes_{k\in\mathbb{N}} L^2(\mathbb{R}, dx)$ and certain $L^2(\mathbb{R}^\infty, d^\infty \mu)$ spaces —"obvious" because the unitary has roughly the same shape as the canonical unitary identifying $\bigotimes_{k=1}^{n} L^2(\mathbb{R}, dx)$ with $L^2(\mathbb{R}^n, d^n x)$. In a second instance, we prove that actually *every* layer of $\bigotimes_{k\in\mathbb{N}} L^2(\mathbb{R}, dx)$ admits such an identification with some $L^2(\mathbb{R}^\infty, d^\infty \mu_{\mathfrak{C}})$. The measures $d^\infty \mu_{\mathfrak{C}}$ of the resulting spaces are given by certain generators of the layers in a natural way. As a consistency check, we find that no matter which generator we employ to construct the measure of a layer, the resulting measures are mut. a.c. Finally, we put everything together and construct a map $\mathscr{W}_{\mathfrak{R}}$ unitarily decomposing $\bigotimes_{k\in\mathbb{N}} L^2(\mathbb{R}, dx)$ into a direct sum of $L^2(\mathbb{R}^\infty, d^\infty \mu_{\mathfrak{C}})$ spaces —one per each layer of the ITP. In doing so, we find an obvious candidate for a "Born rule"-measure on $\bigotimes_{k\in\mathbb{N}} L^2(\mathbb{R}, dx)$ and we prove that it is independent of the generator set $\mathfrak{R}$ used for to build the decomposition map $\mathscr{W}_{\mathfrak{R}}$ —hence, we consider $\mathfrak{R}$ to be a "choice of basis". This yields an obvious structure for pilot-wave theories and Schrödinger QM over $\mathbb{R}^\infty$ and we conjecture that this is precisely what one means by "$\lim_{n\to\infty} L^2(\mathbb{R}^n, d^n x)$" as a wavefunction space.

In **Chapter 5**, we suggest a third way to interpret "$\lim_{n\to\infty} L^2(\mathbb{R}^n, d^n x)$" and then we prove that it coincides with the $\mathscr{W}_{\mathfrak{R}}$ decomposition of Chapter 4. Given $\hat{q}$ is the position operator of $L^2(\mathbb{R}, dx)$, one can see $L^2(\mathbb{R}^n, d^n x)$ as the "joint diagonalization space" of all the "lifted" position operators $Id \otimes \cdots \otimes \hat{q} \otimes Id \otimes \cdots \otimes Id$ acting on $\bigotimes_{k=1}^{n} L^2(\mathbb{R}, dx)$. In order to generalize this idea to $\bigotimes_{k\in\mathbb{N}} L^2(\mathbb{R}, dx)$, we develop a *spectral theorem of joint diagonalization for arbitrarily many self-adjoint operators whose PVMs commute and act on arbitrary Hilbert spaces* —after all, we could not find this either in the literature. Then, we define the "lift" of an unbounded operator to a general ITP, yielding rigorous "lifted" position operators $\widehat{q}_k := Id \otimes \cdots \otimes \hat{q} \otimes Id \otimes \cdots$,

$k \in \mathbb{N}$, for $\bigotimes_{k\in\mathbb{N}} L^2(\mathbb{R}, dx)$. Instead of directly looking for the PVMs of these operators, we first determine the PVMs and functional calculus of $\hat{x}_k$, the obvious position operators of each $L^2(\mathbb{R}^\infty, d^\infty \mu_{\mathfrak{C}})$. Then, we conjecture that these $\hat{x}_k$ are the "push-forwards" by $\mathscr{W}_{\mathfrak{R}}$ of the $\widehat{q}_k$ acting on $\bigotimes_{k\in\mathbb{N}} L^2(\mathbb{R}, dx)$ when restricted to a single layer's $L^2(\mathbb{R}^\infty, d^\infty \mu_{\mathfrak{C}})$ space. To prove it, we develop *a theory to pullback PVMs and functional calculi via* $\mathscr{W}_{\mathfrak{R}}$ *from the* L^2 *spaces to the full ITP*. In doing so, we end up proving that the decomposition of Chapter 4 is precisely the joint diagonal representation of the position operators $\widehat{q}_k$ acting on $\bigotimes_{k\in\mathbb{N}} L^2(\mathbb{R}, dx)$. Namely, $\mathscr{W}_{\mathfrak{R}}$ gives the "position representation" of $\bigotimes_{k\in\mathbb{N}} L^2(\mathbb{R}, dx)$ and the "choice of basis" $\mathfrak{R}$ turns out to be a choice of *spectral basis*. This vindicates the claim that in the context of QM, "$\lim_{n\to\infty} L^2(\mathbb{R}^n, d^n x)$" (or "$L^2(\mathbb{R}^\infty, d^\infty x)$") means the decomposition of $\bigotimes_{k\in\mathbb{N}} L^2(\mathbb{R}, dx)$ given by $\mathscr{W}_{\mathfrak{R}}$.

In **Chapter 6**, we sketch an application of the developed theory. In particular, we prove that there is a canonical way to change the space $\mathbb{R}^\infty$ in "$L^2(\mathbb{R}^\infty, d^\infty x)$" by $\ell^2(\mathbb{N}, \mathbb{R})$. This provides a rigorous way to do Schrödinger QM over $\ell^2(\mathbb{N}, \mathbb{R})$ and as such, over any real separable Hilbert space $\mathcal{K}$ —which can model a field configuration-space. We observe that in the resulting L^2 spaces over $\mathcal{K}$ (known as *wavefunctional spaces*), linear combinations of the lifted position operators $\widehat{q}_k$ act as "multiplication by the argument field" —constituting a rigorous realization of the so-called, *field operators*.

In **Chapter 7**, we observe that von Neumann already anticipated in 1939 that his ITP construction would lead to results akin to ours —namely, in the intersection of rigorous QFT and probability theory. At the same time, we observe that today's community of mathematical physics seems to be deliberately forgetful about the full ITP construction. In particular, von Neumann's objectives for the ITP have not permeated to the modern research in QFT. We provide an (admittedly non-conscientious) bibliographic study partially answering questions like: "what went wrong with the ITP? Why nobody seems to have found our results in all these decades? If they did, why did they discard them? Why nobody uses the ITP in QFT although it seems to be an obvious structure for that?"

Since the ITP is conspicuous by its absence in modern textbooks, in **Appendix A**, we provide a review of its construction following the contents of von Neumann's original paper, but framed in an alternative scaffold. Lastly, in **Appendix B** we expose the obvious representation of the canonical commutation relations (CCR) of $\otimes_{k\in\mathbb{N}} L^2(\mathbb{R}, dx)$, highlighting the relation between Fock space, the CCR representations, the ITP and the "quantum vacuum".

HELP TO READ THE THESIS

In order to ease the reading of this thesis we provide two auxiliary chapters at the very end of the book —such that the reader with a physical copy has them on hand.

In the auxiliary chapter on **Notation** the reader will find a clarification of the employed abbreviations, mathematical conventions and symbols —although the abbreviations and symbols will be introduced in the main text as well. We recommend to read the "Conventions" section before delving into the main matter.

Finally, in the **Index of Results**, we provide the page where each definition, lemma, proposition, theorem and corollary is stated.

1

PHYSICAL MOTIVATION

According to the physics folklore, quantum mechanics (QM) should admit two styles of mathematical formalization. In vague terms, one is a formalization where the central dynamical object is a "wavefunction" (a function over the space of possible configurations of a system and a member of some Hilbert space): the so-called "Schrödinger picture"; the other one is a formalization where the central dynamical objects are "observable operators" (a class of operators acting on the space of "wavefunctions"): the so-called "Heisenberg picture".

Arguably, the latter has been essentially the main focus of the various programs to make quantum field theory (QFT) rigorous. This fact is manifest in the very foundation, after all, Wightman's axioms of what a QFT should be make *"vacuum expectation values of observable operators"* the central object (Streater and Wightman, 1964). In particular, it is clearly manifest in the name of the leading approach, called *algebraic* QFT (Arai, 2018; Baez et al., 2014). There, one abstractly defines algebras of *"observable operators"* without even explicitly alluding to a particular "wavefunction" space until the very end of the construction. The fact that no such approach has so-far successfully built a realistic interacting particle QFT[1] already encourages us to look for approaches based instead on a "Schrödinger picture".

However, the main motivation of the author for such a quest resides in a more basic claim: a satisfactory physical theory should provide a well-defined, coherent and consistent *narrative* to its mathematical structure beyond its *instrumental predictive backbone* —meaning, *beyond* a mere relation between the numbers seen in a measurement device and the mathematical objects of the theory. The reason why such a claim seems obvious to the author is very simple: if a physical theory is only to provide predictions for experimental outcomes, a *neural network* is certainly a more suitable tool. The reason why a physical theory is not thought to be a *"black box" algorithm* for experimental outcome prediction, and physics is not a branch of *machine learning*, is that in addition, physical theories are supposed to provide a *precise explanation, understanding or mechanism* by which the outcome turns out to be what is predicted to be. Surprisingly, this very trivial observation, so obvious to any first year physicist, is a "sacrilege" against the general consensus in physics. The ruling myth, promulgated by very eminent personalities like Richard Feynman, can explicitly be found in the beloved physicist's lectures and outreach books:

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[1] As Strocchi (2013) clearly puts: *"after more than fifty years of QFT we are still in the embarrassing situation of not knowing a single non-trivial (even non-realistic) model of QFT in 3+1 dimensions, allowing a non-perturbative control."*

> *"The more you see how strangely Nature behaves, the harder it is to make a model that explains how even the simplest phenomena actually work. So theoretical physics has given up on that."*
>
> (Feynman, 1966)

> *"No one has found any machinery behind the law. No one can 'explain' any more than we have just 'explained.' No one will give you any deeper representation of the situation. We have no idea about a more basic mechanism from which these results can be deduced."*
>
> (Feynman et al., 1965)

> *"I think I can safely say that nobody understands quantum mechanics. So do not take the lecture [...] feeling that you really have to understand in terms of some model what I am going to describe [...] Do not keep saying to yourself, [...] 'But how can it be like that?' because you will get 'down the drain', into a blind alley from which nobody has escaped. Nobody knows how it can be like that.."*
>
> (Feynman, 1967)

In view of such authoritative assertions, our claim above on what a physical theory should be is rendered as an outdated demand, one that QM forced us to abandon. Fortunately, the fact is that Feynman's claims were already false back when he exposed them: David Bohm (1952a,b) had already provided a simple and clear way to understand in mechanistic terms why quantum particles behave as they do. Now, we should not blame Feynman and his coetaneous for their misunderstanding. They were in their right to be reticent given the novelty of Bohm's discovery and its apparent contradiction of von Neumann's 1932 *no-go theorem*.[2] What is less acceptable is that still today, 60 years later, Feynman's is the mainstream position in academia.[3] Today there is not just one way, but a plethora of mathematically fully rigorous ways to satisfactorily "complete" the sparse ontology[4] of the QM from Feynman's times —as reviewed for instance in (Tumulka, 2022) and (Maudlin, 2019). We forward the reader to these textbooks for the details.

Now, our observation is that perhaps due to the persistent believe on ideas like Feynman's above, in the mainstream approaches to both non-rigorous and rigorous QFT, there is essentially no effort to provide a "complete ontological picture" or an "underlying mechanism". Even worse, a serious explanation of how the image of the world comes about from QFT is rarely given. We claim that such a "blind" quest is the origin of the paramount difficulties found in the field. As such, our aim is to add a contribution to the rigorization of QFT, guided by the quest of an ontologically well-posed theory —inspired by recent proposals with the same guide such as those of Dürr et al. (2005), Struyve (2010) or Deckert et al. (2016), among others.

Our present insight is that almost all "satisfactory" quantum theories in the above sense (e.g., Bohmian mechanics and the GRW "collapse theory") have been formulated in a *Schrödinger*

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[2] In those days, there was a well-known misunderstanding of von Neumann's *no-go theorem* (VonNeumann, 1932) that wrongly led physicists believe there was absolutely no way to explain QM in terms of additional variables, like particle trajectories *à la* Newton. See (Bell, 1966; Hermann, 1935) to understand the precise misunderstanding. Bohmian mechanics is indeed a clearcut counterexample.

[3] One just needs to check Nature's last survey (Gibney, 2025): more than 40% of researchers working on QM believe that the correct way to interpret QM is either the "Copenhagen interpretation" (which provides no definite picture of reality between "measurements", just as Feynman used to claim), either that it is yet unknown or that an interpretation is not even needed at all.

[4] By the *ontology* of a theory we mean what *exists* or *is* according to that theory.

picture: i.e., by representing the wavefunctions over a *configuration-space* (namely, using a so-called *"position representation"* wavefunction, instead of just an abstract vector). Given that the canonical model for such Hilbert spaces is $L^2(\mathbb{R}^n, d^n x)$, our main aim is to clarify which is the analogous model space for settings with countably many degrees of freedom. Namely, we want to make precise sense of what one would like to call "$L^2(\mathbb{R}^\infty, d^\infty x)$", the "limit" of $L^2(\mathbb{R}^n, d^n x)$ as $n \to |\mathbb{N}|$.

1.1 Configuration-Space

The tradition for an ontologically satisfactory quantum theory is to introduce a notion of *primitive ontology* (Allori, 2007), whose dynamics is determined by a "wavefunction" in the Schrödinger picture. To give a primitive ontology is to postulate in addition to a wavefunction (which is a rather abstract mathematical object), the existence of something occurring in 4-dimensional spacetime that explains or mediates our experience of empirical observations; namely, it is the part of the theory's ontology that explicitly provides the manifest image of the world that we see —which a wavefunction on its own cannot do (Tumulka, 2022). In particular, all the approaches described in say, Tumulka's (2022) textbook, do this by introducing a parametrization space for the entities (particles, fields etc) that according to the theory, configure the possible images of the world. It is called *configuration-space* and the minimum number of real-valued parameters needed to fully specify each such configuration is the number of *degrees of freedom.* For example, if the world was "made of" N point-like particles possessing a determinate position in a 3-dimensional space (model-able as $\mathbb{R}^3$), then $3N$ real numbers would be necessary and sufficient to specify the configuration of the world at a given time. We would say that the configuration-space is $\mathbb{R}^{3N}$ and that there are $3N$ degrees of freedom. As an alternative example, if the world was "made of" a continuous distribution of substance with a varying density at each point of space (model-able as $\mathbb{R}^3$), to fully specify the configuration of the world at a fixed time, one would need to prescribe a continuous choice of real number per point in $\mathbb{R}^3$ —namely, a function in $\mathcal{C}^0(\mathbb{R}^3, \mathbb{R})$,[5] known as a "continuous *scalar field"*. In that case, we would say that $\mathcal{C}^0(\mathbb{R}^3, \mathbb{R})$ is the configuration-space and that there are infinitely many degrees of freedom.

From the examples just given, it is clear that a notion of time or simultaneity makes the description simpler. As such, we will defer relativistic discussions for later work and assume for this work that the spacetime we experience is a "classical" spacetime:

Definition 1. We define a *"classical" spacetime* to be a 4-dimensional differentiable manifold M with a preferred choice of foliation $\{\Sigma_t\}_{t\in\mathbb{R}}$ parametrized by $t \in \mathbb{R}$ and equipped with a $(0, 2)$-type tensor field $g \in \Gamma(T^0_2 M)$ that within each leaf Σ_t of the foliation pulls-back to a flat Riemannian metric g_t by the inclusion $i : \Sigma_t \to M$. We assume that the parameter of the foliation t represents a "global time", while each leaf of the foliation, (Σ_t, g_t) is a simultaneous-event plane that we perceive as "3-space" and we call *physical space.*[a] ♦

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[a] Such a "classical" spacetime is exemplified by any relativistic spacetime (which means, a 4-dimensional time orientable and time oriented Lorentzian manifold) when it is globally hyperbolic, has flat leaves and one of its foliations is singled out.

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[5] $\mathcal{C}^0(\mathbb{R}^3, \mathbb{R})$ is the space of continuous maps from $\mathbb{R}^3$ to $\mathbb{R}$.

In this work, we are going to explore configuration-spaces of an infinite but *countable* number of degrees of freedom. Their canonical mathematical model is $\mathbb{R}^\infty$:

Definition 2. Given an arbitrary index set I and a family of sets $\{X_j\}_{j\in I}$, their *Cartesian product* is defined to be:

$$\prod_{j\in I} X_j := \left\{ x : I \to \bigcup_{j\in I} X_j \;\middle|\; x(j) \in X_j \quad \forall j \in I \right\}. \tag{1.1}$$

That is, it is the set of mappings $x : j \in I \longmapsto x(j) \in X_j$. We will denote $x \in \prod_{j\in I} X_j$ also as $(x_j)_{j\in I}$, where $x_j := x(j)$. In particular, one defines the k-th *evaluation map* (or *projection to k-th factor)* as $\pi_k : \prod_{j\in I} X_j \to X_k$, $x \longmapsto \pi_k(x) := x_k$ for each $k \in I$. ◆

Definition 3. We denote by $\mathbb{R}^\infty$ the countable Cartesian product $\mathbb{R} \times \mathbb{R} \times \cdots$. ◆

The positions in 3-dimensional space for a countable number of point-like particles can be parametrized by $\mathbb{R}^\infty$, but also the scalar fields belonging to a real separable Hilbert space: in a separable Hilbert space, there is a countable orthonormal basis (ONB), so, the sequences of possible coefficients $\ell^2(\mathbb{N}, \mathbb{R})$ can be used to fully parametrize such fields, but, $\ell^2(\mathbb{N}, \mathbb{R}) \subseteq \mathbb{R}^\infty$. We will exemplify this in §1.3.

1.2 Pilot-Wave Theories

We guide our quest towards the meaning of "$L^2(\mathbb{R}^\infty, d^\infty x)$" in QM by the *pilot-wave theory* archetype:

Definition 4. The *archetype of a pilot-wave theory* of $n \in \mathbb{N}$ degrees of freedom with global time parameter $t \in \mathbb{R}$ is the following set of items.

(i) The postulate that $\mathbb{R}^n$ (as a topological space[6]) parametrizes the possible configurations of some system (i.e., each point corresponds to a different ontological arrangement of the system at each fixed *time* $t \in \mathbb{R}$).

(ii) A *Schrödinger picture model on* $\mathbb{R}^n$, by which we mean what follows.

(a) As a mathematical structure,

- the specification of a unit vector $\psi_0 \in L^2(\mathbb{R}^n, d^n x)$ called "initial" *wavefunction*, (where $d^n x$ is the Lebesgue measure of $\mathbb{R}^n$) and
- the specification of a self-adjoint densely defined operator $(D(H), H)$ acting on $L^2(\mathbb{R}^n,\ d^n x)$, called *Hamiltonian.* Note that via the functional calculus it generates for each $t \in \mathbb{R}$ an operator $U_t := exp(-\frac{i}{\hbar}Ht)$, constituting a strongly continuous one parameter group[7] of unitary operators (SCOPUG) acting on $L^2(\mathbb{R}^n, d^n x)$.[8] $\hbar$ is a positive constant called *Planck constant.*

(b) As a "law of physics",

- the postulate that this SCOPUG yields the *dynamical law* for ψ, which denotes a time dependent vector in $L^2(\mathbb{R}^n, d^n x)$ called *wavefunction.* Namely, that $\psi : t \in \mathbb{R} \longmapsto$

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[6] We want to assume no more structure in view of the fact that $\mathbb{R}^\infty$ will admit no norm.

[7] By group we mean that $U_t U_s = U_{t+s}$ for all $t, s \in \mathbb{R}$.

[8] Equivalently, by Stone's theorem ——see Theorem 6.2 in (Schmüdgen, 2012)—, one could specify a SCOPUG $\{U_t\}_{t\in\mathbb{R}}$ on $L^2(\mathbb{R}^n, d^n x)$ and uniquely recover a Hamiltonian $(H, D(H))$ generating it as $U_t = exp(-\frac{i}{\hbar}Ht)$.

$\psi_t := U_t\psi_0$. Equivalently (by Proposition 6.5 in (Schmüdgen, 2012)), if $\psi_0 \in D(H)$, ψ is the unique solution to the differential equation $i\hbar\frac{\mathrm{d}}{\mathrm{d}t}\psi_t = H\psi_t$, which is called the *Schrödinger equation (of Hamiltonian H).*

(iii) A trajectory-based *primitive ontology,* by which we mean what follows.

(a) As a mathematical structure,

- the specification of a *guidance law*, i.e., the mapping of each wavefunction $\psi : t \mapsto U_t\psi_0$ to some "flow"

$$\begin{array}{rccc} X^\psi : & \mathbb{R}\times\mathbb{R}^n & \longrightarrow & \mathbb{R}^n \\ & (t, x_0) & \longmapsto & X_t^\psi(x_0) \end{array}$$

satisfying that $\{X_t^\psi(\cdot)\}_{t\in\mathbb{R}}$ is a strongly continuous[9] family of homeomorphisms with $X_0^\psi = Id_{\mathbb{R}^n}$ (the identity of $\mathbb{R}^n$) that is *equivariant* with the measure $|\psi_t|^2 d^n x$, i.e.,

$$\int_{x\in B} |\psi_0|^2(x) d^n x = \int_{x\in X_t^\psi(B)} |\psi_t|^2(x) d^n x, \quad \forall t \in \mathbb{R},\ \forall B \in \mathfrak{B}(\mathbb{R}^n). \tag{1.2}$$

(b) As a "law of physics",

- the postulate that the system has an actual configuration $x_0 \in \mathbb{R}^n$ at $t = 0$ which is unknown to us but is "sampled" from a $|\psi_0|^2 d^n x-$distribution, and the postulate that it follows the deterministic trajectory $t \mapsto X_t^\psi(x_0)$ at all times (so, by equivariance, the configuration of the system is $|\psi_t|^2-$distributed at all $t \in \mathbb{R}$). As such, X^ψ is the *ensemble of possible trajectories of the system.*

(iv) As a corollary, item (iii) explains the main predictive backbone of items (i) and (ii): the so-called *Born Rule.* Namely, that the "probability" that at time $t \in \mathbb{R}$ the system is found in the configuration $x \in B$, for some $B \in \mathfrak{B}(\mathbb{R}^n)$, is is:

$$\text{“}\mathbb{P}(x \in B \ \text{ at } \ t)\text{”} := \int_{x\in B} |\psi_t|^2(x) d^n x. \tag{1.3}$$

◆

It is called a *pilot-wave* theory because it tells us that the possible trajectories of the system are *piloted* or guided by the *wave*function (explaining the wave behavior but "particle look" of fundamental particles in standard QM). The closest analogy is that of classical Hamiltonian mechanics: the wavefunction over configuration-space $\mathbb{R}^n$ is the analogue of a time dependent Hamiltonian over phase space $\mathbb{R}^{2n}$ (Dürr et al., 1995). Also there, one can compute an ensemble of trajectories (namely, the flow lines of the Hamiltonian vector field in phase-space), of which, each one represents a possible trajectory of the system. In that case, the ensemble of trajectories is equivariant with the measure $\rho_t d^{2n}x$ solving the *Liouville equation.*

Now, exactly as happens in Hamiltonian statistical mechanics, our definition requires a clarification: we did not make explicit what the "system" we are talking about represents. If it is to model a laboratory preparation (say, some cold atoms in an isolated vacuum box), then it makes sense to use the words "sampled" and "probability" in a frequentist sense of probability theory. After all, in principle one can repeat the preparation and the experimental determination of its configuration arbitrarily often. Certainly, in such a case, the archetype does not prescribe what happens after we intervene in the evolution of the system —say, in order to determine experimentally the configuration $x \in \mathbb{R}^n$. For that we should include the degrees of freedom of

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[9] Meaning that for any choice $x_0 \in \mathbb{R}^n$, the path $t \mapsto X_t^\psi(x_0)$ is continuous in $\mathbb{R}^n$.

the intervening device in the configuration-space of the "system". It is this why, for a fundamental theory, one considers that the "system" consists of the whole Universe, including us and the measurement devices. But then it is unclear what "sampled" or "probability" in the definition mean because there is only one "repetition" of the system (that we know about). In such a situation, one must change *"sampled from a $|\psi_t|^2 d^n x$ distribution"* (or "$|\psi_t|^2$-distributed") by *"typical according to a measure of typicality given by $|\psi_t|^2 d^n x$"* (or $|\psi_t|^2 d^n x$-typical). Note that the $|\psi_t|^2 d^n x$-notion of typicality is distinguished in a pilot-wave theory because it is time-independent, i.e., it is invariant under the transport by the flow of possible trajectories X^ψ. That is precisely the role played by the equivariance postulate (Dürr et al., 1992).

Example 1. The most famous quantum theory of this type is *Bohmian mechanics*, where one assumes that ontologically, there are $N \in \mathbb{N}$ point-like particles in the Universe, each with a position in physical space at each time. Then, one defines $n := 3N$ and $H := -\sum_{k=1}^{N} \frac{\hbar^2}{2m_k} \nabla_k^2 + V$ (in a suitable domain), where

$$V(x_1, ..., x_N) := \frac{1}{2} \sum_{j=1}^{N} \sum_{k=1,\ k\neq j}^{N} \left(\frac{1}{4\pi\varepsilon_0} \frac{e_j e_k}{|x_j - x_k|} - \frac{G m_j m_k}{|x_j - x_k|} \right), \tag{1.4}$$

represents the so-called Coulomb interaction potential (e_j are the electric charge constants and ε_0 is the electric constant) and Newtonian's gravitational interaction (where m_j are the masses of the particles and G is the gravitation constant).[10] One then defines X_t^ψ to be the flow lines of the time dependent vector field

$$v_t^\psi(x) := \ m^{-1}\ \mathrm{Re}\left\{ \frac{-i\hbar\nabla\psi_t(x)}{\psi_t(x)} \right\}, \qquad m := \begin{pmatrix} m_1 & 0 & & \\ 0 & m_2 & \ddots & \\ & \ddots & \ddots & 0 \\ & & 0 & m_n \end{pmatrix}, \tag{1.5}$$

which naturally satisfy the equivariance condition. As a mere corollary of these definitions, one can derive the rest of assumptions and results of textbook non-relativistic QM, such as: the existence of subsystems with an effective wavefunction that obey Schrödinger equations with different Hamiltonians (say, time dependent ones), the "*collapse* phenomenon" for these subsystem wavefunctions, the phenomenon called "decoherence", the "observable operator" and POVM conundrum etc. See (Dürr et al., 1992; Dürr and Teufel, 2009) for the details.

In this setting, each trajectory $t \mapsto X_t^\psi(x_0)$ automatically satisfies the differential equation:

$$m \frac{\mathrm{d}^2}{\mathrm{d}t^2} X_t^\psi(x_0) = -\nabla\Big(V(x) + \mathcal{Q}^\psi(x)\Big)\Big|_{x=X_t^\psi(x_0)} \qquad \forall t \in \mathbb{R}, \tag{1.6}$$

where $\mathcal{Q}^\psi$ is the so-called *quantum potential*

$$\mathcal{Q}^\psi(x,t) := \sum_{k=1}^{N} \frac{-\hbar^2}{2m_j} \frac{\nabla_j^2 |\psi|(x,t)}{|\psi|(x,t)}. \tag{1.7}$$

Remarkably, equation (1.6) is Newton's second law with an additional (classically unexpected) potential energy term. Hence, not only Bohmian mechanics provides a mechanistic explanation of QM, but it allows us to carry the intuitions we gained in Newtonian mechanics a bit further, together with a clean way to understand why certain objects behave more or less classically. In a sense, in this theory it is crystal clear what one means by the Schrödinger equation being the "quantization of Newtonian mechanics". ♦

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[10] Only in this first chapter, we will denote a multiplication operator with the same symbol as the function by which it multiplies.

1.3 An Example to Motivate our Quest

In this section we will review a quantum theory that is routinely employed by the quantum optics community (Cohen-Tannoudji et al., 1989; Kira and Koch, 2011). It is a theory that has so-far no obvious way to be made rigorous (except using the "cut-off trick" we will mention), and one that could be rigorized if we successfully complete the purpose of the present document. Note that the pilot-wave theory through which we will make it understandable was already given by Bohm (1952b), so technically we will be saying nothing new in this section. Still, it will serve as an example to understand how $\mathbb{R}^\infty$ can also parametrize a field configuration-space.

Note that *we will now turn to a physicist-style rigor until the next subsection.*

Consider the classical description of N point-like particles (say, electrons) of position-velocity coordinates $\{(\vec{r}_j, \dot{\vec{r}}_j)\}_{j=1}^N \subset \mathbb{R}^{6N}$, electric charges $\{e_j\}_{j=1}^N$ and masses $\{m_j\}_{j=1}^N$. Consider also an electromagnetic 4-potential field $(\phi_\cdot, \vec{A}_\cdot) : \mathbb{R}^3 \to \mathbb{R}^4$, composed by fields on the second Sobolev space $H^2(\mathbb{R}^3, \mathbb{R})$, and denote by $(\dot{\phi}_\cdot, \dot{\vec{A}}_\cdot)$ the fields representing their associated velocities.[11]

Denoting by c the speed of light, by ε_0 the electric constant and by μ_0 the magnetic constant, the Lagrangian of such a classical system is given by

$$L\Big((\vec{r}_j, \dot{\vec{r}}_j)_{j=1}^N, \vec{A}_\cdot, \dot{\vec{A}}_\cdot, \phi_\cdot\Big) = \sum_{j=1}^N \left(\frac{1}{2} m_j |\dot{\vec{r}}_j|^{\;2} + \frac{e_j}{c}\, \dot{\vec{r}}_j \cdot \vec{A}_{\vec{r}_j} - e_j \phi_{\vec{r}_j}\right) + \frac{\varepsilon_0}{2} \int_{\vec{r}\in\mathbb{R}^3} \Big(|\vec{E}_{\vec{r}}|^2 - c^2 |\vec{B}_{\vec{r}}|^2\Big) d^3r,$$

where the electric and magnetic fields are defined respectively as $\vec{E}_{\vec{r}} := -\dot{\vec{A}}_{\vec{r}} - \vec{\nabla}\phi_{\vec{r}}$ and $\vec{B}_{\vec{r}} := \vec{\nabla} \times \vec{A}_{\vec{r}}$.[a] The generalized Euler-Lagrange equations[b] of L yield exactly the Maxwell equations (the two that do not follow from the definitions of $\vec{E}$ and $\vec{B}$) and the Newton's second law with the Lorentz and Coulomb forces. The resulting equations are poorly defined because each particle is centered in a singularity of the total force field (among other things). We will fix this issue following the physicist practice (Kira and Koch, 2011). Impose the Coulomb gauge ($\vec{\nabla} \cdot \vec{A} = 0$) and explicitly write $\phi_{\vec{r}} = \sum_{j=1}^N \frac{e_j}{4\pi\varepsilon_0 |\vec{r} - \vec{r}_j|}$, which is the solution to the Gauss equation in Coulomb gauge for a point-charge distribution $\{\vec{r}_j\}_{j=1}^N$. This removes the degrees of freedom in ϕ (now it is fixed). Then, defining $V_{j,\ell}(|\vec{r}_j - \vec{r}_\ell|) := e_j e_\ell / (4\pi\varepsilon_0 |\vec{r}_j - \vec{r}_\ell|)$, the evaluation of ϕ in the Lagrangian above yields a sum $\frac{1}{2}\sum_{j=1}^N \sum_{\ell=1}^N V_{j,\ell}$, which explicitly manifests the singularities mentioned before —exactly in each $j = \ell$ term. One then argues that assuming "there is no self-interaction" or by "re-scaling" the energy, it is possible to omit all the $j = \ell$ potential energy contributions and just leave $\frac{1}{2}\sum_{j=1}^N \sum_{\ell \neq j} V_{j,\ell}$.

Next, one considers a symplectic structure for the composite of the usual phase space of N particles and the infinite dimensional phase space of $\vec{A}_{\vec{r}}$ and its canonically conjugate variable $\vec{\pi}_{\vec{r}} := \dot{\vec{A}}_{\vec{r}}$ (it is possible to do this rigorously —see for instance 9.4.2 in Abraham et al. (2001)).

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[a] Two of the Maxwell equations are engraved in these definitions: evaluating $\vec{A} = \vec{A}(t)$ and $\dot{\vec{A}} = \frac{\mathrm{d}}{\mathrm{d}t}\vec{A}(t)$, we immediately get that $\vec{\nabla} \times \vec{E}_{\vec{r}}(t) \equiv -\frac{\mathrm{d}}{\mathrm{d}t}\vec{B}_{\vec{r}}(t)$ and $\vec{\nabla} \cdot \vec{B}_{\vec{r}}(t) \equiv 0$.

[b] See 2.4C in (Abraham et al., 2001) for a mathematically rigorous way to take functional derivatives and a derivation of the Euler-Lagrange equations with field variables.

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[11] Note that we put the argument of the fields as a subscript $\phi_{\vec{r}}, \vec{A}_{\vec{r}}$, instead of $\phi(\vec{r}), \vec{A}(\vec{r})$ in order to strengthen the analogy between the index j in the particle degrees of freedom $\vec{r}_1, ..., \vec{r}_N$ and the index $\vec{r}$ for the degrees of freedom of the fields $\phi_{\vec{r}}, \vec{A}_{\vec{r}}$.

A generalized Legendre transformation then leaves us with a formal Hamiltonian that (after an integration by parts and a vector calculus identity) looks like:

$$H = \sum_{j=1}^{N} \frac{(\vec{p}_j - e_j \vec{A}_{\vec{r}_j})^2}{2m_j} + \frac{1}{2} \sum_{j} \sum_{\ell \neq j} V_{j,\ell}(|\vec{r}_j - \vec{r}_\ell|) + \frac{\varepsilon_0}{2} \int_{\vec{r} \in \mathbb{R}^3} (|\dot{\vec{A}}|^2_{\vec{r}} - c^2 \vec{A}_{\vec{r}} \cdot \nabla^2 \vec{A}_{\vec{r}}) d^3 r. \tag{1.8}$$

In order to describe electrodynamics in a cavity, the physical-space is considered to be modeled by $\Omega := (-\frac{L}{2}, \frac{L}{2})^3$ for some fixed $L > 0$. Then, one can assume periodic boundary conditions so that essentially any square integrable function inside the box can be expanded using only sinusoids of commensurate wavelengths with the box, $\vec{k} \in (\frac{2\pi}{L})\mathbb{Z}^3$. In a similar vein, one assumes (Kira and Koch, 2011) that there is a countable number of $H^2(\Omega, \mathbb{R}^3)$ solutions —indexed by $\vec{k} \in \mathbb{Z}^3(2\pi/L)$— for the constrained eigenvalue problem

$$\begin{cases} \Delta \vec{u}_{\vec{k}}(\vec{r}) \equiv -|\vec{k}|^2 \vec{u}_{\vec{k}}(\vec{r}) \\ \vec{\nabla} \cdot \vec{u}_{\vec{k}}(\vec{r}) \equiv 0 \end{cases} . \tag{1.9}$$

Presumably, these “modes” $\vec{u}_{\vec{k}}$ (which are essentially plane waves of frequency $\vec{k}$) yield an orthonormal basis (ONB) ranging the space $G := \{\vec{f} \in H^2(\Omega, \mathbb{R}^3) \mid \vec{\nabla} \cdot \vec{f} = 0\}$, namely, those $\vec{A}_\cdot \in H^2(\Omega, \mathbb{R}^3)$ that satisfy the Coulomb gauge. As such, for each $\vec{A}_\cdot \in G$, there exists a unique sequence of coefficients $\{\alpha_{\vec{k}}\}_{\vec{k} \in (\frac{2\pi}{L})\mathbb{Z}^3} \subset \mathbb{C}$ for which[12]

$$\vec{A}_{\vec{r}} \equiv \sum_{\vec{k} \in (\frac{2\pi}{L})\mathbb{Z}^3} \alpha_{\vec{k}} \ \vec{u}_{\vec{k}}(\vec{r}). \tag{1.10}$$

This one-to-one correspondence between $(\alpha_{\vec{k}})_{\vec{k} \in \frac{2\pi}{L}\mathbb{Z}^3} \in \ell^2\Big(\frac{2\pi}{L}\mathbb{Z}^3, \mathbb{C}\Big)$ and $\vec{A} \in H^2(\Omega, \mathbb{R}^3)$ implies we can use the expansion coefficients $\alpha_{\vec{k}}$ as the field’s degrees of freedom instead of the vector field $\vec{A}$. Plugging them in the Hamiltonian (1.8), one obtains[13]

$$H\Big((\vec{r}_j, \vec{p}_j)_{j=1}^N, (\alpha_{\vec{k}}, \dot{\alpha}_{\vec{k}})_{\vec{k} \in \frac{2\pi}{L}\mathbb{Z}^3}\Big) = \sum_{j=1}^{N} \frac{\Big(p_j - e_j \overbrace{\sum_{\vec{k}} \alpha_{\vec{k}} \ \vec{u}_{\vec{k}}(\vec{r}_j)}^{\vec{A}_{\vec{r}_j}} \ \Big)^2}{2m_j} + \frac{1}{2} \sum_{j} \sum_{\ell \neq j} V_{j,\ell}(|\vec{r}_j - \vec{r}_\ell|) + \tag{1.11}$$
$$+ \frac{1}{2} \sum_{\vec{k} \in \frac{2\pi}{L}\mathbb{Z}^3} \Big(\dot{\alpha}^2_{\vec{k}} + c^2 |\vec{k}|^2 \ \alpha^2_{\vec{k}}\Big).$$

Notice that the last term of this Hamiltonian describes a harmonic oscillator for each of the expansion coefficients —interacting non-trivially with the other degrees of freedom via the first term. Since there is a well-known quantization prescription for a system classically behaving as a harmonic oscillator, one can get an ansatz for the quantum counterpart of the theory by the same trick. But, before applying the trick, in the absence of a proper “$L^2(\mathbb{R}^\infty, d^\infty x)$”, one assumes what is called an *“ultraviolet cut-off”*, i.e., that all coefficients $\alpha_{\vec{k}}$ above a certain finite magnitude $|\vec{k}|$ are zero, leaving only, say, $M \in \mathbb{N}$ degrees of freedom $\{\alpha_{\vec{k}_j}\}_{j=1}^M$ to describe the vector field $\vec{A}$.[14] With a slight abuse of notion we will denote $\alpha_{\vec{k}_j}$ as α_j. One then performs a

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[12] In particular, $\vec{A}_{\vec{r}}(t) \equiv \sum_{\vec{k} \in (\frac{2\pi}{L})\mathbb{Z}^3} \alpha_{\vec{k}}(t) \ \vec{u}_{\vec{k}}(\vec{r})$, i.e., if the field depends on time, the expansion coefficients will depend on time.

[13] Now as a function of the expansion coefficients $\alpha_{\vec{k}}$ and their associated velocities $\dot{\alpha}_{\vec{k}}$. Physicists understand this to be a *canonical transformation between infinite dimensional symplectic spaces*.

[14] Equivalently, since the cut-off puts a limit on the frequency of the Fourier modes used to express $\vec{A}$, it supposes a limitation on the spatial variation that a field $\vec{A}$ could take.

canonical quantization *à la* Dirac[15] and the resulting theory is called (non-relativistic) quantum electrodynamics (Cohen-Tannoudji et al., 1989; Kira and Koch, 2011). Each system is described by a wavefunction $\psi_t \in L^2(\mathbb{R}^{3N+M},\ \mathbb{C},\ d^{3N+M}x)$ —of arguments $\psi_t(\vec{r}_1, ...\vec{r}_N, \alpha_1, ..., \alpha_M)$—, that for all $t \in \mathbb{R}$ obeys the following Schrödinger equation:[16]

$$i\hbar\,\frac{\partial}{\partial t}\psi_t(\{\vec{r}_j,\alpha_j\}_j) = \left[\sum_{j=1}^{N}\frac{\left(-i\hbar\vec{\nabla}_{\vec{r}_j} - e_j\vec{A}(\vec{r}_j;\{\alpha_\ell\}_\ell)\right)^2}{2m_j} + \frac{1}{2}\sum_{\substack{j,\ell=1\\ j\neq\ell}}^{N} V_{j,\ell} + \sum_{j=1}^{M}\left(-\frac{\hbar^2}{2}\frac{\partial^2}{\partial\alpha_j^2} + \frac{c^2|\vec{k}_j|^2}{2}\alpha_j^2\right)\right]\psi_t(\{\vec{r}_j,\alpha_j\}_j) \tag{1.12}$$

with $\vec{A}(\vec{r}_j;\{\alpha_\ell\}_\ell) := \sum_{\ell=1}^{M}\alpha_{\vec{k}_\ell}\ \vec{u}_{\vec{k}_\ell}(\vec{r}_j)$.

The motivation for our work is then clear. The given model is cut-off dependent and a priori it is unclear if taking a larger cut-off M makes the model approach any particular limit. One needs a rigorous notion of "$\lim_{M\to\infty} L^2(\mathbb{R}^M, d^M x)$" = "$L^2(\mathbb{R}^\infty, d^\infty x)$" to avoid such artificial choices.

Last but not least, note that $|\psi_t(\{\vec{r}_j,\alpha_j\}_j)|^2$ obeys the continuity equation

$$\frac{\mathrm{d}|\psi_t|^2}{\mathrm{d}t} = -\vec{\nabla}\cdot\left(|\psi_t|^2\ \vec{v}_t\right) \quad \text{for}\ \ \vec{v} := (\vec{v}^{\ r_1}, ...\vec{v}^{\ r_N}, v^{\alpha_1}, ..., v^{\alpha_M}) : \tag{1.13}$$

$$\vec{v}_t^{\ r_l}(\{\vec{r}_j,\alpha_j\}_j) := \mathrm{Re}\left\{\frac{-i\hbar\vec{\nabla}_{\vec{r}_l}\psi_t(\{\vec{r}_j,\alpha_j\}_j)}{m_j\,\psi_t(\{\vec{r}_j,\alpha_j\}_j)}\right\} \quad \text{and} \quad v_t^{\alpha_l}(\{\vec{r}_j,\alpha_j\}_j) := \mathrm{Re}\left\{\frac{-i\hbar\frac{\partial}{\partial\alpha_l}\psi_t(\{\vec{r}_j,\alpha_j\}_j)}{\psi_t(\{\vec{r}_j,\alpha_j\}_j)}\right\}. \tag{1.14}$$

By integrating the continuity equation, one can see that the flow-lines of v_t^ψ make a $|\psi|^2 d^{3N+M}x$-equivariant ensemble of trajectories $X_t^\psi := (\vec{R}_t^1, ...\vec{R}_t^N,\ \Lambda_t^1, ..., \Lambda_t^M)$. With that, as in any pilot-wave theory one can postulate the existence of N point-particles and an actual electromagnetic field, whose trajectory is determined by their joint "initial" configuration $x_0 \in \mathbb{R}^{3M+N}$ as $t \longmapsto (\vec{R}_t^1(x_0), ..., \vec{R}_t^N(x_0))$ and $t \longmapsto \vec{A}_{\vec{r}}^{x_0}(t) := \sum_{j=1}^{M}\Lambda_t^j(x_0)\,\vec{u}_{\vec{k}_j}(\vec{r})$. Note that this would imply the existence of well-defined subordinate electric and magnetic fields $\vec{E}^{x_0}(\vec{r},t), \vec{B}^{x_0}(\vec{r},t)$, given by the formulas of the beginning.

It is worth noting that akin to what happened in Bohmian mechanics, these trajectories satisfy automatically the Newton-Lorentz equation and the Maxwell equations with classically unexpected additional terms:[17]

$$m_j\frac{\mathrm{d}^2\vec{R}_t^{\,j}(x_0)}{\mathrm{d}t^2} = e_j\,\vec{E}^{x_0}(\vec{R}_t^j(x_0),t) + e_j\,\frac{\mathrm{d}\vec{R}_t^{\,j}(x_0)}{\mathrm{d}t}\times\vec{B}^{x_0}(\vec{R}_t^{\,j}(x_0),t) - \vec{\nabla}_{r_j}\mathcal{Q}^\psi|_{\vec{R}_t^j(x_0),\Lambda_t^k(x_0)} \tag{1.15}$$

$$\vec{\nabla}\times\vec{B}^{x_0}(\vec{r},t) = \frac{1}{c^2}\frac{\partial\vec{E}^{\ x_0}(\vec{r},t)}{\partial t} + \mu_0\vec{J}^{x_0}(\vec{r},t) - F(\mathcal{Q}^\psi)|_{\vec{R}_t^{\,k}(x_0),\Lambda_t^k(x_0)} \tag{1.16}$$

where $\vec{J}^{x_0}(\vec{r},t) := \sum_j e_j\delta(\vec{r} - \vec{R}_t^j(x_0))$ is the charge current of classical electrodynamics (with δ the Dirac delta distribution), $\mathcal{Q}^\psi$ is the "quantum potential" (essentially equal to (1.7)), and F is a map related to the inverse Fourier transform.

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[15] Exchanging position and momentum variables by operators satisfying canonical commutation relations.

[16] We denote $\vec{r}_1, ...\vec{r}_N, \alpha_1, ..., \alpha_M$ by $\{\vec{r}_j,\alpha_j\}_j$.

[17] Two of the Maxwell equations are given as in footnote 1. The Gauss equation is "hard-coded" in our explicit solution to the scalar potential $\phi_{\vec{r}}^{x_0}(t) = \sum_k \frac{e_k}{4\pi\varepsilon_0|\vec{r} - \vec{R}_t^k(x_0)|}$. Hence, in this model only the Ampère-Maxwell law has a quantum "perturbation".

What are "Photons"?

In the presented pilot-wave theory, the charged particles are said to have a *particle ontology*, meaning that we postulate there exist material point-like particles representing them. Meanwhile, the electromagnetic (EM) field is said to have a *field ontology*, which means that we postulate the existence of substantive fields taking determinate values at each point of physical space and time, incarnating the EM field. This was precisely also the case in classical electrodynamics. Now, it is often said that quantization shows that the electromagnetic field is composed of light particles called *photons*. But in the presented pilot-wave theory there is nothing like this. How can it be? In understanding how photons can be seen as apparent but fictitious particles in this model —namely, as *virtual particles*—, we will find a connection with the so-called "Fock space" that will illuminate the discussions of Chapter 6 and Appendix B.

If there were no charged particles ($N = 0$), i.e., if we considered a free EM field, then the classical Hamiltonian (1.11) would be reduced to $H(\alpha_1, ..., \alpha_M, \dot{\alpha}_1, ..., \dot{\alpha}_M) = \sum_{j=1}^{M} H_{free}^{\vec{k}_j}(\alpha_j, \dot{\alpha}_j)$ with $H_{free}^{\vec{k}_j}(\alpha_j, \dot{\alpha}_j) := \frac{1}{2}\dot{\alpha}_j^2 + \frac{c^2|\vec{k}_j|^2}{2}\alpha_j^2$. That is, α_j would behave as independent harmonic oscillators of characteristic frequency $c|\vec{k}_j|$. After the obvious quantization over $L^2(\mathbb{R}^M, d^M x)$, the associated operator would be

$$\hat{H} = \sum_{j=1}^{M} \hat{H}_{free}^{\vec{k}_j} \quad \text{with} \quad \hat{H}_{free}^{\vec{k}_j} := \frac{-\hbar^2}{2}\frac{\partial^2}{\partial \alpha_j^2} + \frac{c^2|\vec{k}_j|^2}{2}\alpha_j^2. \tag{1.17}$$

By §7.3.1 in Porta (2019), $\hat{H}_{free}^{\vec{k}}$ is essentially self-adjoint on the Schwartz functions of $L^2(\mathbb{R}, dx)$ and there exists an ONB of its eigenfunctions that we will denote by $\{\phi_n^{\vec{k}}\}_{n\in\mathbb{N}_0} \subset L^2(\mathbb{R}, dx)$. The n-th element is a Gaussian times the n-th Hermite polynomial $h_n(x) := (-1)^n e^{x^2} \frac{\mathrm{d}^n e^{-x^2}}{\mathrm{d}x^n}$,

$$\phi_n^{\vec{k}}(\alpha) := \frac{1}{\sqrt{2^n n! \lambda_{\vec{k}} \sqrt{\pi}}} \, e^{-\frac{1}{2}\left(\frac{\alpha}{\lambda_{\vec{k}}}\right)^2} h_n\Big(\frac{\alpha}{\lambda_{\vec{k}}}\Big) = \frac{1}{\sqrt{n! 2^n \lambda_{\vec{k}} \sqrt{\pi}}} \left(\frac{\alpha}{\lambda_{\vec{k}}} - \lambda_{\vec{k}} \frac{\partial}{\partial \alpha}\right)^n e^{-\frac{1}{2}\left(\frac{\alpha}{\lambda_{\vec{k}}}\right)^2}, \tag{1.18}$$

where $\alpha \in \mathbb{R}$ and $\lambda_{\vec{k}} := \sqrt{\hbar/(c|\vec{k}|)}$. With that, we can get an ONB of the full Hilbert space $L^2(\mathbb{R}^M, d^M x)$ by taking tensor products of the possible combinations: $\{\Psi^{(n_1,...,n_M)}\}_{n_1,...,n_M\in\mathbb{N}_0} \subset L^2(\mathbb{R}^M, d^M x)$, where $\Psi^{(n_1,...,n_M)}(\alpha_1, ..., \alpha_M) := \phi_{n_1}^{\vec{k}_1}(\alpha_1) \cdots \otimes \phi_{n_M}^{\vec{k}_M}(\alpha_M)$. They are eigenvectors of the full Hamiltonian (1.17) —with associated energy eigenvalue $\sum_{\ell=1}^{M} \hbar c|\vec{k}_\ell|(n_\ell + \frac{1}{2})$.

Now, what physicists call *"electromagnetic vacuum"* is the ground state of the quantized free electromagnetic field (Cohen-Tannoudji et al., 1989), explicitly given by the wavefunction

$$\Psi^{(0,...,0)}(\alpha_{k_1}, ..., \alpha_{k_N}) := \phi_0^{\vec{k}_1}(\alpha_{k_1}) \cdots \phi_0^{\vec{k}_M}(\alpha_{k_M}) \overset{(\text{for some } C \in \mathbb{R})}{=} C \, e^{-\frac{1}{2}\sum_{k=1}^{M}\left(\frac{\alpha_k}{\lambda_{\vec{k}}}\right)^2}. \tag{1.19}$$

Because its associated probability density $|\Psi^{(0,...,0)}|^2$ is a Gaussian around $\alpha_{k_j} = 0 \; \forall j$, most[18] pilot-wave trajectories of the expansion coefficients $\{\Lambda_t^k\}_{k=1}^M$ will be close to zero $\forall t \in \mathbb{R}$. Thus, typical worlds will have an electromagnetic potential $\vec{A}_{\vec{r}} \cong 0$ (and hence electric and magnetic fields $\vec{E}, \vec{B} \cong 0$). That explains why we say $\Psi^{(0,...,0)}$ is a "vacuum". However, there is a (small but) non-null set of trajectories $t \mapsto \Lambda_t^k(x_0)$ that are a bit further from the origin, leading to non-zero EM fields. These are the origin of the so-called *"vacuum quantum fluctuations"*: QM allows that a cavity where there are no charged particles to source an EM field evolves into a non-zero EM field.

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[18] Recall that the typicality measure is $|\Psi^{(0,...,0)}|^2 d^M x$.

One says that there is only *one* photon and has frequency $\vec{k}_j$ when the wavefunction is

$$\Psi^{e_j}(\alpha_{k_1}, ..., \alpha_{k_N}) := \phi_0^{\vec{k}_1}(\alpha_{k_1}) \cdots \phi_1^{k_j}(\alpha_{k_j}) \cdots \phi_0^{\vec{k}_M}(\alpha_{k_M}) \propto \alpha_{\vec{k}_j} \; e^{-\frac{1}{2}\sum_{\ell=1}^{M}\left(\frac{\alpha_\ell}{\lambda_{\vec{k}_\ell}}\right)^2}, \tag{1.20}$$

where we denote $e_j := (0, ..., 0, 1, 0, ..., 0)$ (1 in the j-th entry). One says there are two photons of frequency $\vec{k}_j$ if the system has wavefunction

$$\Psi^{2e_j}(\alpha_{k_1}, ..., \alpha_{k_N}) = \phi_0^{\vec{k}_1}(\alpha_{k_1}) \cdots \phi_2^{\vec{k}_j}(\alpha_{k_j}) \cdots \phi_0^{\vec{k}_M}(\alpha_{k_M}) \propto (\alpha_{k_j}^2 - 1) \; e^{-\frac{1}{2}\sum_{\ell=1}^{M}\left(\frac{\alpha_{k_\ell}}{\lambda_{\vec{k}_\ell}}\right)^2}. \tag{1.21}$$

"And so on." That is, a higher excitation of the "harmonic oscillator" in mode α_j corresponds to an additional photon of that j-th frequency. In this sense, photons are nothing but a way of speaking about the excitation levels of each mode. They are called *virtual particles* if one wants to emphasize that they do not really represent particles, but some sort of "excitations".

The Photon Fock Space

Taking photons seriously in the above theory is exactly the way to conceive the so-called *Fock space representation.* Let $\hat{\alpha}_j$ be the operator multiplying by α_j and $\hat{\pi}_j := -i\partial_{\alpha_j}$, both defined in the common dense domain $\mathcal{S}(\mathbb{R}^n)$ (Schwartz functions over $\mathbb{R}^n$). Define for $j \in \{1, .., M\}$ the (scaled) ladder operators

$$\hat{a}_j := \frac{1}{\sqrt{2}}\Big(\frac{\hat{\alpha}_j}{\lambda_{k_j}} + i\lambda_{k_j}\hat{\pi}_j\Big) \quad \text{and} \quad \hat{a}_j^\dagger := \frac{1}{\sqrt{2}}\Big(\frac{\hat{\alpha}_j}{\lambda_{k_j}} - i\lambda_{k_j}\hat{\pi}_j\Big). \tag{1.22}$$

Up to a constant factor, $\hat{a}_\ell^\dagger$ raises the "excitation level" of the ℓ-th mode: $\hat{a}_\ell^\dagger \Psi^{(n_1,...,n_\ell,...,n_M)} = \sqrt{n_\ell + 1}\ \Psi^{(n_1,...,n_\ell+1,...,n_M)}$, i.e.,

$$\hat{a}_\ell^\dagger(\phi_{n_1}^{\vec{k}_1} \otimes \cdots \otimes \phi_{n_\ell}^{\vec{k}_\ell} \otimes \cdots \otimes \phi_{n_M}^{\vec{k}_M}) = \sqrt{n_\ell + 1}\ \phi_{n_1}^{\vec{k}_1} \otimes \cdots \otimes \phi_{n_\ell+1}^{\vec{k}_\ell} \otimes \cdots \otimes \phi_{n_M}^{\vec{k}_M}; \tag{1.23}$$

while $\hat{a}_\ell$ reduces one "excitation level": $\hat{a}_\ell \Psi^{(n_1,...,n_\ell,...,n_M)} = \sqrt{n_\ell}\Psi^{(n_1,...,n_\ell-1,...,n_M)}$, i.e.,

$$\hat{a}_\ell(\phi_{n_1}^{k_1} \otimes \cdots \otimes \phi_{n_\ell}^{k_\ell} \otimes \cdots \otimes \phi_{n_M}^{k_M}) = \sqrt{n_\ell}\ \phi_{n_1}^{k_1} \otimes \cdots \otimes \phi_{n_\ell-1}^{k_\ell} \otimes \cdots \otimes \phi_{n_M}^{k_M}, \tag{1.24}$$

—where we define $\Psi^{(n_1,..,n_M)} := \vec{0}$ if some of the n_k is negative.

That is, $\hat{a}_\ell^\dagger$ *"creates a photon"* of frequency $\vec{k}_\ell$, while $\hat{a}_\ell$ *"annihilates a photon"* of the same frequency. Since $\hat{a}_\ell \Psi^{(0,...,0)} = 0$ for all ℓ, one says that $\hat{a}_\ell$ *"annihilates the vacuum".*

Because of all that, we can obtain any element $\Psi^{(n_1,...,n_M)}$ in the ONB mentioned above by applying the "creation operators" $\hat{a}_\ell^\dagger$ enough times on the "vacuum", namely,

$$\Psi^{(n_1,...,n_M)} = c_{n_1,...,n_M}\ (\hat{a}_1^\dagger)^{n_1} \cdots (\hat{a}_M^\dagger)^{n_M}\ \Psi^{(0,...,0)}, \tag{1.25}$$

where the normalization constant is $c_{(n_1,...,n_M)} = (n_1! \cdots n_M!)^{-1/2}$. Following the above idea of what a photon is, one defines the *m-photon sector* of $L^2(\mathbb{R}^M, d^M x)$ to be

$$\mathcal{H}^{(m)} := \operatorname{span}\Big\{\ \Psi^{(n_1,...,n_M)}\ \Big|\ n_1 + \cdots + n_M = m\ \Big\}, \tag{1.26}$$

such that by the ONB property, one can sectorize:

$$L^2(\mathbb{R}^M,\ d^M x) = \bigoplus_{m=0}^{\infty} \mathcal{H}^{(m)} \quad \text{with} \quad \dim \mathcal{H}^{(m)} = \binom{M+m-1}{m}. \tag{1.27}$$

Now, the m-th sector has exactly the dimensionality of the symmetrized $(\mathbb{C}^M)^{\otimes m}$ space, denoted by $Sym((\mathbb{C}^M)^{\otimes m})$ and defined as follows:

Definition 5. Given a Hilbert space $\mathcal{H}$:

- For $m \in \mathbb{N}$, we define $\mathcal{H}^{\otimes m} := \underbrace{\mathcal{H} \otimes \cdots \otimes \mathcal{H}}_{m}$ and $\mathcal{H}^{\otimes 0} := \mathbb{C}$.
- Given $m \in \mathbb{N}$ and given that S_m is the permutation group of m elements, we denote by $Sym : \mathcal{H}^{\otimes m} \to \mathcal{H}^{\otimes m}$ the unique (see Prop. 2.4 in (Arai, 2018)) bounded linear operator satisfying
$$Sym(f_1 \otimes \cdots \otimes f_m) := \frac{1}{\sqrt{m!}} \sum_{\sigma \in S_m} f_{\sigma(1)} \otimes \cdots \otimes f_{\sigma(m)} \tag{1.28}$$
for all elementary tensor products $f_1 \otimes \cdots \otimes f_m \in \mathcal{H}^{\otimes m}$. ♦

One calls $\mathcal{F}_b(\mathbb{C}^M) := \bigoplus_{m=0}^{\infty} Sym((\mathbb{C}^M)^{\otimes m})$ the *Fock space of M degrees of freedom.*[19] In the Fock space, one calls $\Omega := (1, 0, 0, ...)$ the "vacuum vector" and one defines the normalized symmetric canonical vectors

$$|n_1, ..., n_M\rangle := \frac{1}{\sqrt{n_1! \cdots n_M!}} Sym\Big(e_1^{\otimes n_1} \otimes \cdots \otimes e_M^{\otimes n_M}\Big), \quad n_1, ..., n_M \in \mathbb{N}, \tag{1.29}$$

with $|0, .., 0\rangle := \Omega$. They form an ONB. As such, given the mentioned equal dimensionality, there is a natural unitary identification between $L^2(\mathbb{R}^M, d^M x)$ and $\mathcal{F}_b(\mathbb{C}^M)$ and it is given by identifying $\Psi^{(n_1,...,n_M)}$ with $|n_1, ..., n_M\rangle$,

$$\begin{array}{rccc} U: & L^2(\mathbb{R}^M, d^M x) & \longrightarrow & \mathcal{F}_b(\mathbb{C}^M) \\ \psi = & \displaystyle\sum_{(n_1,..,n_M)\in\mathbb{N}_0^M} \beta_{(n_1,...,n_M)} \Psi^{(n_1,....,n_M)} & \longmapsto & \displaystyle\sum_{(n_1,..,n_M)\in\mathbb{N}_0^M} \beta_{(n_1,...,n_M)} |n_1, ..., n_M\rangle. \end{array} \tag{1.30}$$

This induces in $\mathcal{F}_b(\mathbb{C}^M)$ "creation and annihilation" operators $\hat{A}_\ell := U\ \hat{a}_\ell\ U^{-1}$ and $\hat{A}_\ell^\dagger := U\ \hat{a}_\ell^\dagger\ U^{-1}$ with

$$\hat{A}_\ell^\dagger\, |n_1, ..., n_\ell, ..., n_M\rangle = \sqrt{n_\ell + 1} |n_1, ..., n_\ell{+}1, ..., n_M\rangle, \quad \hat{A}_\ell\, |n_1, ..., n_\ell, ..., n_M\rangle = \sqrt{n_\ell} |n_1, ..., n_\ell{-}1, ..., n_M\rangle \tag{1.31}$$

where we identify $|n_1, ..., n_M\rangle = \vec{0}$ if $n_k < 0$ for some k. Physicists often directly employ this latter representation (called *occupation number representation*) when talking about photons. Within our pilot-wave theory photons are just virtual particles, so although convenient in certain circumstances, the Fock space representation can end up being misleading. Lastly, for completeness, note that the free Hamiltonian (1.17) can be rewritten in terms of the ladder operators as follows:[20]

$$\hat{H} = \sum_{j=1}^{M} \hbar c |\vec{k}_j| \Big(\hat{a}_j^\dagger \hat{a}_j + \frac{1}{2}\Big). \tag{1.32}$$

How a Photon supposes no Contradiction with the Field Ontology

The reader might be surprised that the pilot-wave theory describes the electromagnetic field as a continuum even if electromagnetic energy is known to be absorbed and emitted in discrete packets during atomic experiments. This is no mystery. Assume we put back a particle in the picture in addition to the electromagnetic field, say, an electron in a hydrogen atom. Such electrons, when isolated from the rest of degrees of freedom, have well-defined energy eigenstates called orbitals: denote them by $\varphi_0, \varphi_1, ...$ Now, assume we have sent against the atom a pulse

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[19] We provide a more general definition in §B.5.

[20] This is how one writes the harmonic oscillator Hamiltonian if directly given in Fock representation.

of laser and we have enclosed the system between mirrors. A laser has a very sharp frequency $\vec{k}$, so, as a first approximation, one could leave only one of the expansion modes, say, $\vec{u}_{\vec{k}}$ —hence, only fields like $\vec{A}_{\vec{r}} \cong \alpha_k \vec{u}_{\vec{k}}$, (namely, plane-waves) are considered. If so, one can find that starting an experiment with a joint wavefunction $\psi(\vec{r}, \alpha_k) = \varphi_0(\vec{r})\phi_1^{\vec{k}}(\alpha_k)$ —namely, with a "single photon" state ϕ_1^k (hence, an ontological EM field that is non-zero) and the ground state of the electron φ_0— then, the Schrödinger equation evolves them very close to $\varphi_1(\vec{r})\phi_0^k(\alpha_k)$ —i.e. to a state describing no photon (so, the ontological EM field most likely being zero) but the electron now in the excited state. Next, by the Schrödinger equation, the state will oscillate back to $\varphi_0(\vec{r})\phi_1^k(\alpha_k)$ and forth to $\varphi_1(\vec{r})\phi_0^k(\alpha_k)$. These are the so-called *Rabi oscillations* (Cohen-Tannoudji et al., 1989), which correspond in the pilot-wave theory with an electron absorbing and emitting EM radiation over and over again —it is in between mirrors. If we left the atom in an open environment and not between mirrors, the EM radiation would be absorbed and emitted only once. Now, this oscillation only happens in a meaningful way when the energy eigenvalue difference between ground state and first excited state of the electron matches that of the incoming photon's. But because the latter is exactly proportional to the frequency $\vec{k}$ of the mode (precisely $\frac{3}{2}\hbar c|\vec{k}|$), absorption and emission only happen if the laser with which we have lighted the atom contains a frequency matching the energy interval between two electron orbitals. This explains the spectral absorption lines being so specific. Likewise, one can explain the photoelectric effect (Bohm, 1952b), among others.

Now, note that there was never an "abrupt absorption" in all the explanation: all the ontological elements had a continuous time evolution. What is more, for most times we have an entangled wavefunction between the electron and the modes of the EM field, so there is no way to attribute a wavefunction to the electron and radiation separately.[a] That is, for generic times during light-matter interaction, the notion of photon from above (in terms of excitations of mode wavepackets) becomes highly ambiguous and hand-wavy. On the contrary, the trajectories $\vec{A}_{\vec{r}}^{x_0}(t)$ and $\vec{R}_t^j$ describing the primitive ontology of the pilot-wave theory are well-defined at all times —despite the entanglement or non-triviality of the joint light-matter wavefunction.

And yet, in a photon-detection-screen, point-like detections occur, seemingly contradicting our continuous EM field. The key is to notice that what one sees in such a screen is not the incident light, but the light emitted by an atom in the screen after it absorbed the incident light —with a single Rabi oscillation as above. Since the emission is sourced from essentially a point (the atom), it looks like a point recording to us. This also used to happen in classical electrodynamics, where it was not understood as even the slightest evidence of light being composed of particles. With all, in the pilot-wave theory, light is "absorbed and emitted by the atom" in a non-abrupt manner with an ontological picture that is well-defined at all times, and still, it correctly predicts the virtual appearance of light-packages that is so helpful to explain quantum light-matter interaction.

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[a] Unless one employs *conditional wavefunctions* (Dürr et al., 1992) or reduced density matrices (Cohen-Tannoudji et al., 1989), which is what one should do to talk about absorption and emission more meaningfully.

2

ATTEMPT 1: TAKE THE LIMIT OF THE MEASURE

In order to make reasonable measure theoretic statements about $\mathbb{R}^\infty$, the first thing to be fixed is its σ-algebra. If we eventually give a topology to $\mathbb{R}^\infty$, it seems reasonable to choose the associated Borel σ-algebra (i.e., the smallest σ-algebra that contains all open sets). Indeed, we want to put a topology on $\mathbb{R}^\infty$, for instance, because we wish to talk about ensembles of trajectories in $\mathbb{R}^\infty$ that move continuously. But then, the next question is: which topology should $\mathbb{R}^\infty$ be given? In view of the "cut-off" approach of physicists, we desire to consider "embeddings" of $\mathbb{R}^n$ with arbitrarily big $n \in \mathbb{N}$ inside $\mathbb{R}^\infty$. In particular, eventually, we might want to identify a subset $U \subseteq \mathbb{R}^n$ with $U \times \mathbb{R} \times \mathbb{R} \times \cdots \subseteq \mathbb{R}^\infty$. In order to do this in a topologically coherent manner, since $\mathbb{R}^n$ has an indisputable topology, we would like that whenever $U \subseteq \mathbb{R}^n$ is an open set, then $U \times \mathbb{R} \times \mathbb{R} \times \cdots \subseteq \mathbb{R}^\infty$ is also an open set. The minimal topology satisfying this (namely, making the projections continuous maps) is the *product topology* of $\mathbb{R}^\infty$.

Definition 6. The *product topology* of $\mathbb{R}^\infty$ is the coarsest topology for which

$$\mathfrak{G}_0 := \left\{ U_1 \times \cdots \times U_j \times \mathbb{R} \times \mathbb{R} \times \cdots \;\middle|\; j \in \mathbb{N}, \quad U_j \subseteq \mathbb{R} \text{ open} \right\}, \tag{2.1}$$

is a topological base. We call $\mathfrak{G}_0$, the *finite cylinders of open sets.* ⧫

Hereafter, when it is not specified we will be assuming that $\mathbb{R}^\infty$ has the product topology and is equipped with its Borel σ-algebra, which we will denote by $\mathfrak{B}(\mathbb{R}^\infty)$.

Definition 7. • Given A is a family of subsets of X, the *σ-algebra of X generated by A*, denoted $\sigma(A)$, is the smallest σ-algebra containing all elements of A, namely,

$$\sigma(A) := \bigcap \left\{ C \;\middle|\; A \subseteq C \subseteq \mathcal{P}(X) \text{ and C is a } \sigma\text{-algebra} \right\}. \tag{2.2}$$

- Let τ be a topology for X. We call $\mathfrak{B}(X) := \sigma(\tau)$ the *Borel σ-algebra of* (X, τ).
- Given measurable spaces $(\Sigma_j, X_j)_{j \in I}$ indexed by an arbitrary index set I, define

$$\mathfrak{c}_0 := \left\{ \pi_j^{-1}(E_j) \;\middle|\; j \in I,\ E_j \in \Sigma_j \right\} = \bigcup_{j \in I} \pi_j^{-1}(\Sigma_j). \tag{2.3}$$

We call them the "finite cylinders" of *measurable* sets. Then, the *product σ-algebra* on $X := \prod_{j \in I} X_j$ is defined to be $\sigma(\mathfrak{c}_0)$ and we denote it by $\odot_{j \in I} \Sigma_j$. In particular, it is the coarsest σ-algebra that makes all projections π_k measurable maps (Tao, 2011). ⧫

Alternative Paths that Lead us to the Same Measurable Space

Lemma 1. Given X, Y are topological spaces with $\mathfrak{B}(X), \mathfrak{B}(Y)$ their respective Borel σ-algebras, it can happen that $\mathfrak{B}(X) \odot \mathfrak{B}(Y) \subsetneqq \mathfrak{B}(X \times Y)$. ♦

Proof: See Example 6.4.3 in (Bogachev, 2007). ☐

Although it may be surprising, as Lemma 1 and Proposition 1.4 in (Folland, 1999) together exemplify, given that B is a base of the topology τ on some space X, it can happen that $\sigma(B) \subsetneqq \sigma(\tau) =: \mathfrak{B}(X)$ (i.e., the σ-algebra generated by the topological basis is just a proper subset of the Borel σ-algebra).[a] Hence, it may happen that finite cylinders of *open* sets $\mathfrak{G}_0$ (see Def. 6) generate a strictly smaller σ-algebra than that generated by the product topology. As a result, one could argue that it is less restrictive for our quest to choose $\sigma(\mathfrak{G}_0)$ as sigma algebra despite we give $\mathbb{R}^\infty$ the product topology. In this line, one could claim that it is more convenient to take the σ-algebra generated by some of the following families:

- $\prod_{j\in\mathbb{N}} U_j$ with $U_j = \mathbb{R}$ for all but one $j \in \mathbb{N}$, and $U_j \subset \mathbb{R}$ is open,
- $\prod_{j\in\mathbb{N}} U_j$ with $U_j \subseteq \mathbb{R}$ arbitrary open sets $\forall j$,
- $\prod_{j\in\mathbb{N}} E_j$ with $E_j = \mathbb{R}$ for all but one $j \in \mathbb{N}$, and $E_j \subset \mathbb{R}$ is Borel measurable,
- $\prod_{j\in\mathbb{N}} E_j$ with $E_j \subseteq \mathbb{R}$ arbitrary Borel measurable sets $\forall j$.

Proposition 1. The σ-algebra generated by any family of the above list equals $\odot_{j\in\mathbb{N}}\mathfrak{B}(\mathbb{R})$ and in the case of $\mathbb{R}^\infty$, $\odot_{j\in\mathbb{N}}\mathfrak{B}(\mathbb{R}) = \mathfrak{B}(\mathbb{R}^\infty)$. ♦

Proof: The third item yields exactly the definition of the product σ-algebra $\odot_{j\in I}\mathfrak{B}(\mathbb{R})$ and because there are only countably many factors, by Proposition 1.3 in (Folland, 1999), the fourth point also generates exactly $\odot_{j\in\mathbb{R}}\mathfrak{B}(\mathbb{R})$. Now, by Proposition 1.4 in (Folland, 1999) and because there are countably many factors, the first two also generate $\odot_{j\in I}\mathfrak{B}(\mathbb{R})$. Finally, because $\mathfrak{B}(\mathbb{R})$ is the Borel σ-algebra of a metrizable and separable topological space, by Lemma 1.2 in (Kallenberg, 1997), $\odot_{j\in I}\mathfrak{B}(\mathbb{R}) = \mathfrak{B}(\mathbb{R}^\infty)$. ***o.ε.δ.***

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[a] The key idea is that in order to obtain τ from B, one takes arbitrary unions of elements in B, while a σ-algebra only needs to be closed under *countable* unions.

Note that in order to generalize our results to an uncountable Cartesian product of $\mathbb{R}$, these technicalities would need to be reviewed.

Now, observe that:

Proposition 2. Given an arbitrary measure space $(Q, \Sigma, d\mu)$ (where Q is a set, Σ a σ-algebra and $d\mu$ a *measure* on Σ), the space $L^2(Q, \mathbb{C}, d\mu)$ is well-defined and is a Hilbert space.

Proof: See Theorem 6.6 in Folland (1999).☐

As such, our first attempt to make "$L^2(\mathbb{R}^\infty,\ d^\infty x)$" rigorous will be to look for a measure $d^\infty\mu$ on the Borel σ-algebra of $\mathbb{R}^\infty$, for which $L^2(\mathbb{R}^\infty, d^\infty\mu)$ is a reasonable $n \to \infty$ generalization of $L^2(\mathbb{R}^n, d^n x)$. That is, we would like to choose a $d^\infty\mu$ that is "the limit of" the Lebesgue measures $d^n x$ as $n \to |\mathbb{N}|$. There are two immediate ways to make sense of this —although none of the two will end up working for us. We explain them in the following two sections.

2.1 Take the infinite product measure of Lebesgue measures

Lemma 2. Given two measurable spaces (X, Σ_X) and (Y, Σ_Y), for any pair of respective σ-finite measures μ_X and μ_Y, there *exists a unique* measure $\mu_X \odot \mu_Y$ on $(X \times Y,\ \Sigma_X \odot \Sigma_Y)$ such that $\mu_X \odot \mu_Y(E_X \times E_Y) = \mu_X(E_X) \cdot \mu_Y(E_Y)$ for all $E_X \in \Sigma_X,\ E_Y \in \Sigma_Y$. We call it the *product measure* (of μ_X and μ_Y).

Given n σ-finite measurable spaces $(X_j, \Sigma_j, d\mu_j)_{j=1}^n$, there *exists a unique* measure $d\mu_1 \odot \cdots \odot d\mu_n$ also denoted $\odot_{j=1}^n d\mu_j$ such that for "product cylinders" $E_1 \times \cdots \times E_n \in \odot_{j=1}^n X_j$ with $E_j \in \Sigma_j$,

$$d\mu_1 \odot \cdots \odot d\mu_n(E_1 \times \cdots \times E_n) = d\mu_1(E_1) \cdots d\mu_n(E_n). \tag{2.4}$$

We call it *the product measure of* $(d\mu_j)_{j=1}^n$. ♦

Proof: See Theorem 1.7.11 in (Tao, 2011) for the case $n = 2$. Iterate the construction finitely many times for arbitrary $n \in \mathbb{N}$. ☐

Following Remark 1.7.12 in (Tao, 2011), if μ_x and μ_Y are *not* both σ-finite, there still exists a product measure as in the Lemma above, but it is in general *not* unique.

For example, there is a a unique product measure of n copies of the Lebesgue measure dx (because it is σ-finite). But $d^n x$ (the Lebesgue measure of $\mathbb{R}^n$) satisfies equation (2.4) (with $d\mu_j = dx$), so, $d^n x = \odot_{j=1}^n dx$. Then, the obvious idea is to look for a measure on $\mathbb{R}^\infty$ that acts as an infinite product measure "$\odot_{j=1}^\infty dx$". Let us build it.

As a product measure of dx, we want to impose that whenever we take a cylinder set $E \times \mathbb{R} \times \mathbb{R} \cdots$ with $E \in \mathfrak{B}(\mathbb{R}^n)$,

$$d^\infty x(E \times \mathbb{R} \times \mathbb{R} \times \cdots) \overset{!}{=} d^n x(E) \cdot \prod_{j \in \mathbb{N} \setminus \{1,..,n\}} dx(\mathbb{R}). \tag{2.5}$$

But since $dx(\mathbb{R}) = +\infty$, all such cylinder sets would need to be assigned infinite measure by $d^\infty x$. We will now prove that such a measure $d^\infty x$ does exist but any measurable set in $\mathfrak{B}(\mathbb{R}^\infty)$ will have infinite measure according to it and it will *not* be σ-finite. In fact, we will prove it for $\prod_{j \in I} \mathbb{R}$ with an arbitrary I of infinite cardinality and the product σ-algebra $\odot_{j \in I} \mathfrak{B}(\mathbb{R})$.

Definition 8. Given a family of sets $\{X_j\}_{j \in I}$ of arbitrary index set I, for a subset of indices $A \subseteq I$, define the *partial projection* map

$$\begin{array}{rccc} \pi_{A \leftarrow I}: & \prod_{j \in I} X_j & \longrightarrow & \prod_{j \in A} X_j \\ & (x_j)_{j \in I} & \longmapsto & (x_j)_{j \in A} \end{array}. \tag{2.6}$$

If it is obvious that we are projecting from I, we will also use π_A. Importantly, given $E \subseteq \prod_{j \in A} X_j$ we will denote

$$(\pi_{A \leftarrow I})^{-1}(E) =: E \times \prod_{j \in I \setminus A} X_j. \tag{2.7}$$

Note that trivially, if $J_1 \subseteq J_2 \subseteq I$ then $\pi_{J_2 \leftarrow J_1} \circ \pi_{J_1 \leftarrow I} = \pi_{J_2 \leftarrow I}$. ♦

Lemma 3. Given sets X, Y, and a family $C \subseteq \mathcal{P}(Y)$,[a] for any $f : X \to Y$ it holds that $\sigma(f^{-1}(C)) = f^{-1}(\sigma(C))$. ♦

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[a] Given a set Y, $\mathcal{P}(Y) := \{A \subseteq Y\}$, i.e., it is the family of all subsets of Y.

Proof: [a] First, since $C \subseteq \sigma(C)$, then, $f^{-1}(C) \subseteq f^{-1}(\sigma(C))$ and because $f^{-1}(\sigma(C))$ is a σ-algebra (unions and complements commute with the pre-image), then the smallest σ-algebra containing $f^{-1}(C)$ must be contained in $f^{-1}(\sigma(C))$, i.e., $\sigma(f^{-1}(C)) \subseteq f^{-1}(\sigma(C))$. For the reverse inclusion, consider $\tilde{C} := \{B \subseteq Y \mid f^{-1}(B) \in \sigma(f^{-1}(C))\}$. It is a σ-algebra[b] and $C \subseteq \tilde{C}$. Hence, $\sigma(C) \subseteq \tilde{C}$, which by definition implies that $f^{-1}(\sigma(C)) \subseteq \sigma(f^{-1}(C))$. ***o.ε.δ.***

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[a] Used as a reference: https://proofwiki.org/wiki/Pre-Image_Sigma-Algebra_of_Generated_Sigma-Algebra

[b] Using that complements and countable unions commute with pre-images, let $B \in \tilde{C}$ i.e., $f^{-1}(B) \in \sigma(f^{-1}(C))$, then $f^{-1}(B^c) = f^{-1}(B)^c \in \sigma(f^{-1}(C))$ so by definition $B^c \in \tilde{C}$. On the other hand, given $(B_j)_{j\in\mathbb{N}} \subseteq \tilde{C}$, then $f^{-1}(\cup_{j\in\mathbb{N}} B_j) = \cup_{j\in\mathbb{N}} f^{-1}(B_j) \in \sigma(f^{-1}(C))$ because $f^{-1}(B_j) \in \sigma(f^{-1}(C))$ for all j. Hence, by definition, $\cup_{j\in\mathbb{N}} B_j \in \tilde{C}$.

Proposition 3. For an arbitrary set I, if $(X_j, \Sigma_j)_{j\in I}$ are measurable spaces, then

$$\mathfrak{A}_0 := \left\{ E_J \times \prod_{j\in I\setminus J} X_j \;\middle|\; J \subseteq I \text{ is finite and } E_J \in \odot_{j\in J}\Sigma_j \right\} \tag{2.8}$$

is a *Boolean algebra* of $\prod_{j\in I} X_j$, which means that, $\mathfrak{A}_0 \subseteq \mathcal{P}(\prod_{j\in I} X_j)$ and that it is closed under complement and *finite* union (exactly as a σ-algebra but with just finite unions instead of countable ones), namely

$$\text{(i) if } A \in \mathfrak{A}_0 \text{ then } (\prod_{j\in I} X_j) \setminus A \in \mathfrak{A}_0. \qquad \text{(ii) If } A_1, ..., A_N \in \mathfrak{A}_0, \text{ then } \bigcup_{j=1}^{n} A_j \in \mathfrak{A}_0.$$

Moreover, it is a generating sub-algebra for $\odot_{j\in I}\Sigma_j$, i.e., $\sigma(\mathfrak{A}_0) = \odot_{j\in I}\Sigma_j$. ♦

Proof: We prove that $\mathfrak{A}_0$ is a Boolean algebra:

(i) Any $A \in \mathfrak{A}_0$ is such that $A = E_J \times \prod_{j\in I\setminus J} X_j = \pi_J^{-1}(E_J)$ for some finite $J \subseteq I$ and $E_J \in \odot_{j\in J}\Sigma_j$. Since complement and pre-image commute, $(\prod_{j\in I} X_j) \setminus A \equiv A^c = \pi_J^{-1}(E_J)^c = \pi_J^{-1}((E_J)^c) = \pi_J^{-1}((\prod_{j\in J} X_j) \setminus E_J)$. Then, because $\odot_{j\in J}\Sigma_j$ is a σ-algebra, it is closed under complement so $(\prod_{j\in J} X_j) \setminus E_J \in \odot_{j\in J}\Sigma_j$, which implies $A^c = \pi_J^{-1}((\prod_{j\in J} X_j) \setminus E_J) \in \odot_{j\in I}\Sigma_j$.

(ii) We prove it for $N = 2$ —from which $N \in \mathbb{N}$ follows by finite induction. Let $A_1, A_2 \in \mathfrak{A}_0$, then for $k \in \{1, 2\}$ $A_k = E_{J_k} \times \prod_{j\in I\setminus J_k} X_j$ for some finite $J_k \subseteq I$ and $E_{J_k} \in \odot_{j\in J_k}\Sigma_j$. Let $J := J_1 \cup J_2$ and define $\tilde{E}_{J_k} := E_{J_k} \times \prod_{j\in J\setminus J_k} X_j$, which is an element of $\odot_{j\in J}\Sigma_j$ by definition. Now, $A_k = \tilde{E}_{J_k} \times \prod_{j\in I\setminus J} X_j = \pi_J^{-1}(\tilde{E}_{J_k})$, so, $A_1 \cup A_2 = \pi_J^{-1}(\tilde{E}_{J_1}) \cup \pi_J^{-1}(\tilde{E}_{J_2}) = \pi_J^{-1}(\tilde{E}_{J_1} \cup \tilde{E}_{J_2})$. Being $\odot_{j\in J} X_j$ a σ-algebra, it is closed under union and hence $\pi_J^{-1}(\tilde{E}_{J_1} \cup \tilde{E}_{J_2}) \in \mathfrak{A}_0$.

- Now, if we unfold the definition of $\mathfrak{A}_0$ using Definitions 7 and 8 but in the "pre-image notation" and we use in $(\star)$ that pre-image and union commute:

$$\mathfrak{A}_0 \overset{\text{(by def)}}{=} \bigcup_{\substack{J\subseteq I\\ \text{finite}}} (\pi_{J\leftarrow I})^{-1}\Big(\sigma\Big(\bigcup_{j\in J} (\pi_{j\leftarrow J})^{-1}(\Sigma_j) \Big)\Big) \overset{\text{(Lem. 3)}}{=} \bigcup_{\substack{J\subseteq I\\ \text{finite}}} \sigma\Big((\pi_{J\leftarrow I})^{-1}\Big(\bigcup_{j\in J} (\pi_{j\leftarrow J})^{-1}(\Sigma_j) \Big)\Big) \overset{(\star)}{=}$$

$$= \bigcup_{\substack{J\subseteq I\\ \text{finite}}} \sigma\Big(\bigcup_{j\in J} (\pi_{J\leftarrow I})^{-1} \circ (\pi_{j\leftarrow J})^{-1}(\Sigma_j) \Big) = \bigcup_{\substack{J\subseteq I\\ \text{finite}}} \sigma\Big(\bigcup_{j\in J} (\pi_{j\leftarrow I})^{-1}(\Sigma_j) \Big).$$

But by definition $\odot_{j\in I}\Sigma_j = \sigma\Big(\bigcup_{j\in I} (\pi_{j\leftarrow I})^{-1}(\Sigma_j)\Big)$. Hence, $\mathfrak{A}_0 \subseteq \odot_{j\in I}\Sigma_j$, such that $\sigma(\mathfrak{A}_0) \subseteq \odot_{j\in I}\Sigma_j$. Finally, $\mathfrak{c}_0$ from Def. 7 is a subset of $\mathfrak{A}_0$, so $\sigma(\mathfrak{c}_0) \subseteq \sigma(\mathfrak{A}_0)$. But, $\sigma(\mathfrak{c}_0) =: \odot_{j\in I}\Sigma_j$.

Therefore, $\sigma(\mathfrak{A}_0) = \odot_{j\in I}\Sigma_j$. ***o.ε.δ.***

As mentioned above, now for arbitrary I, we demand that whatever $d^\infty x$ is, on $\mathfrak{A}_0$ it acts as

$$d^\infty x_0\Big(E \times \prod_{j\in I\setminus J} \mathbb{R}\Big) = d^{|J|}x(E) \cdot \prod_{j\in I\setminus J} dx(\mathbb{R}), \quad \forall J \subseteq I \text{ finite}, \quad \forall E \in \odot_{j\in J}\mathfrak{B}(\mathbb{R}). \tag{2.9}$$

which again, equals infinity if I has infinite cardinality.

Proposition 4. The map $d^\infty x_0 : \mathfrak{A}_0 \longrightarrow [0, +\infty]$ such that $d^\infty x_0(E) := +\infty$ for all $E \in \mathfrak{A}_0 \setminus \emptyset$ and $d^\infty x_0(\emptyset) := 0$ is a *pre-measure* on $\mathfrak{A}_0$, i.e.,

(i) $d^\infty x_0(\emptyset) = 0$ **(ii)** if $E, F \in \mathfrak{A}_0 : E_1 \cap E_2 = \emptyset$ then $d^\infty x_0(E_1 \cup E_2) = d^\infty x_0(E_1) + d^\infty x_0(E_2)$

(iii) if $E_1, E_2, ... \in \mathfrak{A}_0$ pairwise disjoint and $\bigsqcup_{j\in\mathbb{N}} E_j \in \mathfrak{A}_0$, then $d^\infty x_0(\bigsqcup_{j\in\mathbb{N}} E_j) = \sum_{j\in\mathbb{N}} d^\infty x_0(E_j)$.

In particular, $d^\infty x_0$ is *not* σ-finite. ♦

Proof: Then, (i) follows by definition. Regarding item (ii), if $E_1, E_2 = \emptyset$, by (i), we get 0 in both sides of the equation to be checked. If $E_k \neq \emptyset$, by definition $d^\infty x_0(E_k) = +\infty$, but also $E_1 \cup E_2 \in \mathfrak{A}_0 \setminus \emptyset$ so by definition $d^\infty x_0(E_1 \cup E_2) = +\infty$ and equality holds. Finally, regarding (iii), if all $E_k = \emptyset$, by (i), we get 0 in both sides. If at least one of them is $E_k \neq \emptyset$, then $\cup_{j=1}^\infty E_j \in \mathfrak{A} \setminus \emptyset$ so $d^\infty x_0(\cup_{j=1}^\infty E_j) = +\infty$ and the equality holds too. ***o.ε.δ.***

Theorem 1 (*Hahn-Kolmogorov Theorem*)**.** Given a set X and a Boolean algebra $\mathcal{A}_0$ on X, for any pre-measure $\mu_0 : \mathcal{A}_0 \to [0, +\infty]$, there *exists* a measure $\mu : \sigma(\mathcal{A}_0) \to [0, +\infty]$ extending μ_0. In particular, μ acts on any $E \in \sigma(\mathcal{A}_0)$ as:

$$\mu(E) \;=\; \inf_{\substack{\{B_j\}_{j\in\mathbb{N}} \subseteq \mathcal{A}_0 \,: \\ E \;\subseteq\; \bigcup_{j\in\mathbb{N}} B_j}} \Big\{ \sum_{j=1}^{\infty} \mu_0(B_j) \Big\}. \tag{2.10}$$

If the measure μ_0 is σ*-finite* in $\mathcal{A}_0$, then μ is the *unique* extension of μ_0 to μ. ♦

Proof: See Theorem 1.14 in (Folland, 1999) or Theorem 1.7.8 in (Tao, 2011). □

Corollary 1. For any infinite I there exists at least one extension of $d^\infty x_0$ to $\odot_{j\in I}\mathfrak{B}(\mathbb{R})$ (hence, by Prop. 1, if $I = \mathbb{N}$, to $\mathfrak{B}(\mathbb{R}^\infty)$) and *it attributes infinite measure to all measurable sets* except to the empty set —to which it gives measure 0. ♦

Proof: By Proposition 3, $\sigma(\mathfrak{A}_0) = \odot_{j\in I}\mathfrak{B}(\mathbb{R})$, so, by Theorem 1 there is a measure on $\odot_{j\in I}\mathfrak{B}(\mathbb{R})$ extending $d^\infty x_0$.

- If $E \in \odot_{j\in I}\mathfrak{B}(\mathbb{R}) \setminus \emptyset$, for any cover of E by elements $A_j \in \mathfrak{A}_0$, at least one of the A_j needs to be non-empty. But by definition $d^\infty x_0(A_j) = \infty$ if A_j is non-empty, so by (2.10), necessarily $d^\infty x(E) = +\infty$. ***o.ε.δ.***

Lemma 4. *Every* extension of $d^\infty x_0$ to $\mathfrak{B}(\prod_{j\in I}\mathbb{R})$ is such that *all open sets have infinite measure.* More generally, they all attribute infinite measure to every Borel measurable neighborhood of every point, i.e., to every Borel measurable set with non-empty interior. ♦

Proof: $\mathfrak{A}_0$ trivially contains the defining base of the product topology $\mathfrak{G}_0$. Let $U \subseteq \prod_{j\in I}\mathbb{R}$ be an arbitrary non-empty open set. By definition of base $U = \cup_{k\in K} B_k$ for some K and some $B_k \in \mathfrak{G}_0 \subseteq \mathfrak{A}_0$. Let $d^\infty x$ be an arbitrary extension of $d^\infty x_0$ to $\mathfrak{B}(\prod_{j\in I}\mathbb{R})$ and take an arbitrary non-empty and at most countable subset $L \subseteq K$. Then,

$$d^\infty x(U) \overset{\text{(additivity)}}{\geq} d^\infty x\Big(\bigcup_{k\in L} B_j\Big) \overset{(\sigma\text{-additivity})}{\geq} \sum_{k\in L} d^\infty x(B_k) \overset{\text{(extension)}}{=} \sum_{k\in L} d^\infty x_0(B_k) \overset{\text{(by def.)}}{=} +\infty.$$

• For the more general statement, note that in any topological space X, a set $A \subseteq X$ is a neighborhood of a point *if and only if* it has non-empty interior.[a] Then, let $E \in \mathfrak{B}(\prod_{j\in I}\mathbb{R})$ be an arbitrary set with non-empty interior. By definition, there exists an open set $U \subseteq E$, so by additivity $\mu(E) \geq \mu(U) = +\infty$.

o.ε.δ.

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[a] Recall, the interior of a set $A \subseteq X$ in a topological space X is the union of all open sets contained in A, which is itself an open set.

The Reasons to Stop Here

We have found two reasons to halt the quest of a literal product measure of Lebesgue measures for now:

- If there is only one extension, certainly it is useless, as we proved that it attributes infinite measure to all measurable sets except the empty set. Hence, it would yield $L^2(\mathbb{R}^\infty, d^\infty x) = \{\vec{0}\}$, since, no other function can be square integrable.
- If there are other extensions, then there is no a priori reason to choose one of those product measures over the rest. There seems to be no reason to consider them together either.[1] Moreover, by Lemma 4 all of them assign infinite measure to every open set of $\mathbb{R}^\infty$, which could be deva stating to reconcile an eventual differential calculus with integration theory.

2.2 Use the Unique Characterizability of the Lebesgue measure

After being introduced to $\mathbb{R}^n$, anybody would expect the existence of a generalized notion of length/area/volume that has the following obvious properties that length/area/volume satisfy: translation invariance (i.e., that the volume of a "solid" is the same regardless of "us moving it around"), σ-additivity (i.e., that if we break a "solid" into countably many pieces and add their volumes, we still obtain the volume of the full "solid") and that the "volume" of the unit cube is 1 (fixing the units of measure). Remarkably, there is a unique such possible notion of volume in $\mathbb{R}^n$: the so-called Lebesgue measure.

Proposition 5. For each $n \in \mathbb{N}$, *there is a unique* measure $\mu : \mathfrak{B}(\mathbb{R}^n) \to [0, +\infty]$ that

(i) is *translation-invariant*, i.e., $\mu(B) = \mu(B + v)$, $\forall v \in \mathbb{R}^n$, $\forall B \in \mathfrak{B}(\mathbb{R}^n)$,

(ii) assigns measure 1 to the unit cube, i.e., $\mu([0,1)^n) = 1$.

This measure μ equals the Lebesgue measure. ♦

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[1] Presumably, most extensions are not σ-finite because $d^\infty x_0$ was so already in $\mathfrak{A}_0$. And a non-σ-finite measure is very pathological because generically, we can no longer employ the Fubini nor Radon-Nikodym theorems, among others. Still, there exists a possibility that one of the extensions is indeed σ-finite or that it satisfies the mentioned theorems.

Proof: It is a well-known result but we could not find it given as a single result, so we sketch the proof.

- Define a *box* to be a set $B \subseteq \mathbb{R}^d$ that is the Cartesian product of n (closed, half-closed or open) intervals $\{I_j\}_{j=1}^n$. We define the volume of the box $B = I_1 \times \cdots \times I_n$ to be the product of the lengths of these intervals, $Vol(B) := \ell(I_1) \cdots \ell(I_n)$, where $\ell(I) := \sup I - \inf I$.
- Define an *elementary set* of $E \subseteq \mathbb{R}^n$ to be the union of finitely many boxes. Note that we can always choose the decomposition to be into disjoint boxes. Denote the set of elementary sets of $\mathbb{R}^n$ by $\varepsilon(\mathbb{R}^n)$ and their complements by $\varepsilon(\mathbb{R}^n)^c := \{\mathbb{R}^n \backslash E \mid E \in \varepsilon(\mathbb{R}^n)\}$.
- The union of elementary sets with their complements, $\overline{\varepsilon}(\mathbb{R}^n) := \varepsilon(\mathbb{R}^n) \cup \varepsilon(\mathbb{R}^n)^c$ is a Boolean algebra by Exercise 1.4.1 in (Tao, 2011).
- The map $m : \overline{\varepsilon}(\mathbb{R}^n) \to [0, +\infty]$ that for each $E \in \varepsilon(\mathbb{R}^n)$ takes a partition into disjoint boxes $E = \sqcup_{j=1}^m B_j$ and assigns $m(E) := \sum_{j=1}^m Vol\,(B_j)$, while for $E \in \varepsilon(\mathbb{R}^n)^c$ assigns $m(E) := +\infty$, is well-defined (independent of the partition and describes all possible E) by Lemma 1.1.2 in (Tao, 2011) and is a *pre-measure* by Exercise 1.7.5 in (Tao, 2011).
- m is σ-finite because we can write $\mathbb{R}^n$ as the countable union of side-1 cubes, $\mathbb{R}^n = \bigcup_{\vec{k} \in \mathbb{Z}^n}((0,1]^n + \vec{k}))$, and all of them have a finite measure 1.
- Consequently, by Theorem 1, there exists a unique measure μ on $\sigma(\overline{\varepsilon}(\mathbb{R}^n))$ extending m.
- In particular, by Exercise 1.4.14 in (Tao, 2011), $\sigma(\overline{\varepsilon}(\mathbb{R}^n)) = \mathfrak{B}(\mathbb{R}^n)$.
- The Lebesgue measure trivially agrees with m on $\overline{\varepsilon(\mathbb{R}^n)}$, so by uniqueness of extension, μ must be the Lebesgue measure on $\mathfrak{B}(\mathbb{R}^n)$, which we denoted by $d^n x$. Moreover, the Lebesgue measure is translation invariant by Theorem 2.42 in (Folland, 1999) and trivially $m([0,1)^n) = 1$. Hence, it satisfies the requirements of the proposition.
- Finally, to prove uniqueness, assume there existed another measure μ' on $\mathfrak{B}(\mathbb{R}^n)$ satisfying as well (i) and (ii). The restriction of μ' to $\overline{\varepsilon}(\mathbb{R}^n)$, denoted m', cannot equal m, because μ was the unique extension of m to $\mathfrak{B}(\mathbb{R}^n)$. At the same time, it must be that $m'|_{\varepsilon(\mathbb{R}^n)} = m|_{\varepsilon(\mathbb{R}^n)}$ because by Exercise 1.1.3 in (Tao, 2011), there is a unique non-negative map $\varepsilon(\mathbb{R}^n) \to [0, +\infty]$ that is finitely additive, translation invariant and assigns value 1 to $[0,1)^n$ and that is $m|_{\varepsilon(\mathbb{R}^n)}$. Therefore, it must be that $m'|_{\varepsilon(\mathbb{R}^n)^c} \neq m|_{\varepsilon(\mathbb{R}^n)^c}$. Namely, there must exist $E \in \varepsilon(\mathbb{R}^n)$ such that for its complement $m'(E^c) < +\infty$. We just need to see that this leads to a contradiction.
- If $m'(E^c) < +\infty$, because $E \in \varepsilon(\mathbb{R}^n)$, $m'(E) = m(E)$, but for all elementary sets $m(E) < +\infty$. Thus, by sigma additivity of μ', $\mu'(\mathbb{R}^n) = \mu'(E \cup E^c) = m'(E) + m'(E^c) < +\infty$. By translation invariance every cube of side 1 has the same measure via μ' and equals 1 by assumption. Since we can cover $\mathbb{R}^n$ with countably many disjoint unit cubes of side 1, i.e., $\mathbb{R}^n = \bigsqcup_{\vec{k} \in \mathbb{Z}^n}((0,1]^n + \vec{k})$, by sigma additivity $\mu'(\mathbb{R}^n) = \sum_{k \in \mathbb{Z}^n} 1 = +\infty$. But we had that $\mu'(\mathbb{R}^n) < +\infty$. Absurd!

o.ε.δ.

It is this why one says that the Lebesgue measure $d^n x$ is *the* notion of "homogeneous" or "uniform measure" in $\mathbb{R}^n$. The translation invariance property gives it the feeling of "uniformity", of not distinguishing any region of $\mathbb{R}^n$ in particular over any other. As opposed to it, a Gaussian measure like $e^{-|x|^2}d^nx$ on $\mathfrak{B}(\mathbb{R}^n)$ would assign more "weight" to a ball around of the origin than the same ball translated anywhere else. But why would we want to use this measure for $L^2(\mathbb{R}^n, d^nx)$'s definition? Perhaps because we expect that physically, $\mathbb{R}^n$ parametrizes what we are modeling equally well if we translate all parameters by a fixed amount. Or perhaps because we do not want to presuppose that a configuration is distinguished over another. Because of all that, we would like to treat all $\mathbb{R}^n$ regions "democratically" and hence we choose to employ d^nx in $L^2(\mathbb{R}^n, d^nx)$. After all, it is the only translation invariant measure. In the same vein, in $\mathbb{R}^\infty$, instead of looking for a product measure of dx's, we could look for a measure that is translation-invariant.

However, this quest is also full of hindrances:

Lemma 5. $\mathbb{R}^\infty = \prod_{j\in\mathbb{N}}\mathbb{R}$ is not *locally compact*, i.e., there is some point that is not in the (topological) interior of any compact set. ♦

Proof: By Theorem 18.6 in (Willard, 2012), a necessary condition for the Cartesian product of topological spaces $\prod_{j\in\mathbb{N}} X_j$ to be locally compact is that all but finitely many factors are compact. This is not the case for $\prod_{j\in\mathbb{N}}\mathbb{R}$ because $\mathbb{R}$ is not compact.

o.ε.δ.

Proposition 6. (i) The product topology of $\mathbb{R}^\infty = \prod_{j\in\mathbb{N}}\mathbb{R}$ is *metrizable,* i.e., $\mathbb{R}^\infty$ admits a distance function inducing the product topology. In particular, for any choice of distance function $d_j : \mathbb{R}\times\mathbb{R} \to [0,+\infty)$ per each factor $\mathbb{R}$, if they are bounded by 1 (i.e. $\sup_{x,y\in\mathbb{R}} d(x,y) \leq 1$), then the map $d : \mathbb{R}^\infty \times \mathbb{R}^\infty \to [0,+\infty)$,

$$d\Big((x_j)_{j\in\mathbb{N}},\ (y_j)_{j\in\mathbb{N}}\Big) := \sum_{k=1}^{\infty} \frac{d_k(x_k,y_k)}{2^k}, \qquad (x_j)_{j\in\mathbb{N}}, (y_j)_{j\in\mathbb{N}} \in \prod_{j\in\mathbb{N}}\mathbb{R}, \tag{2.11}$$

is a distance function on $\mathbb{R}^\infty$ inducing the product topology.

(ii) For any distance function $d : \mathbb{R}\times\mathbb{R} \to [0,+\infty)$, two possible ways to obtain equivalent distance functions that are bounded by 1 are:

$$\tilde{d}(x,y) := \min\{1, d(x,y)\} \quad \textit{and} \quad d'(x,y) := \frac{d(x,y)}{1+d(x,y)}. \tag{2.12}$$

♦

Proof: In (Willard, 2012), see Theorem 22.3 and its proof for (i) and Theorem 22.2 for (ii). □

Note that as a metric space, $\mathbb{R}^\infty$ is Hausdorff. Because $\mathbb{R}^\infty$ is not locally compact, this implies by Theorem 18.2 in (Willard, 2012), that *no point of* $\mathbb{R}^\infty$ *has a compact neighborhood.*

Theorem 2. If X is a metrizable, Cauchy complete and separable topological group,[a] then, there exists a (non-trivial) left-invariant[b] Borel measure that is finite on all compact sets *if and only if* X is locally compact and has no isolated points.[c] ♦

Proof: This is the main result proven in (Oxtoby, 1946). □

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[a] A topological group is a group $(G,\cdot)$ with a topology such that the group operation $(x,y)\in G\times G \mapsto x\cdot y$ and the group inverse $x\in G\mapsto x^{-1}$ are continuous maps.

[b] A Borel measure μ on a topological group $(G,\cdot)$ is left invariant when $\mu(E) = \mu(g\cdot E)\ \forall E\in\mathfrak{B}(G),\ \forall g\in G$.

[c] That is, every point $x\in X$ is a limit point of $X\setminus\{x\}$.

Corollary 2. There is no translation-invariant Borel measure on $\mathbb{R}^\infty$ that is *locally finite*, i.e., that is finite in all compact sets. ⧫

Proof:

- We proved that $\mathbb{R}^\infty$ is metrizable in Prop. 6.
- By Theorem 24.11 in (Willard, 2012), $\mathbb{R}^\infty$ is Cauchy complete because $\mathbb{R}$ is a complete metric space and there are countably many such factors.
- By Theorem 16.4.c of Willard (2012), the product topology of separable topological spaces is separable *if and only if* there are at most $|\mathbb{R}|$ factors. Hence, $\mathbb{R}^\infty$ is separable.
- $\mathbb{R}^\infty$ is also a topological group with the group operation being the addition because any topological vector space is so (see 11.1 in (Folland, 1999)).
- $\mathbb{R}^\infty$ is not locally compact by Lemma 5, so by Theorem 2, there exists no non-trivial left-invariant Borel measure that is finite on compact sets. A left-invariant measure in our case is a μ such that $\mu(v+E)=\mu(E)\quad \forall E \in \mathfrak{B}(\mathbb{R}^\infty)$ and $\forall v \in \mathbb{R}^\infty$, namely, a translation-invariant measure. ***o.ε.δ.***

One could claim that local finiteness can be sacrificed if we still get a well-behaved translation-invariant measure. However, there is a “killer” result:

Theorem 3. Let X be a metrizable, Cauchy complete and separable topological group and μ a (non-zero) left-invariant Borel measure in X. Then, if X is *not* locally compact, every neighborhood $E \subseteq X$ of every point $x \in X$ contains uncountably many disjoint subsets (related to each other by left action of some element of X) with equal finite positive measure. In particular, μ cannot be σ-finite. ⧫

Proof: For the first part see Theorem 1 in (Oxtoby, 1946). The statement on σ-finiteness is Theorem B.1 in (Glasner et al., 2005) —after noting that a topological space is called *Polish* if it is metrizable, Cauchy complete and separable. ***o.ε.δ.***

Corollary 3. *Every non-zero translation invariant Borel measure μ on $\mathbb{R}^\infty$ attributes infinite measure to all open sets.* More generally, it attributes infinite measure to every (Borel) neighborhood of every point, i.e., to every Borel set with non-empty interior. In particular, μ is *not σ-finite.* ⧫

Proof: $\mathbb{R}^\infty$ is a metrizable, complete and separable topological group as proven in Cor. 2. Let μ be a (non-zero) translation-invariant measure μ. This exactly means that μ is left (and right) invariant. In any topological space X, a set $A \subseteq X$ is a neighborhood of a point *if and only if* it has non-empty interior. Let $E \in \mathfrak{B}(\mathbb{R}^\infty)$ be any set with non-empty interior. Since by Lemma 5, $\mathbb{R}^\infty$ is not locally compact, Theorem 3 implies there exist countably many measurable sets $\{B_j\}_{j\in\mathbb{N}}$ that are mutually disjoint and $\mu(B_j)=\alpha \in (0,+\infty)\ \forall j \in \mathbb{N}$ with $B_j \subset E\ \forall j \in \mathbb{N}$. But then,

$$\mu(E) \overset{\text{(additivity)}}{\geq} \mu\Big(\bigsqcup_{j\in\mathbb{N}} B_j\Big) \overset{(\sigma\text{-additivity})}{=} \sum_{j\in\mathbb{N}} \mu(B_j) = \sum_{j\in\mathbb{N}} \alpha \overset{(\alpha>0)}{=} +\infty. \tag{2.13}$$

- In particular, any open set is a neighborhood of any point it contains, so the statement about open sets follows from this. It also follows from Theorem 3 that μ is not σ-finite. ***o.ε.δ.***

The Reasons to Stop Here

We have proven that not only every open set of $\mathbb{R}^\infty$ would have infinite measure (making the link between integration and an eventual theory of calculus harder) but in particular, no translation invariant measure $d^\infty\mu$ is σ-finite. Non-σ-finite measures are usually claimed to be very pathological because, typically, one can no longer employ big theorems like Fubini's or Radon-Nikodym's. Interestingly, in the literature, one can find very natural proposals of translation invariant measures, a prime example being the one in (Baker, 1991, 2004) —where they even prove Fubini-like theorems for the proposed measure. We plan to discuss their relation to our approach elsewhere. For now, we stop here our quest for a literal generalization of the Lebesgue measure for a simple reason: as we will see now, the use of the Lebesgue measure was really not necessary for any crucial purpose in QM! In particular, under this observation the path to be followed turns out to be simpler than embracing a non-σ-finite measure all at once.

2.3 In QM the Lebesgue Measure is Replaceable

As we said before, it is convenient to use d^nx in $L^2(\mathbb{R}^n, d^nx)$ to avoid "presuppositions" about particular regions of $\mathbb{R}^n$. But, was such a "uniformity" really necessary in QM? After all, every element $\psi \in L^2(\mathbb{R}^n, d^nx)$ is already forced to decay and thus, the measure $|\psi|^2 d^nx$ (which is what will have an experimental correlate in QM —unlike d^nx itself) will never be "uniform".

Definition 9. Given a measurable space (X, Σ), a measurable function $\rho : X \to [0, +\infty]$ and a measure $d\mu$, we denote by $\rho d\mu$ the mapping $\Sigma \to [0, +\infty]$ such that

$$\rho\, d\mu(B) := \int_{x\in B} \rho(x) d\mu\ , \quad B \in \Sigma. \tag{2.14}$$

(Note that the "d" in front of a measure has no mathematical meaning, it is just notationally convenient in the context of Lebesgue integration.) ♦

Lemma 6. In the above situation, whichever the measurable function $\rho : X \to [0, +\infty]$ is, $d\nu := \rho d\mu$ is *still a measure* on (X, Σ).

We call ρ the *density* of the measure $d\nu$ with respect to $d\mu$ and denote it by $\frac{d\nu}{d\mu}$. ♦

Proof: First, $d\nu(\emptyset) = \int_\emptyset \rho(x)d\mu = 0$. Second, given pairwise disjoint sets $(B_j)_{j\in\mathbb{N}} \subseteq \Sigma$, define $s_N(x) := \sum_{k=1}^N \mathbb{1}_{B_k}(x)$ —where $\mathbb{1}_{B_k}(x)$ denotes the *indicator function* of B_k (which equals 1 if $x \in B_k$ and 0 otherwise). s_N are monotonously increasing and point-wise, $\lim_{k\to\infty} \mathbb{1}_{B_k}(x) = \mathbb{1}_{\cup_{j\in\mathbb{N}} B_k}(x)$. Hence, using the monotone convergence theorem (Thm. 2.8.2 in (Bogachev, 2007)) in $(\star)$,

$$\sum_{k=1}^\infty d\nu(B_k) \overset{\text{(by def)}}{=} \lim_{N\to\infty} \sum_{k=1}^N \int_{x\in X} \mathbb{1}_{B_k}(x)\rho(x)d\mu = \lim_{N\to\infty} \int_{x\in X} s_N(x)\rho(x)d\mu \overset{(\star)}{=} \int_{x\in X} \mathbb{1}_B(x)\rho(x)d\mu = d\nu(B).$$

o.ε.δ.

Let us sketch heuristically the reason why *per se,* we were not interested on $d^n x$ to weight measures of subsets of $\mathbb{R}^n$. Instead, we were just interested on the measure $|\psi|^2 d^n x$ given by each $\psi \in L^2(\mathbb{R}^n, d^n x)$.

Often, a measure on $\mathbb{R}^n$, say $|\psi|^2 d^n x$, can be rewritten in terms of another measure $d^n \mu$ and another density $|\phi|^2$ as $|\phi|^2 d^n \mu$. For example, consider $\psi(x) := (\pi\sigma^2)^{-n/4} exp(-\frac{|x|^2}{2\sigma^2})$ (it is in $L^2(\mathbb{R}^n, d^n x)$). Its associated measure $d^n \mu := |\psi|^2 d^n x$ is a Gaussian centered in the origin. $\phi(x) \equiv 1$ is a vector in $L^2(\mathbb{R}^n, d^n \mu)$ and (quite trivially) it has an associated measure $|\phi|^2 d^n \mu$ that equals $|\psi|^2 d^n x$. The key remark is that even if according to $d^n \mu$ the unit ball around $x = 0$ has a bigger measure than the unit ball around $x = (1, 0, ..., 0)$, i.e., even if $d^n \mu$ is highly "non-uniform", we can express the same measure as $d^n x$ could via L^2-densities. The key is that the density $|\phi|^2$ that multiplies $d^n \mu$ can "compensate" for the "presuppositions" of the background measure $d^n \mu$ and make it "as flexible as the Lebesgue measure". That is, when it comes to wavefunctions and the Born rule, *the Lebesgue measure's role was to be a background measure* relative to which we could express other measures via densities like $|\psi|^2$. "Uniformity" was not a requirement for that, so, a priori, one could choose alternative background measures. But, which would be the other "allowed" background measures? Certainly, if we took $d^n \tilde{\mu} := \mathbb{1}_{|x|\geq 1} d^n \mu$ as background measure, there would be no way to describe the Gaussian measure $|\psi|^2 d^n x$ from above, because $d^n \tilde{\mu}$ assigns measure 0 to the unit ball around the origin no matter which density we "multiply it with". As such, by "allowed" background measure we mean one that is as "flexible" as the Lebesgue measure in the sense that we can reach the same measures by multiplying them with densities. The necessary and sufficient condition for that (as we prove below) is that the candidate background measure has exactly the same zero-measure sets than the Lebesgue measure —a condition known as *mutual absolute continuity.*

Now, one could point out that in a pilot-wave theory $d^n x$ plays a more "foreground" role. For instance, one can make assertions of the style "independently of ψ, for almost every initial configuration..." or "almost every trajectory...", which pretend to use $d^n x$ with no multiplying $|\psi|^2$. And yet, in any such case, what one really cares about is the zero-measure sets (*null sets*) and not the particular weight of non-null sets. Thus, as before, a mut. a.c. measure would serve equally well. As a side-note, changing the "background measure" $L^2(\mathbb{R}^n, d^n x) \mapsto L^2(\mathbb{R}^n, d^n \mu)$ would require to change the wavefunction $\psi_t \mapsto \widetilde{\psi}_t$ ensuring $|\psi_t|^2 d^n x = |\widetilde{\psi}_t|^2 d^n \mu$ —in order to ensure the same probabilistic predictions result from both theories. Likewise, one would need to "push-forward" the guidance law to the new picture in $L^2(\mathbb{R}^n, d^n \mu)$, such that the same ensemble of trajectories results from the identified wavefunctions.

That said, the key is to note that the measure $|\psi|^2 d^n x$ (via the configuration-space Born rule) gives *all* the textbook experimental predictions of QM (see e.g., §5.2 in (Teufel, 2021)). Hence, every experimental prediction is "independent of the background measure" —or more precisely, experiments do not impose any background measure. Similarly, note that the acceptable trajectory ensembles for a mechanistic understanding of QM (via pilot-wave theories) are only constrained by the evolution of the measure $|\psi_t|^2 d^n x$ (and never of $d^n x$ itself). Hence, the ontology does not either impose any background measure. What is more, we will see that in changing the wavefunction space from $L^2(\mathbb{R}^n, d^n x)$ to $L^2(\mathbb{R}^n, d^n \mu)$, the "push-forwarded" theory is still governed by a Schrödinger equation!

With all, after the push-forward to a space with a different background measure, nothing in a pilot-wave theory seems to change. In what follows, we make all these ideas rigorous.

Definition 10. Given a measurable space (X, Σ) and two measures $d\mu_1, d\mu_2$:

- $d\mu_1$ is *absolutely continuous* (a.c.) with respect to $d\mu_2$, denoted $d\mu_1 \ll d\mu_2$, when $d\mu_2(B) = 0 \Longrightarrow d\mu_1(B) = 0 \quad \forall B \in \Sigma$. Equivalently $d\mu_1(B) > 0 \Longrightarrow d\mu_2(B) > 0 \quad \forall B \in \Sigma$.
- $d\mu_1,\ d\mu_2$ are *mutually absolutely continuous* (mut. a.c.), denoted $d\mu_1 \sim d\mu_2$, when $d\mu_1 \ll d\mu_2$ and $d\mu_2 \ll d\mu_1$, i.e., when they agree on which sets have zero and non-zero measure.
- $d\mu_1,\ d\mu_2$ are *mutually singular*, denoted $d\mu_1 \perp d\mu_2$ when $\exists B \in \Sigma$ such that $d\mu_1(B) = 0$ and $d\mu_2(X \setminus B) = 0$ (i.e., they have disjoint supports). ♦

Theorem 4 (*Radon-Nikodym*)**.** Given a measurable space (X, Σ) and two *σ-finite* measures $d\mu_1, d\mu_2$:

$$d\mu_1 \ll d\mu_2 \quad \Longleftrightarrow \quad \exists \rho : X \to [0, +\infty] \ \text{ measurable s.th. } \ d\mu_1 = \rho \, d\mu_2.$$

Moreover, ρ is *unique* (up to changes over $d\mu_2$-null sets). We denote $\rho =: \frac{d\mu_1}{d\mu_2}$. ♦

Proof: See Theorem 2.10 in Kallenberg (1997). □

Corollary 4. Given a measurable space (X, Σ) and two σ-finite measures $d\mu_1, d\mu_2$,

$$d\mu_1 \sim d\mu_2 \quad \Longleftrightarrow \quad \exists \rho : X \to (0, +\infty] \ \text{ measurable s.th. } \ d\mu_1 = \rho \, d\mu_2.$$

In particular, $d\mu_2 = \frac{1}{\rho} \, d\mu_1$. ♦

Proof: ($\Longleftarrow$) By Theorem 4, this implies $d\mu_1 \ll d\mu_2$. But, $1/\rho$ is well-defined everywhere and hence, for any $B \in \Sigma$:

$$\frac{1}{\rho} d\mu_1(B) = \int_{x \in B} \frac{1}{\rho(x)} d\mu_1 = \int_{x \in B} \frac{1}{\rho(x)} \rho(x) d\mu_2 = d\mu_2(B),$$

i.e., $d\mu_2 = \frac{1}{\rho} d\mu_1$. By Theorem 4, this means that $d\mu_2 \ll d\mu_1$.

($\Longrightarrow$) By Theorem 4, $\exists \eta : X \to [0, +\infty]$ such that $d\mu_1 = \eta d\mu_2$. We claim that $\eta(x) > 0$ for $d\mu_2$-almost every $x \in X$. This would imply that changing the value of η in the $d\mu_2$-null set where it is 0 by a non-zero value, one could get a measurable $\rho : X \to (0, +\infty]$ with $\rho d\mu_2 = \eta d\mu_2 = d\mu_1$, (proving the statement).

To prove the claim, assume it was false. Then, there would exist a non-null set (for $d\mu_2$) $B \in \Sigma$ (i.e., $d\mu_2(B) \neq 0$) such that $\int_{x \in B} \eta(x) d\mu_2 = 0$, i.e., $\eta d\mu_2(B) = d\mu_1(B) = 0$. But by mut. a.c., this implies $d\mu_2(B) = 0$ which we said to be non-zero. Absurd! ***ο.ε.δ.***

Proposition 7. Given a measurable space (X, Σ) and two σ-finite measures $d\mu_1,\ d\mu_2$,

$$\left\{ \rho \ d\mu_1 \ \middle| \ \begin{matrix} \rho : X \to [0, +\infty] \\ \text{is measurable} \end{matrix} \right\} = \left\{ \rho \ d\mu_2 \ \middle| \ \begin{matrix} \rho : X \to [0, +\infty] \\ \text{is measurable} \end{matrix} \right\} \Longleftrightarrow \ d\mu_1 \sim d\mu_2.$$

(Informally, *mut. a.c. is exactly the condition so that two measures can express the same measures by multiplication with a density.*)

In particular, the correspondence is $\rho(x) \, d\mu_1 = \frac{d\mu_1}{d\mu_2}(x) \, \rho(x) \, d\mu_2$. ♦

Proof: ($\Longleftarrow$) By Theorem 4, $\exists \eta : X \to [0,+\infty]$ measurable such that $d\mu_1 = \eta d\mu_2$. Then, given the measure $\nu := \rho d\mu_1$ for an arbitrary measurable $\rho : X \to [0,+\infty]$, we can also obtain ν as $\tilde{\rho} d\mu_2$ by taking $\tilde{\rho} := \rho \cdot \eta$. By mutual absolute continuity we can also do the same exchanging 1 and 2.

($\Longrightarrow$) $\rho(x) \equiv 1$ is measurable so $\rho d\mu_1 = d\mu_1$ is in the l.h.s[a] set. By the hypothesis, it is also a member of the r.h.s, meaning there exists some $\eta : X \to [0,+\infty]$ such that $\eta d\mu_2 = d\mu_1$. But then, by Theorem 4, $d\mu_1 \ll d\mu_2$. We can do the same argument exchanging 1 and 2. ***o.ε.δ.***

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[a] We denote *l.h.s:=* "left hand side" and *r.h.s:=* "right hand side".

For our claim we need something stronger than this. We only want to impose "equal flexibility" for L^1-densities (so the ones that decay fast enough). Will mut. a.c. still be the necessary and sufficient condition for that?

Lemma 7. Given a measurable space (X, Σ) and an arbitrary σ-finite measure $d\mu$, $\exists \rho \in L^1(X, d\mu)$ with $\rho(x) > 0$ for almost every $x \in X$. ♦

Proof: By σ-finiteness, there exist $\{B_j\}_{j\in\mathbb{N}} \subseteq \Sigma$ with $d\mu(B_j) \in (0,+\infty)$ $\forall j$ and $X = \cup_{j\in\mathbb{N}} B_j$. Then, we claim that $\rho := \sum_{k=1}^{\infty} \alpha_k \mathbb{1}_{B_k}$ with $\alpha_k := \frac{1}{k^2\ d\mu(B_k)} > 0$ satisfies the lemma. Indeed, because every $x \in X$ is at least in some B_k, say, in B_{k_0}, $\rho(x) \geq \alpha_{k_0} > 0$. But also, $\rho \in L^1(X, d\mu)$ because

$$\int_{x\in X} |\rho|(x) d\mu = \int_{x\in X} \sum_{k=1}^{\infty} \alpha_k \mathbb{1}_{B_k}(x) d\mu \leq \sum_{k=1}^{\infty} \alpha_k\ d\mu(B_k) = \sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6} < +\infty.$$

o.ε.δ.

Proposition 8. Given a measurable space (X, Σ) and two σ-finite measures $d\mu_1$, $d\mu_2$,

$$\left\{ |\psi|^2\ d\mu_1\ \middle|\ \psi \in L^2(X, d\mu_1) \right\} = \left\{ |\phi|^2\ d\mu_2\ \middle|\ \phi \in L^2(X, d\mu_2) \right\} \iff d\mu_1 \sim d\mu_2.$$

(Informally, *mut. a.c. is exactly the condition so that two measures can express the same measures by multiplication with* $|\psi|^2$ *for square integrable* ψ's.)

In particular, the correspondence is $|\psi|^2(x)\, d\mu_1 = |\tilde{\psi}|^2(x)\, d\mu_2$ with $\tilde{\psi}(x) := e^{i\theta(x)} \sqrt{\frac{d\mu_1}{d\mu_2}(x)}\ \psi(x)$ for any $\theta : X \to \mathbb{R}$ measurable. ♦

Proof: ($\Longleftarrow$) By Theorem 4, $\exists \eta : X \to [0,+\infty]$ measurable such that $d\mu_1 = \eta d\mu_2$. Then, given the measure $\nu := |\psi|^2 d\mu_1$ for an arbitrary $\psi \in L^2(X, d\mu_1)$, we can also obtain ν as $|\tilde{\psi}|^2 d\mu_2$ by taking $\tilde{\psi} := \psi \cdot \sqrt{\eta}$ since then $|\tilde{\psi}|^2 d\mu_2 = |\psi|^2 \eta d\mu_2 = |\psi|^2 d\mu_1$. By mutual absolute continuity we can also do the same exchanging 1 and 2.

($\Longrightarrow$) By Lemma 7, there exists a $\rho \in L^1(X, d\mu_1)$ such that $\rho(x) > 0$ for $d\mu_1$-almost every $x \in X$. We can make it non-zero in the $d\mu_1$-null sets to make it $\rho(x) > 0$ everywhere. Then, trivially $\psi := \sqrt{\rho} \in L^2(X, d\mu_1)$ and by hypothesis there exists some $\tilde{\psi} \in L^2(X, d\mu_2)$ such that $|\psi|^2 d\mu_1 = |\tilde{\psi}|^2 d\mu_2$. Because $|\psi|^2(x) > 0$, $\frac{1}{|\psi|^2}$ is everywhere well-defined and

$$\frac{|\tilde{\psi}|^2}{|\psi|^2} d\mu_2(B) = \int_{x\in B} \frac{|\tilde{\psi}|^2(x)}{|\psi|^2(x)} d\mu_2 = \int_{x\in B} \frac{|\psi|^2(x)}{|\psi|^2(x)} d\mu_1 = d\mu_1(B) \quad \forall B \in \Sigma.$$

Hence, $d\mu_1 = \frac{|\tilde{\psi}|^2}{|\psi|^2} d\mu_2$. But then, by Theorem 4, $d\mu_1 \ll d\mu_2$ and $\frac{d\mu_1}{d\mu_2} = \frac{|\tilde{\psi}|^2}{|\psi|^2}$. Repeat the argument exchanging $1, 2$. ***o.ε.δ.***

Proposition 9 (*$d^n x$ was a replaceable background measure in QM over $\mathbb{R}^n$*)**.** Fix an arbitrary $n \in \mathbb{N}$ and let $d^n\mu$ be a Borel measure on $\mathbb{R}^n$. Then, $d^n\mu \sim d^n x$ *if and only if* a pilot-wave theory where we substitute $L^2(\mathbb{R}^n, d^n x)$ by $L^2(\mathbb{R}^n, d^n\mu)$ yields the *same quantum theory* in the following sense: there exists a linear isomorphism $W : L^2(\mathbb{R}^n, d^n x) \longrightarrow L^2(\mathbb{R}^n, d^n\mu)$ such that,

1. there is an identification of the wavefunctions

$$\psi \in L^2(\mathbb{R}^n, d^n x) \iff \widetilde{\psi} := W\psi \in L^2(\mathbb{R}^n, d^n\mu).$$

2. The "Born rule" in both theories yields exactly the same probabilities (hence both quantum theories *share their experimental predictions*): for all $B \in \mathfrak{B}(\mathbb{R}^n)$ and all $\psi \in L^2(\mathbb{R}^n, d^n x)$

$$\mathbb{P}\left(\begin{array}{c}\text{the system is found in}\\ \text{configuration } x \in B \text{ if des-}\\ \text{cribed by } \psi \text{ in } d^n x\text{-theory}\end{array}\right) := \int_{x\in B} |\psi|^2(x) d^n x = \int_{x\in B} |\widetilde{\psi}|^2(x) d^n\mu =: \mathbb{P}\left(\begin{array}{c}\text{the system is found in}\\ \text{configuration } x \in B \text{ if des-}\\ \text{cribed by } \widetilde{\psi} \text{ in } d^n\mu\text{-theory}\end{array}\right)$$

which is the same as saying that $|\psi|^2 d^n x = |\widetilde{\psi}|^2 d^n\mu$ for all $\psi \in L^2(\mathbb{R}^n, d^n x)$.

These two items imply that

3. a time dependent wavefunction $t \mapsto \psi_t$ is governed by a Schrodinger equation in one of the theories *if and only if* the identified $t \mapsto \widetilde{\psi}_t$ is governed by a Schrodinger equation in the other theory (hence, both quantum theories *share the "type of fundamental law" for the wavefunction*). More concretely,

$$\left(\begin{array}{c}\psi_0 \in L^2(\mathbb{R}^n, d^n x) \text{ evolves as } t \longmapsto \psi_t = U_t\psi_0\\ \text{for some SCOPUG } \{U_t\}_{t\in\mathbb{R}} \text{ on } L^2(\mathbb{R}^n, d^m x), \text{ i.e.,}\\ \exists (H, D(H)) \text{ self-adjoint operator on } L^2(\mathbb{R}^n, d^m x)\\ \text{s.th. } U_t = exp(\frac{-i}{\hbar} Ht) \text{ is the dynamical law:}\\ \forall \psi_0 \in D(H), \psi_t := U_t\psi_0 \text{ is the unique solution of}\\ i\hbar \frac{\mathrm{d}}{\mathrm{d}t}\psi_t = H\psi_t \;\; \forall t \in \mathbb{R}\end{array}\right) \iff \left(\begin{array}{c}\widetilde{\psi}_0 \in L^2(\mathbb{R}^n, d^n \mu) \text{ evolves as } t \longmapsto \widetilde{\psi}_t = \widetilde{U}_t\psi_0\\ \text{for some SCOPUG } \{\widetilde{U}_t\}_{t\in\mathbb{R}} \text{ on } L^2(\mathbb{R}^n, d^n \mu), \text{ i.e.,}\\ \exists (\widetilde{H}, D(\widetilde{H})) \text{ self-adjoint operator on } L^2(\mathbb{R}^n, d^n \mu)\\ \text{s.th. } \widetilde{U}_t = exp(\frac{-i}{\hbar} \widetilde{H}t) \text{ is the dynamical law:}\\ \forall \widetilde{\psi}_0 \in D(\widetilde{H}), \widetilde{\psi}_t := \widetilde{U}_t\widetilde{\psi}_t \text{ is the unique solution of}\\ i\hbar \frac{\mathrm{d}}{\mathrm{d}t}\widetilde{\psi}_t = \widetilde{H}\widetilde{\psi}_t \;\; \forall t \in \mathbb{R}\end{array}\right).$$

In particular, $\widetilde{U}_t = WU_tW^{-1}$ and $\widetilde{H} = WHW^{-1}$, $D(\widetilde{H}) = WD(H)$.

4. They allow a common ensemble of trajectories $Q_t : \mathbb{R}^n \to \mathbb{R}^n$ for the primitive ontology (hence both theories *share their ontological predictions*), i.e., an homeomorphism family $Q_t : \mathbb{R}^n \to \mathbb{R}^n$, $t \in \mathbb{R}$ satisfies

$$\Big(t \mapsto Q_t \text{ is equivariant wrt } t \mapsto \psi_t\Big) \iff \Big(t \mapsto Q_t \text{ is equivariant wrt } t \mapsto \widetilde{\psi}_t\Big)$$

$$\left(\int_{x\in B} |\psi_0|^2 d^n x = \int_{x\in Q_t(B)} |\psi_t|^2 d^n x \;\; \forall t, B\right) \iff \left(\int_{x\in B} |\widetilde{\psi}_0|^2 d^n \mu = \int_{x\in Q_t(B)} |\widetilde{\psi}_t|^2 d^n \mu \;\; \forall t, B\right).$$

If such an isomorphism W exists, it is unitary and it is *unique* (up to a "gauge" phase): it is the one acting as $W\psi := \frac{\psi}{\sqrt{\rho}} e^{i\theta}$ on $\psi \in L^2(\mathbb{R}^n, d^n x)$, where $\rho : \mathbb{R}^n \to (0, +\infty]$ is the unique measurable function such that $d^n\mu = \rho \; d^n x$ (given by Corollary 4) and $\theta : \mathbb{R}^n \to [-\pi, \pi)$ is an arbitrary measurable function that we call *"gauge" phase.* ♦

To prove it, we prove a more general claim that trivially particularizes to Proposition 9 (and which will be very convenient later).

Theorem 5. Let there be a measurable space (X, Σ) and two σ-finite measures $d\mu, d\nu$.

1. $$d\mu \sim d\nu \iff \exists \text{ isomorphism } W : L^2(X, d\nu) \to L^2(X, d\mu) \text{ s.th. } |\psi|^2 d\nu = |W\psi|^2 d\mu \quad \forall \psi \in L^2(X, d\nu). \tag{2.15}$$

- If such an isomorphism W exists, it is unitary and it is *unique* (up to a "gauge" phase): it is the one acting as $W\psi := \frac{\psi}{\rho} e^{i\theta}$ on $\psi \in L^2(X, d\nu)$, where $\rho^2 : X \to (0, +\infty]$ is the unique measurable function s.th. $d\mu = \rho^2 \; d\nu$ (given by Corollary 4) and $\theta : X \to [-\pi, \pi)$ is an arbitrary measurable function that we call a *"gauge" phase.*

2. The condition to the r.h.s. of (2.15) holds *if and only if*, there exists an isomorphism $W : L^2(X, d\nu) \to L^2(X, d\mu)$ such that the "Born rules" according to each pair of wavefunctions identified by W agree with each other, i.e., such that for all $\Psi \in L^2(X, d\nu)$ and $B \in \Sigma$

$$\text{“}\mathbb{P}\begin{pmatrix} \text{system is found in} \\ x \in B \text{ if described} \\ \text{by } \Psi \text{ in } d\nu\text{-theory} \end{pmatrix}\text{''} := \int_{x\in B} |\Psi|^2(x) d\nu = \int_{x \in B} |W\Psi|^2(x) d\mu =: \text{“}\mathbb{P}\begin{pmatrix} \text{system is found in} \\ x \in B \text{ if described} \\ \text{by } W\Psi \text{ in } d\mu\text{-theory} \end{pmatrix}\text{''}.$$

If any of these equivalent conditions holds, then the following also ones hold:

3. If there is a SCOPUG $\{U_t\}_{t\in\mathbb{R}}$ on $L^2(X, d\nu)$ telling the time evolution for an initial $\Psi_0 \in L^2(X, d\nu)$ as $t \in \mathbb{R} \mapsto \Psi_t := U_t \Psi_0$, then the family $\widetilde{U}_t := W U_t W^{-1}$ is telling how to evolve the identified wavefunction $W\Psi_0$ on $L^2(X, d\mu)$ (so that it keeps being the W-identified vector at all times), i.e., $W(U_t \Psi_0) = \widetilde{U}_t(W\Psi_0)$. In particular, $\{\widetilde{U}_t\}_{t\in\mathbb{R}}$ is still a SCOPUG.

Equivalently, by Stone's theorem (see Theorem 6.2 in (Schmüdgen, 2012)), the dynamics of wavefunctions in both spaces are generated via a Schrödinger equation by self-adjoint operators $(H, D(H))$ and $(\widetilde{H}, D(\widetilde{H}))$ (i.e., $U_t = e^{-\frac{i}{\hbar}Ht}$ and $\widetilde{U}_t = e^{-\frac{i}{\hbar}\widetilde{H}t}$) and

$$(\widetilde{H}, D(\widetilde{H})) = (WHW^{-1}, \; WD(H)). \tag{2.16}$$

4. If X is a topological space and $\Sigma = \mathfrak{B}(X)$ with $d\nu, d\mu$ Borel, an homeomorphism family $Q_t : X \to X$, $t \in \mathbb{R}$ satisfies

$$\Big(t \mapsto Q_t \text{ is equivariant wrt } t \mapsto \Psi_t\Big) \iff \Big(t \mapsto Q_t \text{ is equivariant wrt } t \mapsto W\Psi_t\Big)$$

$$\left(\int_{x\in B} |\Psi_0|^2 d^n x = \int_{x \in Q_t(B)} |\Psi_t|^2 d^n x \quad \forall t, B\right) \iff \left(\int_{x\in B} |W\Psi_0|^2 d^n \mu = \int_{x\in Q_t(B)} |W\Psi_t|^2 d^n \mu \quad \forall t, B\right).$$

With all, we say that mutual absolute continuity is the exact condition that allows the *same* pilot-wave theory to be expressed in two different L^2-spaces: the experimental predictions agree by 2, the dynamical equations are both Schrödinger equations by 3 and the ontological predictions agree by 4.

◆

Proof: **Item 1 :** For ($\Longleftarrow$) note that the hypothesis trivially implies the l.h.s of Proposition 8, which is what we wanted to prove by its r.h.s.

For the ($\Longrightarrow$) implication, take the $\rho^2 : X \to (0, +\infty]$ measurable s.th. $d\mu = \rho^2 \; d\nu$ and $d\nu = \frac{1}{\rho^2} d\mu$, which exists and is unique (up to null-set changes) by Corollary 4. Then, defining $\rho := \sqrt{\rho^2}$ and $W\psi := \psi/\rho$ for arbitrary $\psi \in L^2(X, d\nu)$, we get that, for any $B \in \Sigma$,

$$|\psi|^2 d\nu(B) = \int_{x\in B} |\psi|^2(x)d\nu = \int_{x\in B} |\psi|^2(x)\frac{1}{\rho^2(x)}d\mu = \int_{x\in B} |W\psi(x)|^2 d\mu = |W\psi|^2 d\mu(B).$$

Hence, $|\psi|^2 d\nu = |W\psi|^2 d\mu$. This also proves that W is a well-defined map $L^2(X,d\nu) \to L^2(X,d\mu)$ because putting $B = X$, $\|\psi\|^2_{L^2(X,d\nu)} = |\psi|^2 d\nu(X) = |W\psi|^2 d\mu(X) = \|W\psi\|^2_{L^2(X,d\mu)}$. Moreover, this proves that W is an isometry. It is obvious that W is linear too, so it is injective. We just miss to prove that it is surjective. For that, given an arbitrary $\tilde{\psi} \in L^2(X,d\mu)$, note that $(\rho\ \tilde{\psi}) \in L^2(X,d\nu)$ because

$$\left\|\rho\ \tilde{\psi}\right\|^2_{L^2(X,d\nu)} = \int_{x\in X} |\tilde{\psi}|^2(x)\rho^2(x)d\nu = \int_{x\in X} |\tilde{\psi}|^2(x)d\mu = \left\|\tilde{\psi}\right\|^2_{L^2(X,d\mu)} < +\infty.$$

But trivially, $W(\rho\tilde{\psi}) = \tilde{\psi}$. Hence, W is the linear bijection we wanted and it is unitary.

- For the uniqueness (up to "gauge"), consider an arbitrary $\psi \in L^2(X,d\nu)$. The constraint $|\psi|^2 d\nu = |W\psi|^2 d\mu$ together with $d\nu = \frac{1}{\rho^2}d\mu$ imply that $\frac{|\psi|^2}{\rho^2}d\mu = |W\psi|^2 d\mu$, which implies that $\frac{|\psi|}{\rho}(x) = |W\psi|(x)$ for $d\nu$-almost every $x \in X$. But this holds for every W satisfying $W\psi(x) \equiv \frac{\psi(x)}{\rho(x)}e^{i\theta(x;\psi)}$, with a potentially ψ-dependent $\theta(x;\psi) \in [-\pi,\pi)$. Let us prove that in fact, the linearity of W forbids θ from depending on ψ.

Consider arbitrary $\psi,\phi \in L^2(\mathbb{R},d\nu)$, $\alpha \in \mathbb{C}\setminus\{0\}$ and $x \in ess\,supp(\psi)$. Then,

$$\alpha\frac{\psi(x)}{\rho(x)}e^{i\theta(x;\ \psi)} + \frac{\phi(x)}{\rho(x)}e^{i\theta(x;\ \phi)} = \alpha W(\psi)(x) + W(\phi)(x) \overset{(W \text{ linear})}{=}$$

$$= W(\alpha\psi+\phi)(x) = \frac{(\alpha\psi+\phi)(x)}{\rho(x)}e^{i\theta(x;\ \alpha\psi+\phi)} \quad \Longrightarrow$$

$$\Longrightarrow \quad \left|\alpha\psi(x)e^{i\theta(x,\psi)} + \phi(x)e^{i\theta(x,\phi)}\right|^2 = \left|\alpha\psi(x) + \phi(x)\right|^2 \quad \overset{(|z+w|^2=|z|^2+|w|^2+2\,\mathrm{Re}\{z\overline{w}\}\quad \forall z,w\in\mathbb{C})}{\Longrightarrow}$$

$$\Longrightarrow \quad \mathrm{Re}\Big\{\alpha\psi(x)\overline{\phi(x)}e^{i\Delta_{\psi,\phi}(x)}\Big\} = \mathrm{Re}\Big\{\alpha\psi(x)\overline{\phi(x)}\Big\}, \tag{2.17}$$

where $\Delta_{\psi,\phi}(x) := \theta(x,\psi) - \theta(x,\phi)$. The $\alpha = 1$ and $\alpha = i$ cases of (2.17) prove that the real and imaginary parts of $\psi(x)\overline{\phi(x)}e^{i\Delta_{\psi,\phi}(x)}$ equal those of $\psi(x)\overline{\phi(x)}$, such that

$$\psi(x)\overline{\phi(x)}e^{i\Delta_{\psi,\phi}(x)} = \psi(x)\overline{\phi(x)} \quad \Longrightarrow \quad \Delta_{\psi,\phi}(x) = 0 \quad \Longrightarrow \quad \theta(x,\psi) = \theta(x,\phi).$$

- With all, only $W\psi(x) = \frac{\psi(x)}{\rho(x)}e^{i\theta(x)}$ with an arbitrary measurable $\theta : X \to [-\pi,\pi)$ satisfies what we claimed.

Item 2: It is just a "fancy looking" restatement.

Item 3 : It follows from the fact that W is unitary. Let $\psi_0 \in L^2(X,d\nu)$ be arbitrary and define $t \in \mathbb{R} \longmapsto \psi_t := U_t\psi_0$ for some SCOPUG $\{U_t\}_{t\in\mathbb{R}}$ acting on $L^2(X,d\nu)$. Then, if $\widetilde{\psi}_t := W\psi_t$ is the identified path on $L^2(X,d\mu)$, the operators $\widetilde{U}_t := WU_tW^{-1}$ immediately satisfy that $\widetilde{\psi}_t = \widetilde{U}_t\widetilde{\psi}_0$. Now, $\{\widetilde{U}_t\}_{t\in\mathbb{R}}$ also turns out to be a SCOPUG, but this time in $L^2(X,d\mu)$. To see why: "U" because a composition of unitaries is unitary; "G" because $\widetilde{U}_s\widetilde{U}_t = WU_sW^{-1}WU_tW^{-1} = WU_sU_tW^{-1} = WU_{s+t}W^{-1} = \widetilde{U}_{t+s}$; and, "SC" because: let $\widetilde{\psi} \in L^2(X,d\mu)$ arbitrary, then we know that $t \in \mathbb{R} \mapsto U_t(W^{-1}\widetilde{\psi})$ is continuous since $\{U_t\}_{t\in\mathbb{R}}$ is SC, but then, since W is a continuous map (it has operator bound 1) $t \in \mathbb{R} \mapsto W \circ U_t(W^{-1}\widetilde{\psi}) = \widetilde{U}_t\widetilde{\psi}$ is continuous too.

Finally, we prove that the generator of $\widetilde{U}_t$, denoted $\widetilde{H}$, can be obtained from that of U_t as $\widetilde{H} = WHW^{-1}$ with $D(\widetilde{H}) = WD(H)$. Let $\psi \in D(\widetilde{H})$, then

$$\widetilde{H}\psi \overset{(\star)}{=} i\hbar\left(\frac{\mathrm{d}}{\mathrm{d}t}\widetilde{U}_t\psi\right)\Big|_{t=0} = i\hbar \overset{\|\cdot\|}{\lim_{h\to 0}} \left(\frac{\widetilde{U}_h\psi-\psi}{h}\right) \overset{(\star\star)}{=} i\hbar W\left(\lim_{h\to 0}\frac{U_hW^{-1}\psi - W^{-1}\psi}{h}\right) =$$

$$= i\hbar W\left(\frac{\mathrm{d}}{\mathrm{d}t}U_tW^{-1}\psi\right)\Big|_{t=0} \overset{(\star)}{=} WHW^{-1}\psi.$$

In $(\star)$ we employed the definition of generator of a SCOPUG, while in $(\star\star)$ we used the sequential continuity of W together with $\tilde{U}_t = WU_tW^{-1}$.

Item 4 : Use that by 2, $|\psi_0|^2 d\nu = |\widetilde{\psi}_0|^2 d\mu$ and $|\psi_t|^2 d\nu = |\widetilde{\psi}_t|^2 d\mu$. Then the right hand sides of the two equivariance conditions are exactly equal to each other. Likewise for the left ones.

ο.ε.δ.

2.4 Take an Infinite Product Measure of *Probability* Measures

With all, we proved that the same Schrodinger QM can be done using, instead of $d^n x$, any mutually absolutely continuous measure $d^n\mu$. The key realization is then the following one: unlike the pathological non-uniqueness and non-sigma-finiteness we found when trying to make sense of a literal limit of $d^n x$, there is a class of measures $d^n\mu$ for which a limit product measure $d^\infty\mu$ exists uniquely and is very well-behaved. Those are the probability measures.

Theorem 6 (*Kolmogorov's Extension Theorem*)**.** Let $(X_j)_{j\in I}$ be a family of topological spaces indexed by an arbitrary set I. For each *finite* subset of indices $K \subseteq I$, consider an inner regular probability measure μ_k on $\odot_{j\in K}\mathfrak{B}(X_j)$ obeying the *compatibility condition* that whenever K_1, K_2 are two nested finite subsets of indices $K_1 \subseteq K_2 \subseteq I$, the measure of $B \in \odot_{j\in K_1} X_j$, given by $\mu_{K_1}(B)$ is the same as that given by $\mu_{K_2}(B \times \prod_{j\in K_2\setminus K_1} X_j)$, namely, that

$$\mu_{K_1} = \mu_{K_2} \circ (\pi_{K_1 \leftarrow K_2})^{-1}. \tag{2.18}$$

Then, there *exists a unique probability* measure μ_I on $\odot_{j\in I}\mathfrak{B}(X_j)$ that "restricts" to the provided measures in finite cylinder sets, i.e., $\mu_K = \mu_I \circ (\pi_{K \leftarrow I})^{-1}$. Moreover, it is built using Theorem 1 such that it acts for $E \in \odot_{j\in\mathbb{N}}\mathfrak{B}(X_j)$ as

$$\mu_I(E) = \inf_{\substack{\{\,A_k\,\}_{k\in\mathbb{N}} \subseteq \mathfrak{A}_0 \,: \\ A_k \;=\; E_{J_k} \times \prod_{j\in I\setminus J_k} X_j \text{ for finite } J_k \subseteq I \\ E \;\subseteq\; \bigcup_{k\in\mathbb{N}} A_k}} \left\{\sum_{k=1}^{\infty} \mu_{J_k}(E_{J_k})\right\} \tag{2.19}$$

where $\mathfrak{A}_0$ is the Boolean algebra of Prop. 3 (with $\Sigma_j = \mathfrak{B}(X_j)$). ♦

Proof: See Theorem 2.4.3 and its proof in (Tao, 2011). □

Corollary 5 (*Existence of unique product measures for probability measures*)**.** Let $(X_j)_{j\in I}$ be a family of locally compact, second countable and Hausdorff spaces.[2] Then, for any choice of Borel probability measure $d\mu_j$ on each X_j, there *exists a unique product measure* $d^I\mu \equiv \odot_{j\in I} d\mu_j$, i.e., a unique measure on $\odot_{j\in I}\mathfrak{B}(X_j)$ such that for all finite $J \subseteq I$ and all $E_J \in \odot_{j\in J}\mathfrak{B}(X_j)$

$$d^I\mu(E_J \times \prod_{j\in I\setminus J} X_j) = (d^J\mu)(E_J), \tag{2.20}$$

where $d^J\mu := \odot_{j\in I} d\mu_j$ (the finite product measure of $\{d\mu_j\}_{j\in J}$). In particular, for any $J \subseteq I$ finite and any $E_j \in \mathfrak{B}(X_j)$, $d^I\mu(\prod_{j\in J} E_j \times \prod_{j\in I\setminus J} X_j) = \prod_{j\in J} d\mu_j(E_j)$.

Moreover, $d^I\mu$ is itself a *probability* measure (and hence, σ-finite). ♦

Proof: • For all finite $J \subseteq I$, the space $\prod_{j\in J} X_j$ with product topology is still locally compact (by Theorem 18.6 in (Willard, 2012)), second countable (by Theorem 16.2 in (Willard, 2012)) and Hausdorff (by Theorem 13.8 (b)).

- By Theorem 7.8 in (Folland, 1999), any finite Borel measure on a locally compact, second countable and Hausdorff space is regular (even Radon) and in particular is inner regular. Therefore, for every finite $J \subseteq I$, the finite product measure $d^J\mu := \odot_{j\in J} d\mu_j$ is inner regular.

- We prove that finite product measures satisfy the compatibility condition (2.18). Given finite nested sets $J_1 \subseteq J_2 \subseteq I$, let $E_j \in \mathfrak{B}(X_j)$ for each $j \in J_1$. Defining $d^{J_1}\mu := \odot_{j\in J_1} d\mu_j$, by definition of finite product measure (Lemma 2),

$$d^{J_1}\mu\Bigg(\prod_{j\in J_1} E_j\Bigg) = \prod_{j\in J_1} d\mu_j(E_j). \tag{2.21}$$

Similarly, defining $d^{J_2}\mu := \odot_{j\in J_2} d\mu_j$,

$$\begin{aligned} d^{J_2}\mu \circ (\pi_{J_1\leftarrow J_2})^{-1}\Big(\prod_{j\in J_1} E_j\Big) = d^{J_2}\mu\Big(\prod_{j\in J_1} E_j \times \prod_{j\in J_2\setminus J_1} X_j\Big) \overset{(\text{Lem. }2)}{=} \prod_{j\in J_1} d\mu_j(E_j) \prod_{j\in J_2\setminus J_1} \cancelto{1}{d\mu_j(X_j)} = \\ = \prod_{j\in J_1} d\mu_j(E_j) \overset{(2.21)}{=} d^{J_1}\mu\Big(\prod_{j\in J_1} E_j\Big). \end{aligned} \tag{2.22}$$

But, $d^{J_2}\mu \circ (\pi_{J_1\leftarrow J_2})^{-1}$ is trivially a measure on[a] $\odot_{j\in J_2}\mathfrak{B}(X_j)$. Hence, because we just proved that it agrees with $d^{J_1}\mu$ on product cylinders, the uniqueness statement of Lemma 2 implies that $d^{J_2}\mu \circ (\pi_{J_1\leftarrow J_2})^{-1} = d^{J_1}\mu$ —which is the compatibility condition (2.18).

- Therefore, the hypotheses of Theorem 5 are satisfied and it proves the corollary. ***o.ε.δ.***

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[a] One just needs to use that pre-images and countable unions commute.

Corollary 6. Given a sequence of Borel probability measures on $\mathbb{R}$, $(d\mu_j)_{j\in\mathbb{N}}$, there exists a unique product measure $d^\infty\mu$ on $\mathfrak{B}(\mathbb{R}^\infty)$ and it is moreover, a probability measure. ♦

Proof: $\mathbb{R}$ is locally compact because for any $x_0 \in \mathbb{R}$, the closed ball $\{x \in \mathbb{R} \ : \ |x - x_0| \leq 1\}$ is compact by the Heine Borel property (of $\mathbb{R}$) and its interior trivially contains x_0.In addition, $\mathbb{R}$ is separable, so, by Theorem 16.11 in Willard (2012), it is second countable. Finally, it is a metric space, so it is also Hausdorff. With all, one can apply Corollary 5. ***o.ε.δ.***

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[2] Or any other topological spaces with enough properties to ensure that the finite product of Borel probability measures is *inner regular*.

Theorem 7 (*Tonelli's theorem).* Let $(X, \Sigma_X, \mu_X), (Y, \Sigma_Y, \mu_Y)$ be σ-finite measure spaces and let $f : X \times Y \to [0, +\infty]$ be measurable with respect to $\Sigma_X \odot \Sigma_Y$. Then, $x \mapsto \int_{y\in Y} f(x, y) d\mu_Y$ and $y \mapsto \int_{x\in X} f(x, y) d\mu_X$ are measurable functions (on Σ_X and Σ_Y respectively), and

$$\int_{(x,y)\in X\times Y} f(x, y)\ d\mu_X \odot d\mu_Y = \int_{x\in X} \Big(\int_{y\in Y} f(x, y)\ d\mu_Y \Big) d\mu_X = \int_{y\in Y} \Big(\int_{x\in X} f(x, y)\ d\mu_X \Big) d\mu_Y .$$

◆

Proof: See Theorem 1.7.15 in (Tao, 2011). ☐

Lemma 8. Given $n \in \mathbb{N}$, measurable spaces (X_j, Σ_j) for $j \in \{1, ..., n\}$, and σ-finite measure pairs $d\mu_j, d\nu_j$ on Σ_j, if $d\nu_j \ll d\mu_j\ \forall j \in \{1, ..., n\}$, then the product measures $d^n\mu := \odot_{j=1}^n d\mu_j$, $d^n\nu := \odot_{j=1}^n d\nu_j$ also satisfy $d^n\nu \ll d^n\mu$.

Moreover, if $d\nu_j = \rho_j\ d\mu_j$ for $\rho_j := \frac{d\nu_j}{d\mu_j} : X \to (0, +\infty]$ (given by Cor. 4), then

$$\frac{d^n\nu}{d^n\mu}(x_1, ..., x_n) = \rho_1(x_1) \cdots \rho_n(x_n).$$

(*"The product of Radon-Nikodym derivatives is the derivative of the product measure"*). ◆

Proof: We prove the lemma for the case $n = 2$. The rest of cases follow by finite iteration.

Take an arbitrary $E \in \Sigma_1 \odot \Sigma_2$. $\mathbb{1}_E$ is non-negative and trivially measurable in $\Sigma_1 \odot \Sigma_2$, so we can apply (the Tonelli) theorem 7 to it, such that

$$d^2\nu(E) = \int_{(x_1,x_2)\in X_1\times X_2} \mathbb{1}_E(x_1, x_2)\ d^2\nu \overset{\text{(Tonelli)}}{=} \int_{x_1\in X_1} \Big(\int_{x_2\in X_1} \mathbb{1}_E(x_1, x_2)\ d\nu_2 \Big)\ d\nu_1 \overset{(\nu_2 \ll d\mu_2)}{=}$$

$$= \int_{x_1\in X_1} \Big(\int_{x_2\in X_1} \mathbb{1}_E(x_1, x_2)\ \rho_2(x_2) d\mu_2 \Big)\ d\nu_1 \overset{(\nu_1 \ll d\mu_1)}{=} \int_{x_1\in X_1} \Big(\int_{x_2\in X_1} \mathbb{1}_E(x_1, x_2)\ \rho_2(x_2) d\mu_2 \Big)\ \rho_1(x_1) d\mu_1 =$$

$$\left(\begin{array}{c} f(x_1, x_2) := \mathbb{1}_E(x_1, x_2)\rho_1(x_1)\rho_2(x_2) \text{ is non-negative and measurable too —it is a product} \\ \text{of measurable functions—, so we can apply to it the Tonelli theorem as well} \end{array} \right)$$

$$= \int_{(x_1,x_2)\in X_1\times X_2} \mathbb{1}_E(x_1, x_2)\ \rho_1(x_1)\rho_2(x_2) d^2\mu = \rho_1\rho_2 d^2\mu(E).$$

That is, $d^2\nu = \rho_1\rho_2 d^2\mu$. By Theorem 4, this implies that $d^2\nu \ll d^2\mu$ and that $\frac{d^2\nu}{d^2\mu}(x_1, x_2) = \rho_1(x_1) \cdot \rho(x_2)$. ***o.ε.δ.***

By Lemma 8, given an arbitrary sequence $(d\mu_j)_{j\in\mathbb{N}}$ of Borel probability measures such that[3] $d\mu_j \sim dx_j$, fixing any $n \in \mathbb{N}$, $d^n\mu := \otimes_{j=1}^n d\mu_j \sim d^n x$. Therefore, by Prop. 9, a pilot-wave theory on $\mathbb{R}^n$ using $L^2(\mathbb{R}^n, d^n x)$ or using $L^2(\mathbb{R}^n, d^n\mu)$ will be equivalent for all $n \in \mathbb{N}$. But, while §2.2 and 2.3 showed that there is no reasonable $n \to \infty$ limit for $d^n x$, by Cor. 6, there is a unique limit measure $d^\infty\mu := \odot_{j\in\mathbb{N}} d\mu_j$ on $\mathfrak{B}(\mathbb{R}^\infty)$ and it is σ-finite. Hence, unlike "$L^2(\mathbb{R}^\infty, d^\infty x)$", there is a well-behaved space $L^2(\mathbb{R}^\infty, d^\infty\mu)$ and it yields the standard quantum theory for finite n cut-offs $L^2(\mathbb{R}^n, d^n\mu)$! Moreover:

Lemma 9. $L^2(\mathbb{R}^\infty, d^\infty\mu)$ in the above notation is a *separable* Hilbert space. ◆

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[3] Note that $dx_j \equiv dx$ for all j. The subindex j is employ for mere notational convenience.

Proof: By Theorem 4.13 in (Brezis, 2011), if Q is a second countable metrizable topological space and $d\mu$ is a σ-finite Borel measure of Q, then $L^2(Q, \mathbb{C}, d\mu)$ is separable.

As we saw, by Theorem 16.4.c in Willard (2012), $\mathbb{R}^\infty$ is separable. But, by Theorem 16.11 in Willard (2012), any separable metric space is second countable, so $\mathbb{R}^\infty$ is second countable. We already found it is metrizable in Prop. 6. Therefore, by the above, $L^2(\mathbb{R}^\infty, \mathbb{C}, d\mu)$ is separable.

o.ε.δ.

Therefore, we would still be in the familiar setting of a quantum theory possessing countable ONBs. With all, everything points at $L^2(\mathbb{R}^\infty, d^\infty\mu)$ as the rigorous limit "$\lim_{n\to\infty} L^2(\mathbb{R}^n, d^n x)$". However, there is a subtlety that needs a clarification. It is true that no matter which $d\mu_j \sim dx_j$ or $d\nu_j \sim dx_j$ we choose for each $j \in \mathbb{N}$, at finite cut-offs, the resulting measures $d^n\mu \sim d^n\nu \sim d^n x$ and hence that they will all give the "same" Schrodinger quantum theory. One would naively expect that in the limit they will still satisfy $d^\infty\mu \sim d^\infty\nu$. Unfortunately, this is *not* necessarily true, as we proceed to prove now. In Chapter 4 we will give a very general characterization by Kakutani for when such measures are mut. a.c. For now, we will prove this claim by giving a particular counterexample —which will also exemplify how to employ results from probability theory in our setting.

Terminology from Probability Theory useful to understand $d^\infty\mu$ on $\mathbb{R}^\infty$

A priori, in the context of the infinite product of probability measures from above, one would prefer not to employ probability theory terminology because we are not interpreting the measures $d^n\mu, d^\infty\mu$ in any probability-theoretic sense. For us, the fact that the background measures $d^n\mu$ or $d^\infty\mu$ take value 1 in the full set is a mere matter of convenience allowing the usage of Kolmogorov's extension theorem. Even still, it is good to know the terminology of probability theory in order to be able to employ its related results.

In probability theory, given a measurable space (Ω, Σ) and a probability measure μ (i.e., a measure with $\mu(\Omega) = 1$), instead of "sets" $B \in \Sigma$ one says *events*, and their elements are called *elementary events* —as such, (Ω, Σ) is called the *event space*. Instead of "measure" one uses the word *probability*, e.g., $\mu(B)$ is the *probability of the event* B or *the probability that an elementary event in* B *occurs*. One calls any measurable function $Y : \Omega \to \mathbb{R}$ a *random variable* and calls the push-forwarded measure on $(\mathbb{R}, \mathfrak{B}(\mathbb{R}))$, $\mathbb{P} := \mu \circ Y^{-1}$, the *probability distribution of* Y. One writes for $B \in \mathfrak{B}(\mathbb{R})$, $\mathbb{P}(Y \in B) := \mu(Y^{-1}(B))$ saying this is "the probability that the variable Y takes a value in B". Note that usually one abuses notation employing the same $\mathbb{P}$ for different random variables Y. In fact, one often writes $\mathbb{P}$ instead of μ, altogether. One calls the $L^1(\Omega, \mu)$-norm of Y its *expectation value*, denoting $\mathbb{E}(Y) := \int_{x\in\Omega} Y(x)\ \mu$ —it can equally be computed as $\int_{y\in\mathbb{R}} y\ \mathbb{P}$. Also, one defines the *variance of* Y as $\mathrm{Var}(Y) := \mathbb{E}((Y - \mathbb{E}(Y))^2)$, and it is equally computed as $\mathbb{E}(Y^2) - \mathbb{E}(Y)^2$. In general, one calls the norm $\|Y\|^p_{L^p(\Omega,\mu)}$ the *p-th moment of* Y (sometimes with no absolute value).

If given a set of random variables $\{Y_j\}_{j=1}^n$, one can consider an $\mathbb{R}^n$-valued measurable function $Y : \Omega \to \mathbb{R}^n, w \mapsto (Y_1(w), ..., Y_n(w))$ that is called *random vector*, of *probability distribution* $\mu \circ Y^{-1}$ on $\mathfrak{B}(\mathbb{R}^n)$. In this context, the distributions of each component, $\mu \circ Y_k^{-1}$, are called *marginal distributions*. Finally, note that one says *almost surely* instead of *almost everywhere*.

As an example, the "Born rule" consists on treating $\mu_t := |\psi_t|^2 d^n x$ as a time dependent probability measure on $(\Omega, \Sigma) = (\mathbb{R}^n, \mathfrak{B}(\mathbb{R}^n))$, for which the elementary events are the possible system configurations $x \in \mathbb{R}^n$. Each projection map $\pi_k : \mathbb{R}^n \to \mathbb{R}$ then gives a random variable the distribution of which, $\mathbb{P}(x_k \in A) := (|\psi_t|^2 d^n x) \circ \pi_k^{-1}(A)$, tells the probability that the k-th degree of freedom takes value in $A \in \mathfrak{B}(\mathbb{R})$. Together, the projection maps π_k form a trivial random vector $(\pi_1, ..., \pi_n) = Id$, the distribution of which is exactly μ_t.

One last terminological paragraph. Each random variable Y induces in Ω a σ-algebra by pulling back the σ-algebra on $\mathbb{R}$: $\sigma(Y) := Y^{-1}(\mathfrak{B}(\mathbb{R}))$ —note that it is a subset of Σ. These are the events that matter if one is solely interested on Y's outcomes. Then, given a set of random variables $(Y_j)_{j=1}^n$, one says that they are *independent* when $\mu(\bigcap_{j=1}^n B_j) = \prod_{j=1}^n \mu(B_j)$ for all choices of $B_j \in \sigma(Y_j)$. That is, when $\mathbb{P}\Big(\bigcap_{j=1}^n \{Y_j \in A_j\}\Big) = \prod_{j=1}^n \mathbb{P}(Y_j \in A_j)$ for all $A_j \in \mathfrak{B}(\mathbb{R})$. In words, when the probabilities of outcomes for each random variable do not depend on —are independent of— what the other variable's outcomes are. One can prove (see Lemma 4.10 in (Kallenberg, 1997)) that $(Y_j)_{j=1}^n$ are independent *if and only if* the distribution of the random vector they form is exactly the product measure of their marginal distributions: $\mu \circ (Y_1, ..., Y_n)^{-1} = \mu \circ Y_1^{-1} \odot \cdots \odot \mu \circ Y_n^{-1}$. In a similar fashion, one says that a *sequence* of random variables $(Y_j)_{j\in\mathbb{N}}$ (or *random sequence*) is *independent* if any finite subfamily of the sequence is independent. Finally, one says $(Y_j)_{j\in\mathbb{N}}$ are *equally distributed* if all marginal distributions yield the same $\mathfrak{B}(\mathbb{R})$-measure: $\mu \circ Y_k^{-1} = \mu \circ Y_j^{-1}\ \forall j, k \in \mathbb{N}$.

In our case, we took $(\mathbb{R}^\infty, \mathfrak{B}(\mathbb{R}^\infty))$ (now seen as an event space) and equipped it with the infinite product of probability measures $d^\infty\mu$ for the sequence of probability measures $(d\mu_j)_{j\in\mathbb{N}}$ on $\mathfrak{B}(\mathbb{R})$. Each projection map $\pi_k : \mathbb{R}^\infty \to \mathbb{R}$ defines a random variable on the probability space $(\mathbb{R}^\infty, \mathfrak{B}(R^\infty), d^\infty\mu)$ so $(\pi_k)_{k\in\mathbb{N}}$ is a random sequence. It is a very trivial random sequence: on the one hand, by definition of $d^\infty\mu$ (Cor. 5), the marginal distribution of π_k is $d\mu_k$, i.e., $d^\infty\mu \circ \pi_k^{-1} = d\mu_k$. Moreover, for all finite $J \subset \mathbb{N}$, the distribution of the random vector $(\pi_k)_{k\in J}$ satisfies (by Cor. 5), $d^\infty\mu \circ (\pi_{J\leftarrow\mathbb{N}})^{-1} = \odot_{k\in J} d\mu_k$. As such, $(\pi_k)_{k\in\mathbb{N}}$ is a sequence of *independent* random variables. In particular, they will be equally distributed whenever $d\mu_j = d\mu_k\ \forall j, k \in \mathbb{N}$.

Proposition 10 (*Kolmogorov's Strong Law of Large Numbers*)**.** Given a probability space $(\Omega, \Sigma, d\mu)$ and a sequence of independent random variables $(Y_j)_{j\in\mathbb{N}} \subseteq L^2(\Omega, d\mu)$,

$$\sum_{j=1}^\infty \frac{\mathrm{Var}(Y_j)}{j^2} < +\infty \implies \lim_{N\to\infty} \sum_{j=1}^N \frac{Y_j - \mathbb{E}(Y_j)}{N} = 0 \quad \text{almost surely.}$$

♦

Proof: See 10.12 in (Folland, 1999). □

Proposition 11 (*Kolmogorov-Khinchin Two-Series Theorem*)**.** Given a probability space $(\Omega, \Sigma, d\mu)$ and a sequence of independent random variables $(Y_j)_{j\in\mathbb{N}}$,

$$\sum_{j=1}^\infty \mathbb{E}(Y_j) < +\infty \;\text{ and }\; \sum_{j=1}^\infty \mathrm{Var}(Y_j) < +\infty \implies \sum_{j=1}^\infty Y_j < +\infty \text{ almost surely.}$$

If $\mathbb{P}(|Y_j| \leq c) = 1$ for some $c > 0$, the reverse implication also holds. ♦

Proof: See Theorem 2 in Sect. 2, Chap. 4 of (Shiryaev, 2016)'s Vol. 2. □

Lemma 10. There exist choices of Borel probability measures on $\mathbb{R}$, $(d\mu_j)_{j\in\mathbb{N}}$, $(d\nu_j)_{j\in\mathbb{N}}$ such that $d\mu_j \sim d\nu_j \sim dx$ for all $j\in\mathbb{N}$ but such that $d^\infty\mu \not\sim d^\infty\nu$.

♦

Proof: For each $j\in\mathbb{N}$, consider $d\mu_j := (2\pi)^{-1/2}exp(-x_j^2/2)dx$ and $d\nu_j := j(2\pi)^{-1/2}exp(-j^2x_j^2/2)$ dx, which are normalized Gaussians with mean 0 and respective standard deviations 1 and $1/j$. First, all $d\mu_j, d\nu_j$ are mut. a.c. to the Lebesgue measure dx because they are given by a strictly positive density (see Cor. 4). Also, they are probability measures because the densities are in $L^1(\mathbb{R}, dx)$. Next, consider $\ell^2(\mathbb{N},\mathbb{R}) := \{\alpha\in\mathbb{R}^\infty \mid \sum_{k=1}^\infty \alpha_k^2 < +\infty\}$. As we will prove in Lemma 28, it is a Borel measurable set, i.e., $\ell^2(\mathbb{N})\in\mathfrak{B}(\mathbb{R}^\infty)$. Now, we claim that $d^\infty\mu(\ell^2(\mathbb{N})) = 0$ and $d^\infty\nu(\ell^2(\mathbb{N})) = 1$. If so, they cannot be mut. a.c.

- To prove that $d^\infty\mu(\ell^2(\mathbb{N})) = 0$ we employ Prop. 10. Define the random variable $Y_j := (\pi_j)^2$ for each $j\in\mathbb{N}$. Note that by definition, the distribution of π_j for each $j\in\mathbb{N}$ is $d\mu_j$ (a normal of expectation 0 and variance 1 —which is in L^p for all $p\in\mathbb{N}$), as such, $\pi_j\in L^4(\mathbb{R}^\infty, d^\infty\mu)$ and $Y_j\in L^2(\mathbb{R}^\infty, d^\infty\mu)$ Together, $(\pi_k)_{k\in\mathbb{N}}$ are independent random variables as proven in the box above, so, $(Y_j)_{j\in\mathbb{N}}$ must be so. With all, $(Y_j)_{j\in\mathbb{N}}$ are independent, identically distributed and in L^2. In particular:

$$\mathbb{E}(Y_j) = \mathbb{E}((\pi_j)^2) \overset{\mathbb{E}(\pi_j)=0}{=} \mathrm{Var}(\pi_j) = 1, \tag{2.23}$$

$$\mathrm{Var}(Y_j) = \mathbb{E}(Y_j^2) - \cancelto{1}{\mathbb{E}(Y_j)^2} = \mathbb{E}((\pi_k)^4) - 1 = \frac{4!}{4\cdot 2!} - 1 = 2, \tag{2.24}$$

where we used that for a random variable Z with a Gaussian distribution of 0 mean and σ standard deviation, $\mathbb{E}(Z^{2m}) = (2m)!\sigma^{2m}/(2^m m!)$, $m\in\mathbb{N}\cup\{0\}$. But then, $\sum_{j=1}^\infty \mathrm{Var}(Y_j)/j^2 = \sum_{j=1}^\infty 2/j^2 = \pi^2/3$, so by Prop. 10, $\lim_{n\to\infty}\frac{1}{n}\sum_{j=1}^n Y_j = \mathbb{E}(Y_j) = 1$ almost surely. Using that $Y_j(\alpha) = \pi_j^2(\alpha) = \alpha_k^2$ for $\alpha\in\mathbb{R}^\infty$ and the "analyst jargon", this is to say that

$$\lim_{n\to\infty}\frac{1}{n}\sum_{j=1}^n \alpha_k^2 = 1 \text{ for almost every } \alpha\in\mathbb{R}^\infty,$$

which means that

$$d^\infty\mu\Big(\Big\{\alpha\in\mathbb{R}^\infty \mid \lim_{N\to\infty}\frac{1}{n}\sum_{j=1}^n \alpha_k^2 = 1\Big\}\Big) = d^\infty\mu(\mathbb{R}^\infty). \tag{2.25}$$

But now, for $\alpha\in\mathbb{R}^\infty$,

$$\lim_{n\to\infty}\frac{1}{n}\sum_{j=1}^n \alpha_k^2 = 1 \quad\Longrightarrow\quad \lim_{n\to\infty} n\Big(\frac{1}{n}\sum_{j=1}^n \alpha_k^2\Big) = +\infty \quad\Longleftrightarrow\quad \sum_{j=1}^\infty \alpha_k^2 = +\infty.$$

Hence, as sets,

$$\Big\{\alpha\in\mathbb{R}^\infty \mid \lim_{N\to\infty}\frac{1}{n}\sum_{j=1}^n \alpha_k^2 = 1\Big\} \subseteq \Big\{\alpha\in\mathbb{R}^\infty \mid \sum_{j=1}^\infty \alpha_j = +\infty\Big\} = \mathbb{R}^\infty\setminus\ell^2(\mathbb{N},\mathbb{R}). \tag{2.26}$$

Altogether,

$$d^\infty\mu(\mathbb{R}^\infty) \overset{(2.25)}{=} d^\infty\mu\Big(\Big\{\alpha\in\mathbb{R}^\infty \mid \lim_{N\to\infty}\frac{1}{n}\sum_{j=1}^n \alpha_k^2 = 1\Big\}\Big) \overset{(2.26)}{\leq} d^\infty\mu\Big(\mathbb{R}^\infty\setminus\ell^2(\mathbb{N})\Big).$$

Therefore, $d^\infty\mu(\mathbb{R}^\infty) = d^\infty\mu(\mathbb{R}^\infty\setminus\ell^2(\mathbb{N}))$ and, a fortiori, $d^\infty\mu(\ell^2(\mathbb{N})) = 0$.

- Now, to prove that $d^\infty\nu(\ell^2(\mathbb{N})) = 1$ we employ Prop. 11. Again, we define the random variables $Y_j := (\pi_j)^2$ for $j \in \mathbb{N}$, which by the same reasons as above form a sequence of independent random variables (no longer identically distributed). In particular,

$$\mathbb{E}(Y_j) = \mathbb{E}((\pi_j)^2) \overset{\mathbb{E}(\pi_j)=0}{=} \mathrm{Var}(\pi_j) = \frac{1}{j^2} \implies \sum_{j=1}^{\infty} \mathbb{E}(Y_j) = \sum_{j=1}^{\infty} \frac{1}{j^2} = \frac{\pi^2}{6}, \tag{2.27}$$

$$\mathrm{Var}(Y_j) = \mathbb{E}((\pi_k)^4) - \mathbb{E}((\pi_k)^2)^2 = \frac{4!}{4 \cdot 2!}\frac{1}{j^4} - \frac{1}{j^4} = \frac{2}{j^4} \implies \sum_{j=1}^{\infty} Var(Y_j) = \sum_{j=1}^{\infty} \frac{2}{j^4} = \frac{\pi^4}{45}. \tag{2.28}$$

Then, by Prop. 11, $\sum_{j=1}^{\infty} Y_j < +\infty$ almost surely, i.e., $\sum_{j=1}^{\infty} \pi_k^2(\alpha) = \sum_{j=1}^{\infty} \alpha_k^2 < +\infty$ for almost every $\alpha \in \mathbb{R}^\infty$, which is to say that $d^\infty\nu(\ell^2(\mathbb{N})) = d^\infty\nu(\mathbb{R}^\infty) = 1$.

o.ε.δ.

Corollary 7. There exist choices of Borel probability measures on $\mathbb{R}$, $(d\mu_j)_{j\in\mathbb{N}}$, $(d\nu_j)_{j\in\mathbb{N}}$ with $d\mu_j \sim d\nu_j \sim dx$ for all $j \in \mathbb{N}$ —hence, for every $n \in \mathbb{N}$, $L^2(\mathbb{R}^n, d^n\mu)$ and $L^2(\mathbb{R}^n, d^n\mu)$ allow the construction of quantum theories equivalent (in the sense of Theorem 5) to those over $L^2(\mathbb{R}^n, d^n x)$—, *but*, such that $L^2(\mathbb{R}^\infty, d^\infty\mu)$ and $L^2(\mathbb{R}^\infty, d^\infty\nu)$ do *not* allow equivalent quantum theories over them. ♦

Conclusion

What seemed to be a great idea turned out to be as catastrophic as the previous ones: there is no obvious choice of probability measures $d\mu_j \sim dx_j$ and the resulting $L^2(\mathbb{R}^\infty, d^\infty\mu)$ spaces do not allow the construction of equivalent QM in the sense of Theorem 5! And yet, we will come back to this section, because it will turn out that when mixed with the idea of next chapter, a solution to the problem at hands —“who is ‘$L^2(\mathbb{R}^\infty, d^\infty x)$’ in QM?”— will emerge.

3

ATTEMPT 2: TAKE THE LIMIT OF THE TENSOR PRODUCT

We have seen that the path of interpreting "$\lim_{n\to\infty} L^2(\mathbb{R}^n, d^n x)$" as taking the limit of the measure leads, even in the best-case scenario, to different possible limit spaces. In this chapter we will try an alternative route in view of the following lemma.

Lemma 11. Canonically, $L^2(\mathbb{R}^n, d^n x)$ is identified with the tensor product of Hilbert spaces $L^2(\mathbb{R}, dx)\otimes\cdots\otimes L^2(\mathbb{R}, dx)$ via the unique unitary isomorphism $U : L^2(\mathbb{R}, dx)\otimes\cdots\otimes L^2(\mathbb{R}, dx) \longrightarrow L^2(\mathbb{R}^n, d^n x)$ that maps elementary tensor products $f_1 \otimes \cdots \otimes f_n$ to the (classes of) factorized functions $(x_1, ..., x_n) \mapsto f_1(x_1)\cdots f_n(x_n)$ —that we denote $f_1 \cdots f_n$ when there is no risk of confusion. ♦

Theorem 8. Let $Z \subseteq X$ be a dense subspace of a normed vector space X and let Y be a Banach space. Let $L : Z \to Y$ be a linear and bounded operator. Then, there *exists a unique bounded linear extension* $\widetilde{L} : X \to Y$ (i.e., $\widetilde{L}_{\restriction Z} = L$) and it has the same operator norm, $\|L\|_{\mathcal{L}(Z,Y)} = \left\|\widetilde{L}\right\|_{\mathcal{L}(X,Y)}$.

In particular, for an arbitrary $x \in X$, given an arbitrary sequence $(z_n)_{n\in\mathbb{N}} \subset Z$ with $\lim_{n\to\infty} z_n = x$ (which exist by density of $Z \subseteq X$), $\lim_{n\to\infty} L(z_n)$ exists and equals $\widetilde{L}(x)$.

Proof: See 3.28 in (Teufel, 2021). □

Proof of Lemma 11: Let $(\phi_j)_{j\in\mathbb{N}} \subset L^2(\mathbb{R}, dx)$ be an orthonormal basis (ONB) of $L^2(\mathbb{R}, dx)$. Then, respectively, $\mathfrak{B}_1 := \{\phi_{j_1} \otimes \cdots \otimes \phi_{j_n}\}_{j_1,...,j_n\in\mathbb{N}}$ and

$$\mathfrak{B}_2 := \left\{\phi_{j_1}\cdots\phi_{j_n} : (x_1, ..., x_n) \mapsto \phi_{j_1}(x_1)\cdots\phi_{j_n}(x_n)\right\}_{j_1,...,j_n\in\mathbb{N}}$$

make ONBs of $L^2(\mathbb{R}, dx)\otimes\cdots\otimes L^2(\mathbb{R}, dx)$ and $L^2(\mathbb{R}^n, d^n x)$ (see e.g., 5.5 in (Teufel, 2021)). Given the constraint that elementary tensor products must be mapped to their respective factorized functions in a linear way, this leaves a unique choice of map between the spans of B_1 and B_2:

$$\begin{array}{cccc} U : & \operatorname{span}(\mathfrak{B}_1) & \longrightarrow & \operatorname{span}(\mathfrak{B}_2) \\ & \Psi = \sum_{j_1,...,j_n=1}^{m} \alpha_{j_1,...,j_n}\ \phi_{j_1}\otimes\cdots\otimes\phi_{j_n} & \longmapsto & \sum_{j_1,...,j_n=1}^{m} \alpha_{j_1,...,j_n}\ \phi_{j_1}\cdots\phi_{j_n} \end{array} . \tag{3.1}$$

U is clearly a bounded linear operator (it has operator norm 1) mapping from a dense subspace of $L^2(\mathbb{R},dx)\otimes\cdots L^2(\mathbb{R},dx)$ to $L^2(\mathbb{R}^n,d^nx)$. By Theorem 8, there exists a unique bounded extension. But, the following one is an extension:

$$\begin{array}{rccc} \widetilde{U}: & L^2(\mathbb{R},dx)\otimes\cdots\otimes L^2(\mathbb{R},dx) & \longrightarrow & L^2(\mathbb{R}^n,d^nx) \\ \Psi = & \sum\limits_{j_1,\dots,j_n=1}^{\infty} \alpha_{j_1,\dots,j_n}\ \phi_{j_1}\otimes\cdots\otimes\phi_{j_n} & \longmapsto & \sum\limits_{j_1,\dots,j_n=1}^{\infty} \alpha_{j_1,\dots,j_n}\ \phi_{j_1}\cdots\phi_{j_n} \end{array} \tag{3.2}$$

and it is trivially a unitary isomorphism of Hilbert spaces. ***o.ε.δ.***

Hence, $L^2(\mathbb{R}^n,d^nx)\cong\otimes_{j=1}^n L^2(\mathbb{R},dx)$, so, what if we interpret "$\lim_{n\to\infty} L^2(\mathbb{R}^n,d^nx)$" as the limit of the *tensor product* instead of the *measure*? —i.e., what if we say that "$\lim_{n\to\infty} L^2(\mathbb{R}^n,d^nx)$" $=\lim_{n\to\infty}\bigotimes_{k=1}^n L^2(\mathbb{R},dx)$? If so, because there is (essentially) a unique notion of infinite tensor product of Hilbert spaces —developed in detail by von Neumann (1939)—, we could get rid of the "jungle of choices" found in Chapter 2 (recall that it was the main issue we found there).

3.1 The Space $L^2(\mathbb{R},dx)\otimes L^2(\mathbb{R},dx)\otimes\cdots$

Due to the fact that von Neumann's theory to build arbitrary tensor products of Hilbert spaces is rarely present in modern textbooks, we provide a detailed review of his construction in Appendix A. In the main text, we will only introduce the minimal details required for the narrative cohesion. For all the claims made with no proof in this section, the reader will find a proof (or a reference to a proof) in the mentioned Appendix

As a starting point, for a tentative vector $\psi_1\otimes\psi_2\otimes\cdots$ with $\psi_j\in L^2(\mathbb{R},dx)$, the obvious norm generalizing the one in finite tensor products would be $\|\psi_1\otimes\psi_2\otimes\cdots\|=\prod_{j\in\mathbb{N}}\|\psi_j\|$. But this only makes sense when the infinite product[1] exists. As such, not all imaginable elementary tensor products will be meaningful.

Definition 11. We call a sequence $(\psi_j)_{j\in\mathbb{N}}=(\psi_1,\psi_2,...)\in\prod_{j\in\mathbb{N}}L^2(\mathbb{R},dx)$ such that $\prod_{j\in\mathbb{N}}\|\psi_j\|$ converges, a *$\mathscr{C}$-sequence*. Denote the subset they constitute in $\prod_{j\in\mathbb{N}}L^2(\mathbb{R},dx)$ by $\mathscr{C}$. ♦

Definition 12. Let $\widetilde{\mathcal{L}}$ denote the vector space of all conjugate-multilinear forms[2] $\Phi:\mathscr{C}\longrightarrow\mathbb{C}$, $(\phi_1,\phi_2,...)\longmapsto\Phi(\phi_1,\phi_2,...)$.

1. For each $(\psi_j)_{j\in\mathbb{N}}\in\mathscr{C}$ define $\bigotimes_{j\in\mathbb{N}}\psi_j\equiv\psi_1\otimes\psi_2\otimes\cdots$ to be the vector of $\widetilde{\mathcal{L}}$ that acts on each $(\phi_j)_{j\in\mathbb{N}}\in\mathscr{C}$ as
$$\psi_1\otimes\psi_2\otimes\cdots(\phi_1,\phi_2....):={}^{q}\prod_{j\in\mathbb{N}}\langle\psi_j,\,\phi_j\rangle:=\begin{cases}\prod\limits_{j\in\mathbb{N}}\langle\psi_j,\phi_j\rangle & \text{if it converges}\\ 0 & \text{if it does not converge}\end{cases}. \tag{3.3}$$
We call them the *elementary tensor products*.
2. Define $V:=span\{\psi_1\otimes\psi_2\cdots\mid(\psi_j)_{j\in\mathbb{N}}\in\mathscr{C}\}$. It is a complex vector subspace of $\widetilde{\mathcal{L}}$.
3. Define the map $\langle\cdot,\cdot\rangle:V\times V\to\mathbb{C}$ by extending
$$\Big\langle\phi_1\otimes\phi_2\otimes\cdots,\ \psi_1\otimes\psi_2\otimes\cdots\Big\rangle:=\psi_1\otimes\psi_2\otimes\cdots(\phi_1,\phi_2,...). \tag{3.4}$$
sesquilinearly to finite linear combinations (conjugate-linear in the *first* slot). It is an inner product on V so $\|\cdot\|:=\sqrt{\langle\cdot,\cdot\rangle}$ is a norm on V.

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[1] See Appendix A to learn the meaning of an infinite product of (possibly complex) numbers.

[2] That is, those forms such that $\Phi(\psi_1,...,\lambda\psi_j+\mu\phi_j,...)=\overline{\lambda}\Phi(\psi_1,...)+\overline{\mu}\Phi(\psi_1,...,\phi_j,\psi_{j+1},...)$ for each $j\in\mathbb{N}$, $(\psi_j)_{j\in\mathbb{N}}\in\mathscr{C}$, $\phi_k\in L^2(\mathbb{R},dx)$ and $\mu,\lambda\in\mathbb{C}$.

4. We define $\bigotimes_{j\in\mathbb{N}} L^2(\mathbb{R},dx) \equiv L^2(\mathbb{R},dx)\otimes L^2(\mathbb{R},dx)\otimes\cdots$ to be the set of

   point-wise limits of $||\cdot||$*-Cauchy sequences in* V

   (i.e.,*point-wise limits of linear combinations of elementary tensor products)*

   They are a subspace of $\widetilde{\mathcal{L}}$. Explicitly, $\Psi\in\bigotimes_{j\in\mathbb{N}} L^2(\mathbb{R},dx)$ *if and only if* $\exists(\Phi_n)_{n\in\mathbb{N}}\subset V$:

   $$(i)\ \Phi(\psi_1,\psi_2,...)=\lim_{n\to\infty}\Phi_n(\psi_1,\psi_2,...)\ \ \forall(\psi_j)_{j\in\mathbb{N}}\in\mathscr{C} \qquad (ii)\ \lim_{N\to\infty}\sup_{n,m\geq N}\|\Phi_n-\Phi_m\|=0. \tag{3.5}$$

5. Equip $\bigotimes_{j\in\mathbb{N}} L^2(\mathbb{R},dx)$ with the inner product that is the natural extension of $\langle\cdot,\cdot\rangle$ (via Cauchy sequences in V).[3] It makes $\bigotimes_{j\in\mathbb{N}} L^2(\mathbb{R},dx)$ a Hilbert space for which V is a dense subspace. We call $(\bigotimes_{j\in\mathbb{N}} L^2(\mathbb{R},dx),\langle\cdot,\cdot\rangle)$ the (proper) *infinite tensor product* (ITP). ♦

By Theorem 24, this is the *unique* way to build the ITP in the following sense. Let $\mathcal{H}$ be a Hilbert space satisfying the following two conditions (that "any" generalization of the tensor product is expected to satisfy):

(a) For every $(\psi_j)_{j\in\mathbb{N}},(\phi_j)_{j\in\mathbb{N}}\in\mathscr{C}$ corresponding vectors $\psi_1\widetilde{\otimes}\psi_2\widetilde{\otimes}\cdots$, $\phi_1\widetilde{\otimes}\phi_2\widetilde{\otimes}\cdots\in\mathscr{H}$ with inner product ${}^q\prod_{j\in\mathbb{N}}\langle\psi_j,\phi_j\rangle$ can be designated (*so, elementary tensor products are defined also in* $\mathscr{H}$).

(b) The span of elementary tensor products is dense in $\mathscr{H}$

Then, there *exists a unique* unitary isomorphism between $\mathscr{H}$ and $\bigotimes_{j\in\mathbb{N}} L^2(\mathbb{R},dx)$ that identifies the elementary tensor products $\psi_1\widetilde{\otimes}\psi_2\widetilde{\otimes}\cdots\longleftrightarrow\psi_1\otimes\psi_2\otimes\cdots$ with each other.

Definition 13. **(i)** A $\mathscr{C}_0$*-sequence* is a tuple $(\psi_j)_{j\in\mathbb{N}}\in\prod_{j\in\mathbb{N}} L^2(\mathbb{R},dx)$ for which $\sum_{j=1}^{\infty}|\|\psi_j\|-1|$ exists. We denote the set they compose by $\mathscr{C}_0$.

Note that $\mathscr{C}_0\subseteq\mathscr{C}$ and that the rest of $\mathscr{C}$-sequences $(\psi_j)_{j\in\mathbb{N}}\in\mathscr{C}\setminus\mathscr{C}_0$ are all such that $\bigotimes_{j\in\mathbb{N}}\psi_j=0$. Hence, by taking $\mathscr{C}_0$ instead of $\mathscr{C}$ one only looses tuples that have a $\vec{0}$ elementary tensor product[4] (this is a necessary refinement for the following to be an equivalence relation).

(ii) For $(\psi_j)_{j\in I},(\phi_j)_{j\in I}\in\mathscr{C}_0$ define the *equivalence relation*

$$(\psi_j)_{j\in\mathbb{N}}\approx(\phi_j)_{j\in\mathbb{N}} \qquad :\Longleftrightarrow \qquad \left(\ \sum_{j=1}^{\infty}|\langle\psi_j,\phi_j\rangle-1| \text{ exists }\right). \tag{3.6}$$

The set of all $\approx$-equivalence classes will be denoted by Γ and the equivalence class of $(\psi_j)_{j\in\mathbb{N}}$ by $[(\psi_j)_{j\in\mathbb{N}}]$. Note that we will abuse notation saying that $\bigotimes_{j\in\mathbb{N}}\psi_j$ is a $\mathscr{C}_0$-sequence or it belongs to some $\approx$-class whenever $(\psi_j)_{j\in\mathbb{N}}$ does so. ♦

Lemma 12. Given $(f_j)_{j\in\mathbb{N}},(g_j)_{j\in\mathbb{N}}\in\mathscr{C}_0$,

$$\begin{cases} \bullet\ (f_j)_{j\in\mathbb{N}}\approx(g_j)_{j\in\mathbb{N}}\Longrightarrow\displaystyle\prod_{j\in\mathbb{N}}\langle f_j,g_j\rangle \text{ converges. It is } \neq 0 \text{ unless } \langle f_j,g_j\rangle=0 \text{ for some } j\in\mathbb{N}. \\ \bullet\ (f_j)_{j\in\mathbb{N}}\not\approx(g_j)_{j\in\mathbb{N}}\Longrightarrow \textit{either } \displaystyle\prod_{j\in\mathbb{N}}\langle f_j,g_j\rangle \text{ converges to 0, } \textit{or} \text{ does not converge. Hence, } \bigotimes_{j\in\mathbb{N}} f_j\perp\bigotimes_{j\in\mathbb{N}} g_j. \end{cases}$$

♦

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[3] For $\Psi,\Phi\in\otimes_{j\in\mathbb{N}}L^2(\mathbb{R},dx)$, we define $\langle\Psi,\Phi\rangle:=\lim_{j\to\infty}\langle\Psi_j,\Phi_j\rangle$, where $(\Psi_j)_j,(\Phi_j)_{\subseteq}V$ are Cauchy sequences whose strong limits yield Ψ,Φ respectively.

[4] The theorem proving uniqueness of ITP also holds if we only look at $\mathscr{C}_0$-sequences.

In a sense, the equivalence relation $\approx$ captures when elementary tensor products lay "close to each other" in the Hilbert space. As we will see in a moment, they have to be "finite perturbations" of each other to be in the same equivalence class. Most importantly, this equivalence relation allows a very convenient "dissection" of the ITP (the physical interpretation of which is clarified in Appendix B):

Definition 14. For each class $\mathfrak{C} \in \Gamma$, denote the closed vector subspace of $\bigotimes_{j\in\mathbb{N}} L^2(\mathbb{R}, dx)$ *generated by* the elementary tensor products in $\mathfrak{C}$ as

$$\otimes^{\mathfrak{C}}_{j\in\mathbb{N}} L^2(\mathbb{R}, dx) := \overline{\operatorname{span}\Big\{\psi_1 \otimes \psi_2 \otimes \cdots \ \big|\ (\psi_j)_{j\in\mathbb{N}} \in \mathfrak{C}\Big\}}. \tag{3.7}$$

We call it the $\mathfrak{C}$-th *improper ITP* or in short, the $\mathfrak{C}$-th *layer (of the ITP)*. ♦

Theorem 9. **(i)** For every class $\mathfrak{C} \in \Gamma$ there exist multiple $\bigotimes_{j\in\mathbb{N}} \psi^o_j \in \bigotimes_{j\in\mathbb{N}} L^2(\mathbb{R}, dx)$ with $\left\|\psi^o_j\right\| = 1\ \forall j \in \mathbb{N}$. We call them *generators* of the $\mathfrak{C}$-th layer because each of them satisfies:

$$\overline{\operatorname{span}\Big\{\phi_1 \otimes \cdots \otimes \phi_n \otimes \psi^o_{n+1} \otimes \psi^o_{n+2} \otimes \cdots \ \big|\ n \in \mathbb{N},\ \phi_j \in L^2(\mathbb{R}, dx)\Big\}} = \otimes^{\mathfrak{C}}_{j\in\mathbb{N}} L^2(\mathbb{R}, dx). \tag{3.8}$$

Informally, one can obtain the whole $\mathfrak{C}$-th layer by considering the span of all the elementary tensor products that are finitely many factor-replacements away from $\psi^o_1 \otimes \psi^o_2 \otimes \cdots$ (i.e., those that share the asymptotic tail of $\otimes_{j\in\mathbb{N}}\psi^o_j$).

- **(ii)** If $\mathfrak{C}, \mathfrak{D} \in \Gamma : \mathfrak{C} \neq \mathfrak{D}$, then $\otimes^{\mathfrak{C}}_{j\in\mathbb{N}} L^2(\mathbb{R}, dx) \perp \otimes^{\mathfrak{D}}_{j\in\mathbb{N}} L^2(\mathbb{R}, dx)$, i.e., vectors from different layers are orthogonal to each other.
- **(iii)** $\otimes_{j\in\mathbb{N}} L^2(\mathbb{R}, dx)$ equals the closure of finite linear combinations of vectors from all the layers, i.e.,

$$\otimes_{j\in\mathbb{N}} L^2(\mathbb{R}, dx) = \overline{\operatorname{span}\Big(\bigcup_{\mathfrak{C}\in\Gamma} \otimes^{\mathfrak{C}}_{j\in\mathbb{N}} L^2(\mathbb{R}, dx)\Big)}. \tag{3.9}$$

(Points (ii) and (iii) suggest that $\bigotimes_{k\in\mathbb{N}} L^2(\mathbb{R}, dx) =$ "$\bigoplus_{\mathfrak{C}\in\Gamma}$" $(\otimes^{\mathfrak{C}}_{j\in\mathbb{N}} L^2(\mathbb{R}, dx))$. We will make this precise in the next section.)

- **(iv)** The number of layers $|\Gamma|$ is *uncountable.*
- **(v)** Each layer $\otimes^{\mathfrak{C}}_{j\in\mathbb{N}} L^2(\mathbb{R}, dx)$ is a *separable* Hilbert space (with the induced inner product). Hence, $\otimes_{j\in\mathbb{N}} L^2(\mathbb{R}, dx)$ is a *non-separable* Hilbert space. ♦

Lemma 13. Given an ONB $\{\phi_n\}_{n\in\mathbb{N}}$ of $L^2(\mathbb{R}, dx)$,

- **(i)** the set of all elementary tensor products formed by elements of the ONB,

$$\Big\{\phi_{n_1} \otimes \phi_{n_2} \otimes \cdots \ \big|\ n_k \in \mathbb{N} \ \ \forall k \in \mathbb{N}\Big\} \tag{3.10}$$

is an orthonormal family but is *not* an ONB of $\bigotimes_{k\in\mathbb{N}} L^2(\mathbb{R}, dx)$.

- **(ii)** Fixing a $\phi_{k_1} \otimes \phi_{k_2} \otimes \cdots$ for some $(k_\ell)_{\ell\in\mathbb{N}} \in \prod_{\ell\in\mathbb{N}} \mathbb{N}$, the countable set

$$\Big\{\phi_{n_1} \otimes \cdots \otimes \phi_{n_N} \otimes \phi_{k_{N+1}} \otimes \phi_{k_{N+2}} \otimes \cdots \ \big|\ N \in \mathbb{N},\ n_1, ..., n_n \in \mathbb{N}\Big\} \tag{3.11}$$

belongs to a common layer of the ITP and makes an ONB of that layer.

Furthermore, for any layer $\mathfrak{C} \in \Gamma$, one can find an ONB as follows. Given an arbitrary generator $\psi_1^o \otimes \psi_2^o \otimes \cdots$ of $\mathfrak{C}$, build for each $j \in \mathbb{N}$ build an ONB $(\phi_j^n)_{n\in\mathbb{N}_0}$ of $L^2(\mathbb{R}, dx)$ that has as zeroth element ψ_j^o, i.e., $\phi_j^0 := \psi_j^o$. Then,

$$\left\{\phi_1^{n_1} \otimes \cdots \otimes \phi_N^{n_N} \otimes \psi_{N+1}^o \otimes \psi_{N+2}^o \otimes \cdots \mid N \in \mathbb{N},\ k_1, ..., k_N \in \mathbb{N}_0\right\} \tag{3.12}$$

is an ONB of $\otimes_{j\in\mathbb{N}}^{\mathfrak{C}} L^2(\mathbb{R}, dx)$. ♦

3.2 The Arbitrary Direct Sum of Hilbert Spaces

In order to talk about the "layerization" suggested by Theorem 9.(iii) more explicitly, namely, in order to be able to write

$$\bigotimes_{k\in\mathbb{N}} L^2(\mathbb{R}, dx) = \text{“}\bigoplus_{\mathfrak{C}\in\Gamma}\text{”}\ (\otimes_{j\in\mathbb{N}}^{\mathfrak{C}} L^2(\mathbb{R}, dx)), \tag{3.13}$$

one needs a notion of direct sum $\oplus_{j\in I}\mathcal{H}_j$ of Hilbert spaces $\mathcal{H}_j$ for uncountable index sets I. Since we found no reference where this is defined explicitly in a rigorous way, we proceed to do so in this section.

Definition 15. Given an arbitrary index set I and a family of arbitrary Hilbert spaces $\{\mathcal{H}_j\}_{j\in I}$, define

$$\bigoplus_{j\in I} \mathcal{H}_j := \left\{(f_j)_{j\in J} \in \prod_{j\in I} \mathcal{H}_j \;\middle|\; \sum_{j\in I} \|f_j\|^2 \text{ converges (in the sense of Def. 30)}\right\}. \tag{3.14}$$

♦

Lemma 14. Any vector $(f_j)_{j\in I} \in \bigoplus_{j\in I} \mathcal{H}_j$ is such that $f_j = 0$ for all but, at most, countably many $j \in I$. ♦

Proof: By Prop. 43, if $\sum_{j\in I} \|f_j\|^2$ is finite then $\|f_j\| = 0$ for all but, at most, countably many $j \in \mathbb{N}$. Since $\|\cdot\|$ is a norm for each $\mathcal{H}_j$, $\|f_j\| = 0 \Longleftrightarrow f_j = 0$.

o.ε.δ.

Proposition 12. **(i)** $\bigoplus_{j\in I} \mathcal{H}_j$ has a well-defined complex vector space structure given by

$$\lambda_1 (f_j)_{j\in I} + \lambda_2 (g_j)_{j\in I} := (\lambda_1 f_j + \lambda_2 g_j)_{j\in I}, \tag{3.15}$$

for $(f_j)_{j\in I}, (g_j)_{j\in I} \in \bigoplus_{j\in I} \mathcal{H}_j$ and $\lambda_1, \lambda_2 \in \mathbb{C}$.

(ii) The form $\langle\cdot,\cdot\rangle : \bigoplus_{j\in I} \mathcal{H}_j \times \bigoplus_{j\in I} \mathcal{H}_j \longrightarrow \mathbb{C}$

$$\Big\langle (f_j)_{j\in I},\ (g_j)_{j\in I} \Big\rangle := \sum_{j\in I} \langle f_j, g_j\rangle \tag{3.16}$$

is well-defined and is an *inner product.* ♦

Proof: For the proof we will assume $|I|$ is *infinite* (for the finite case the whole thing follows from the usual direct sum construction).

Given arbitrary $(f_j)_{j\in I}, (g_j)_{j\in I} \in \bigoplus_{j\in I} \mathcal{H}_j$, by Lemma 14, $f_j = 0$ for all $j \in I \setminus \{i_1, i_2, ...\}$ and $g_j = 0$ for all $j \in I \setminus \{k_1, k_2, ...\}$, for some $i_\ell, k_\ell \in I$, $\ell \in \mathbb{N}$. Define $J := \{i_\ell\}_{\ell\in\mathbb{N}} \cup \{k_\ell\}_{\ell\in\mathbb{N}}$. J is countable, so, we can enumerate its elements as $J = \{j_1, j_2, ...\}$.

(i) For arbitrary $\lambda_1, \lambda_2 \in \mathbb{C}$,

$$\sum_{j\in I}\|\lambda_1\, f_j+\lambda_2\, g_j\|^2=\sum_{k=1}^{\infty}\|\lambda_1\, f_{j_k}+\lambda_2\, g_{j_k}\|^2=\lim_{N\to\infty}\sum_{k=1}^{N}\|\lambda_1\, f_{j_k}+\lambda_2\, g_{j_k}\|^2\leq$$

$$\left(\begin{array}{c}\text{using that: }\ \|\lambda_1\, f_{j_k}+\lambda_2\, g_{j_k}\|^2 \overset{\text{(trgl.inq.)}}{\leq} (\ |\lambda_1|\|f_{j_k}\|+|\lambda_2|\|g_{j_k}\|\ )^2= \\ =|\lambda_1|^2\|f_{j_k}\|^2+|\lambda_2|^2\|g_{j_k}\|^2+2|\lambda_1||\lambda_2|\|f_{j_k}\|\|g_{j_k}\|) \overset{\text{(Peter Paul)}}{\leq} 2|\lambda_1|^2\|f_{j_k}\|^2+2|\lambda_2|^2\|g_{j_k}\|^2\end{array}\right) \tag{3.17}$$

$$\leq \lim_{N\to\infty}\left(2|\lambda_1|^2\sum_{k=1}^{N}\|f_{j_k}\|^2+2|\lambda_2|^2\sum_{k=1}^{N}\|g_{j_k}\|^2\right)=2|\lambda_1|^2\sum_{j=1}^{\infty}\|f_j\|^2+2|\lambda_2|^2\sum_{j=1}^{\infty}\|g_j\|^2,$$

which, taking the zeros into account again, is equal to $2|\lambda_1|^2\sum_{j\in I}\|f_j\|^2+2|\lambda_2|^2\sum_{j\in I}\|g_j\|^2$. But this is finite because $(f_j)_{j\in I}, (g_j)_{j\in I} \in \bigoplus_{j\in I}\mathcal{H}_j$.

(ii) $\sum_{j\in I}|\langle f_j, g_j\rangle|$ is finite because:

$$\sum_{j\in I}|\langle f_j, g_j\rangle|=\sum_{k=1}^{\infty}|\langle f_{j_k}, g_{j_k}\rangle| \overset{\text{(Cauchy Sch.)}}{\leq} \sum_{k=1}^{\infty}\|f_{j_k}\|\|g_{j_k}\| \overset{\text{(Peter Paul)}}{\leq} \frac{1}{2}(\sum_{k=1}^{\infty}\|f_{j_k}\|^2+\sum_{k=1}^{\infty}\|g_{j_k}\|^2),$$

which, taking the zeros into account again, is equal to $\frac{1}{2}(\sum_{j\in I}\|f_{j_k}\|^2+\sum_{j\in I}\|g_{j_k}\|^2)$. But this is finite because $(f_j)_{j\in I}, (g_j)_{j\in I} \in \bigoplus_{j\in I}\mathcal{H}_j$. Hence, by Prop. 43, $\sum_{j\in I}\langle f_j, g_j\rangle$ is convergent. This proves that the form $\langle\cdot,\cdot\rangle$ is well-defined.

- Now we prove it is an inner product. First, it is conjugate linear in the first argument: given $\lambda_1, \lambda_2 \in \mathbb{C}$, $(h_j)_{j\in I} \in \bigoplus_{j\in I}\mathcal{H}_j$,

$$\Big\langle \lambda_1(f_j)_{j\in I}+\lambda_2(h_j)_{j\in I},\ (g_j)_{j\in I}\Big\rangle=\sum_{j\in I}\Big\langle \lambda_1 f_j+\lambda_2 h_j,\ g_j\Big\rangle=\sum_{k=1}^{\infty}\langle \lambda_1 f_{j_k}+\lambda_2 h_{j_k},\ g_{j_k}\rangle= \tag{3.18}$$

$$\overset{\text{(+ is conts)}}{=} \overline{\lambda_1}\sum_{k=1}^{\infty}\langle f_{j_k},\ g_{j_k}\rangle+\overline{\lambda_2}\sum_{k=1}^{\infty}\langle h_{j_k},\ g_{j_k}\rangle=\overline{\lambda_1}\sum_{j\in I}\langle f_j,\ g_j\rangle+\overline{\lambda_2}\sum_{j\in I}\langle h_j,\ g_j\rangle,$$

which equals $\overline{\lambda_1}\langle (f_j)_{j\in I}, (g_j)_{j\in I}\rangle+\overline{\lambda_2}\langle (h_j)_{j\in I}, (g_j)_{j\in I}\rangle$. The linearity in the second argument is proven analogously. We prove that $\langle\cdot,\cdot\rangle$ is conjugate symmetric:

$$\Big\langle (f_j)_{j\in I},\ (g_j)_{j\in I}\Big\rangle=\sum_{j\in I}\langle f_j, g_j\rangle=\sum_{k=1}^{\infty}\langle f_{j_k}, g_{j_k}\rangle=\sum_{k=1}^{\infty}\overline{\langle g_{j_k}, f_{j_k}\rangle}=\overline{\sum_{k=1}^{\infty}\langle g_{j_k}, f_{j_k}\rangle}=\overline{\Big\langle (g_j)_{j\in I},\ (f_j)_{j\in I}\Big\rangle}$$

- Next, note that

$$\Big\langle (f_j)_{j\in I},\ (f_j)_{j\in I}\Big\rangle=\sum_{j\in I}\|f_j\|^2=\sum_{k=1}^{\infty}\|f_{j_k}\|^2\geq 0. \tag{3.19}$$

Moreover, as soon as some $f_{j_k}\neq 0$, $\|f_{j_k}\|>0$ so, unless $f_j=0$ for all $j\in I$, it must be that $\sum_{k=1}^{\infty}\|f_{j_k}\|^2>0$. Thus, $\langle\cdot,\cdot\rangle$ is also positive definite.

ο.ε.δ.

Proposition 13. The topology induced by $\langle\cdot,\cdot\rangle$ on $\bigoplus_{j\in I}\mathcal{H}_j$ is *Cauchy complete.* ♦

Proof: Let there be for $k \in \mathbb{N}$ a Cauchy sequence $F^k := (f_j^k)_{j\in I} \in \bigoplus_{j\in I} \mathcal{H}_j$, i.e., such that uniformly in k, ℓ, $\left\|F^k - F^\ell\right\| \xrightarrow{k,\ell\to\infty} 0$. By sequential continuity of $x \in \mathbb{R} \mapsto x^2$, uniformly in k, ℓ, $\left\|F^k - F^\ell\right\|^2 \xrightarrow{k,\ell\to\infty} 0$. In particular,

$$\left\|F^k - F^\ell\right\|^2 = \langle F^k - F^\ell, F^k - F^\ell\rangle = \sum_{j\in I}\langle f_j^k - f_j^\ell, f_j^k - f_j^\ell, \rangle = \sum_{j\in I}\left\|f_j^k - f_j^\ell\right\|^2.$$

But then, its uniform limit to 0 in k, ℓ means by definition that $\forall\varepsilon > 0 \;\; \exists N^\varepsilon \in \mathbb{N}$ such that

$$\sum_{j\in I}\left\|f_j^k - f_j^\ell\right\|^2 \leq \varepsilon \quad \forall k, \ell \geq N^\varepsilon. \tag{3.20}$$

Each term $\left\|f_j^k - f_j^\ell\right\|^2$ is non-negative, so by Lemma 34, this means that $\forall\varepsilon > 0, \;\; \exists N^\varepsilon$ such that

$$\sum_{j\in J}\left\|f_j^k - f_j^\ell\right\|^2 \leq \varepsilon \quad \forall J \subseteq I \text{ finite and } \forall k, \ell \geq N^\varepsilon. \tag{3.21}$$

The $|J| = 1$ case tells us that $\forall\varepsilon > 0 \;\; \exists N^\varepsilon \in \mathbb{N}$ such that $\left\|f_j^k - f_j^\ell\right\| \leq \varepsilon \quad \forall k, f \geq N^{\varepsilon^2}$, i.e., that the sequence $(f_j^k)_{k\in\mathbb{N}} \subseteq \mathcal{H}_j$ is a Cauchy sequence. But $\mathcal{H}_j$ is a Hilbert space so $\exists g_j \in \mathcal{H}_j$ that is its limit, i.e., $\lim_{k\to\infty}\left\|f_j^k - g_j\right\| = 0$.

Now, define $G := (g_j)_{j\in I}$. We proceed to prove that $G \in \bigoplus_{j\in I}\mathcal{H}_j$ and that it is the limit of $(F^k)_{k\in\mathbb{N}}$. For all $J \subseteq I$ finite,

$$\sum_{j\in J}\left\|g_j - f_j^k\right\|^2 \overset{\text{(by def)}}{=} \sum_{j\in J}\left\|\overset{\mathcal{H}_j}{\lim_{\ell\to\infty}} f_j^\ell - f_j^k\right\|^2 \overset{\left(\substack{\mathcal{H}_j \text{ continuous} \\ \& \; J \text{ finite}}\right)}{=} \lim_{\ell\to\infty}\sum_{j\in J}\left\|f_j^\ell - f_j^k\right\|^2 \overset{(3.21)}{\leq} \varepsilon \quad \forall k \geq N^\varepsilon. \tag{3.22}$$

This leaves an upper bound that is independent of J.

$$\sup_{J\subseteq I \text{ finite}}\sum_{j\in J}\left\|g_j - f_j^k\right\|^2 \leq \varepsilon \;\; \forall k \geq N^\varepsilon \overset{\text{(Lemma 34)}}{\Longrightarrow} \underbrace{\sum_{j\in I}\left\|g_j - f_j^k\right\|^2}_{\left\|G-F^k\right\|^2} \leq \varepsilon \;\; \forall k \geq N^\varepsilon. \tag{3.23}$$

Fixing any $k_0 \geq N^\varepsilon$ this means

$$\sqrt{\varepsilon} \geq \left\|G - F^{k_0}\right\| \overset{\text{(rev. triang. ineq.)}}{\geq} \left|\|G\| - \left\|F^{k_0}\right\|\right| \implies \|G\| \leq \sqrt{\varepsilon} + \left\|F^{k_0}\right\| < \infty. \tag{3.24}$$

Hence, $G \in \bigoplus_{j\in I}\mathcal{H}_j$. But the last result of (3.23) means, by definition, that $\lim_{k\to\infty}\left\|G - F^k\right\|^2 = 0$. Hence, $(F^k)_{k\in\mathbb{N}}$ converges to G. Since $(F^k)_{k\in\mathbb{N}}$ was an arbitrary Cauchy sequence, $\bigoplus_{j\in I}\mathcal{H}_j$ is Cauchy complete. ***o.ε.δ.***

Corollary 8. $(\bigoplus_{j\in I}\mathcal{H}_j, \; \langle\cdot,\cdot\rangle)$ is a Hilbert space. We call it the *arbitrary direct sum of* $(\mathcal{H}_j)_{j\in I}$. ♦

Proposition 14. The set of vectors for which only finitely many sectors are non-zero, namely

$$\oplus_{j\in I}^{\mathscr{F}}\mathcal{H}_j := \left\{(f_j)_{j\in I} \in \prod_{j\in I}\mathcal{H}_j \;\middle|\; \exists J \subseteq I \text{ finite with } f_j = 0 \;\; \forall j \notin J\right\}, \tag{3.25}$$

is a *dense subspace* of $\bigoplus_{j\in I}\mathcal{H}_j$. ♦

Proof: That $\oplus^{\mathscr{F}}_{j\in I}\mathcal{H}_j$ is a vector subspace of $\bigoplus_{j\in I}\mathcal{H}_j$ is trivial. We prove denseness. Let $F := (f_j)_{j\in I} \in \bigoplus_{j\in I}\mathcal{H}_j$ arbitrary. Then, by Lemma 14, $f_j = 0\ \forall j \in I$ except for (at most) countably many indices, which we denote by $J := \{j_k\}_{k\in\mathbb{N}} \subseteq I$. Let $F^k := (h^k_j)_{j\in I}$ where

$$h^k_j := \begin{cases} f_j & \text{if } j = j_\ell \in J \text{ and } \ell \leq k \\ 0 & \text{else.} \end{cases} \tag{3.26}$$

Trivially, $F^k \in \oplus^{\mathscr{F}}_{j\in I}\mathcal{H}_j$. But moreover,

$$\lim_{k\to\infty}\left\|F - F^k\right\|^2 = \lim_{k\to\infty}\sum_{j\in I}\left\|f_j - h^k_j\right\|^2 = \lim_{k\to\infty}\sum_{\ell=k+1}^{\infty}\|f_{j_\ell}\|^2 \overset{\left(\pm\sum_{\ell=1}^{k}\|f_{j_\ell}\|^2\right)}{=} \tag{3.27}$$

$$= \lim_{k\to\infty}\left(\sum_{\ell=1}^{\infty}\|f_{j_\ell}\|^2 - \sum_{\ell=1}^{k}\|f_{j_\ell}\|^2\right) = \sum_{\ell=1}^{\infty}\|f_{j_\ell}\|^2 - \sum_{\ell=1}^{\infty}\|f_{j_\ell}\|^2 = \|F\|^2 - \|F\|^2 = 0.$$

Hence, for arbitrary $F \in \bigoplus_{j\in I}\mathcal{H}_j$, there exists a sequence in $\oplus^{\mathscr{F}}_{j\in I}\mathcal{H}_j$ converging to F. ***o.ε.δ.***

Proposition 15. Let $\mathcal{H}$ be an arbitrary Hilbert space and let $\{\mathcal{H}_j\}_{j\in I}$ be a family of closed vector subspaces of $\mathcal{H}$ that are pairwise orthogonal and have the property that $\mathcal{H} = \overline{\operatorname{span}(\cup_{j\in I}\mathcal{H}_j)}$. Then, there is a canonical unitary identification

$$\bigoplus_{j\in I}\mathcal{H}_j \cong \mathcal{H} \qquad \text{given by} \qquad (f_j)_{j\in I} \cong \sum_{\ell=1}^{\infty} f_{j_\ell}. \tag{3.28}$$

where $\{j_\ell\}_{\ell\in\mathbb{N}} \subseteq I$ is the countable set outside which $f_{j_\ell} = 0$, as given by Lemma 14. ◆

Proof: Define

$$\begin{array}{rccc} U : & \oplus^{\mathscr{F}}_{j\in I}\mathcal{H}_j & \longrightarrow & \mathcal{H} \\ & (f_j)_{j\in I} \text{ with } f_j \neq 0 \text{ only for } j \in \{j_1, ..., j_m\} \subseteq I & \longmapsto & \sum_{\ell=1}^{m} f_{j_\ell}. \end{array} \tag{3.29}$$

It is obviously linear but it is also isometric, since:

$$\|U((f_j)_{j\in I})\|^2 = \left\|\sum_{\ell=1}^{n} f_{j_\ell}\right\|^2 \overset{(\mathcal{H}_j\perp\mathcal{H}_k)}{=} \sum_{\ell=1}^{n}\|f_{j_\ell}\|^2 = \|(f_j)_{j\in I}\|^2. \tag{3.30}$$

Hence, in particular, it is bounded and has operator norm 1. By Props. 14 and 13 it maps from a dense subspace to a Banach space. Therefore, by Theorem 8, there exists a unique linear extension $\widetilde{U} : \bigoplus_{j\in I}\mathcal{H}_j \to \mathcal{H}$ with the same operator norm.

- Any map that satisfies (3.28), must agree with $\widetilde{U}$ in $\oplus^{\mathfrak{F}}_{j\in I}\mathcal{H}_j$ so, by uniqueness, it will equal $\widetilde{U}$ everywhere. Now we prove that $\widetilde{U}$ is a unitary.

- Given an arbitrary $\Psi \in \bigoplus_{j\in I}\mathcal{H}_j$, by density, there exists a sequence $(\Psi^k)_{k\in\mathbb{N}} \subseteq \oplus^{\mathscr{F}}_{j\in I}\mathcal{H}_j$ with $\Psi_k \xrightarrow[k\to\infty]{} \Psi$. But then, by Theorem 8, $\lim_{k\to\infty} U(\Psi^k)$ exists and $\widetilde{U}(\Psi) = \lim\limits_{k\to\infty} U(\Psi^k)$. This implies:

$$\left\|\widetilde{U}(\Psi)\right\| = \left\|\lim_{k\to\infty} U(\Psi^k)\right\| \overset{(\|\cdot\| \text{ is continuous})}{=} \lim_{k\to\infty}\left\|U(\Psi^k)\right\| \overset{(3.30)}{=} \lim_{k\to\infty}\left\|\Psi^k\right\| = \|\Psi\|, \tag{3.31}$$

i.e., $\widetilde{U}$ is an isometry. Hence, it is also injective by definiteness of the norm.

- By hypothesis, $\text{span}(\cup_{j\in I}\mathcal{H}_j)$ is dense in $\mathcal{H}$ so, for an arbitrary $\Psi \in \mathcal{H}$ there exists a sequence of linear combinations $(\sum_{\ell=1}^{N_n}\Psi^n_{j^n_\ell})_{n\in\mathbb{N}}$ in $\mathcal{H}$ with $\Psi^n_{j^n_\ell} \in \mathcal{H}_{j^n_\ell}$ for some $j^n_\ell \in I$ and $N_n \in \mathbb{N}$, whose limit is Ψ as $n \to \infty$. For each element in that sequence, define $(\Phi^n_j)_{j\in\mathbb{N}}$ to be the vector of $\oplus^{\mathscr{F}}_{j\in I}\mathcal{H}_j$ that is zero for all $j \in I$ except for $j \in \{j^n_1, ..., j^n_{N_n}\}$, where it takes the value $\Phi^n_j := \Psi^n_j$. Then, by definition, $U((\Phi^n_j)_{j\in I}) = \sum_{\ell=1}^{N_n}\Psi^n_{j^n_\ell}$. But if so, $(\Phi^n_j)_{j\in\mathbb{N}}$ is a Cauchy sequence, because

$$\left\|(\Phi^n_j)_{j\in\mathbb{N}} - (\Phi^m_j)_{j\in\mathbb{N}}\right\| \overset{(\widetilde{U}\text{ linear isometry})}{=} \left\|U((\Phi^n_j)_{j\in\mathbb{N}}) - U((\Phi^m_j)_{j\in\mathbb{N}})\right\| = \left\|\sum_{\ell=1}^{N_n}\Psi^n_{j^n_\ell} - \sum_{\ell=1}^{N(m)}\Psi^m_{j^m_\ell}\right\|, \tag{3.32}$$

which limits to 0 uniformly in n, m because $(\sum_{\ell=1}^{N_n}\Psi^n_{j^n_\ell})_{n\in\mathbb{N}}$ is convergent and hence Cauchy. Therefore, $(\Phi^n_j)_{j\in\mathbb{N}}$ converges to some $\Phi \in \bigoplus_{j\in I}\mathcal{H}_j$ because $\bigoplus_{j\in I}\mathcal{H}_j$ is Cauchy complete by Prop. 13. With all:

$$\widetilde{U}(\Phi) = \widetilde{U}(\lim_{n\to\infty}\Phi^n_j) \overset{(U\text{ is continuous})}{=} \lim_{n\to\infty}\widetilde{U}(\Phi^n_j) = \lim_{n\to\infty}\sum_{\ell=1}^{N_n}\Psi^n_{j^n_\ell} = \Psi. \tag{3.33}$$

Hence, $\widetilde{U}$ is surjective too. Altogether, $\widetilde{U}$ is a unitary isomorphism. ***o.ε.δ.***

3.3 The Direct Sum Decomposition of $\otimes_{k=1}^{\infty} L^2(\mathbb{R}, dx)$ into $\approx$-Layers

Corollary 9. Precisely as anticipated, $\bigotimes_{j\in\mathbb{N}} L^2(\mathbb{R}, dx) \cong \bigoplus_{\mathfrak{C}\in\Gamma}\Big(\otimes^{\mathfrak{C}}_{j\in\mathbb{N}} L^2(\mathbb{R}, dx)\Big)$. ♦

Proof: By Theorem 9, $\{\otimes^{\mathfrak{C}}_{j\in\mathbb{N}}L^2(\mathbb{R}, dx)\}_{\mathfrak{C}\in\Gamma}$ is a family of subspaces of $\bigotimes_{j\in\mathbb{N}} L^2(\mathbb{R}, dx)$ that are closed, pairwise orthogonal and the family's span's closure equals $\bigotimes_{j\in\mathbb{N}} L^2(\mathbb{R}, dx)$. Therefore, by Prop. 15, $\bigotimes_{j\in\mathbb{N}} L^2(\mathbb{R}, dx) = \bigoplus_{\mathfrak{C}\in\Gamma}\Big(\otimes^{\mathfrak{C}}_{j\in\mathbb{N}} L^2(\mathbb{R}, dx)\Big)$. ***o.ε.δ.***

In order to make this identification more natural, let us now introduce the notion of arbitrary sum for vectors that generalizes von Neumann's arbitrary sum of complex numbers —explained in Appendix A.

Definition 16. Given a Hausdorff[5] topological vector space V and an arbitrary index set I, a family of vectors $(v_j)_{j\in I} \subseteq V$ is said to have a well-defined sum $v \in \mathcal{H}$, denoted $\sum_{j\in I} v_j = v$, when the net[6] of finite partial sums $(\sum_{j\in J} v_j)_{J\in\mathcal{P}_0(I)}$ converges to v (see Def. 28 for $\mathcal{P}_0(I)$). ♦

For Hilbert spaces, one can rephrase this without the net theoretic jargon as follows.

Lemma 15. Let $(\mathcal{H}, \|\cdot\|)$ be a normed vector space and let $f, (f_j)_{j\in I} \subseteq \mathcal{H}$ be arbitrary. Then, $\sum_{j\in I} f_j = f$ *if and only if* $\forall\varepsilon > 0$ there exists a finite subset of indices $I^\varepsilon \subseteq I$ such that for all finite super-set $J \supseteq I^\varepsilon$, $J \subseteq I$,

$$\left\| f - \sum_{j\in J} f_j \right\| \le \varepsilon. \tag{3.34}$$

(Informally *when finite partial sums become arbitrarily close to some fixed vector.*) ♦

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[5] That way, by Prop. 42, a convergent net in V has a unique limit vector.

[6] See the gray box around Def. 25 in Appendix A for the basics of net theory.

Proof: By definition, $\sum_{j\in I} f_j = f$ holds *if and only if* $\sum_{j\in J} f_j \xrightarrow[J\to\mathcal{P}_0(I)]{} f \in \mathcal{H}$ holds. By Definitions 27 and 28, this holds *if and only if* for each neighborhood $\mathscr{U}$ of f,

$$\exists \text{ a finite } I_0 \subseteq I \quad \text{such that} \quad \sum_{j\in J} f_j \in \mathscr{U} \quad \forall \text{ finite } J \supseteq I_0,\ J \subseteq I. \tag{3.35}$$

Now, each neighborhood $\mathscr{U}$ of f contains by definition an open neighborhood of f, and in the case of a metric space like $\mathcal{H}$, all such open neighborhoods contain a metric ball around f. Since balls are also neighborhoods of f, (3.35) holds for arbitrary neighborhoods of f *if and only if* it holds for open balls B of arbitrary radius ε around f.

o.ε.δ.

Following the use of this notion of arbitrary sum (in the "ε-form") by Halmos (1957), the next proposition characterizing arbitrary sums becomes a mere corollary of his text.

Proposition 16. Let $\mathcal{H}$ be a Hilbert space, I an arbitrary index set, $(f_j)_{j\in I}, (g_j)_{j\in I} \subseteq \mathcal{H}$ arbitrary families of vectors and $f, g \in \mathcal{H}$.

(i) If I is finite, the notion in Def. 16 coincides with the usual finite sum.

(ii) If I is countable, say, if $I = \{j_k\}_{k\in\mathbb{N}}$, then

$$\left(\sum_{j\in I} f_j = f\right) \iff \left(\lim_{n\to\infty}\left\|\sum_{k=1}^{n} f_{j_k} - f\right\| = 0\right).$$

If so, then $\sum_{j\in I} f_j = \sum_{k=1}^{\infty} f_{j_k}$, where the r.h.s is the usual notion of infinite sum of vectors ($\lim_{N\to\infty}\sum_{k=1}^{N} f_{j_k}$).

(iii) If I is uncountable,

$$\left(\sum_{j\in I} f_j = f\right) \iff \left(\exists \text{ a finite or countable } \ J\subseteq I : \sum_{j\in J} f_j = f \ \text{ and } \ f_j = 0\ \forall j \in I\backslash J\right).$$

(iv) If $\sum_{j\in J} f_j = f$ and $\sum_{j\in J} g_j = g$ then, for all $\alpha \in \mathbb{C}$,

$$(a)\ \sum_{j\in J} \alpha f_j = \alpha f \qquad (b)\ \sum_{j\in J}(f_j + g_j) = f + g \qquad (c)\ \sum_{j\in J}\langle g,\ f_j\rangle = \langle g,\ f\rangle.$$

(v) If $(f_j)_{j\in I}$ is an orthogonal family, $\sum_{j\in I} f_j$ exists *if and only if* $\sum_{j\in I}\|f_j\|^2 < +\infty$.

(vi) Given a possibly uncountable ONB $(\phi_j)_{j\in I} \subseteq \mathcal{H}$, for each vector $f \in \mathcal{H}$ there exist unique expansion coefficients $(\alpha_j)_{j\in I} \subseteq \mathbb{C}$ such that $f = \sum_{j\in I}\alpha_j\phi_j$. In particular, $\alpha_j = \langle\phi_j, f\rangle$ and at most countably many α_j are non-zero.

♦

Proof: **Item (i):** Just take $I^\varepsilon = I$ for any $\varepsilon > 0$.

Item (ii): ($\Rightarrow$) For each $\varepsilon > 0$, take I^ε as in Lemma 15 and let $\{j_{k_1}, ..., j_{k_n}\} := I^\varepsilon$. Define $N^\varepsilon := \max\{k_1, ..., k_n\}$. Then, for any $n \geq N^\varepsilon$ we have that $\{j_1, ..., j_n\}$ is a finite super-set of I^ε and thus, by hypothesis,

$$\left\|\sum_{k=1}^{n} f_{j_k} - f\right\| \leq \varepsilon. \tag{3.36}$$

But by definition, this implies that $\lim_{N\to\infty}\sum_{k=1}^{N} f_{j_k} = f$.

(⇐) For each $\varepsilon > 0$, $\exists N^{\varepsilon}$ such that (3.36) holds for all $n \geq N^{\varepsilon}$. Hence, taking $I^{\varepsilon} = \{j_1, ..., j_{N^{\varepsilon}}\}$, we obtain what we wanted to show.

Item (iii): (⇒) See Theorem 8.1 in (Halmos, 1957) for a proof that the cardinality of the set $J \subseteq I$ such that $f_j \neq 0$ if $j \in J$ must be at most countable. Then, since for all finite $K \subseteq I$, $\sum_{j\in K} f_j = \sum_{j \in K \cap J} f_j$, for each $\varepsilon > 0$ one can take the I^{ϵ} for $\sum_{j\in I} f_j$'s convergence and use $I^{\epsilon} \cap J$ as the ε-set for the convergence of $\sum_{j\in J} f_j$. In particular, $\sum_{j \in J} f_j = f$.

(⇐) Take the I^{ϵ} of the convergence of $\sum_{j\in J} f_j$ as the I^{ϵ} for the convergence of $\sum_{j\in I} f_j$. Any super-set that gets out of J just adds zeros in the partial sums.

Item (iv): See Theorems 7.1, 7.2 and 7.3 in (Halmos, 1957).

Item (v): See Theorem 8.2 in (Halmos, 1957).

Item (vi): See Theorem 14.1.(iv) in (Halmos, 1957) for a proof that $\sum_{j\in I}\langle \phi_j, f\rangle \phi_j = f$ —such that indeed, $\alpha_j = \langle \phi_j, f\rangle$. For the sum to exist, a necessary condition was that at most countably many $\langle \phi_j, f\rangle\phi_j \neq 0$, but this implies, since $\phi_j \neq 0$, that $\langle \phi_j, f\rangle = 0$ for all j except for countably many of them. Finally, to prove uniqueness of expansion coefficients, assume there were some other coefficients $(\beta_j)_{j\in I}$ such that $\sum_{j\in I} \beta_j \phi_j = f$. Then, using item (iv).(c), $\langle \phi_k, f\rangle = \sum_{j\in J} \langle \phi_k, \beta_j \phi_j\rangle = \beta_k$. Therefore, $\beta_k = \alpha_k$. ***o.ε.δ.***

With all, one trivially gets that:

Corollary 10. The isomorphism of Prop. 15 identifies $(f_j)_{j\in I} \cong \sum_{j\in I} f_j$. ⧫

Corollary 11. For each $\mathfrak{C} \in \Gamma$ denote by $P^{\mathfrak{C}}$ the orthogonal projector to the layer $\otimes_{j\in\mathbb{N}}^{\mathfrak{C}} L^2(\mathbb{R}, dx)$. Then, by Corollaries 9 and 10, for each $\Psi \in \otimes_{j\in\mathbb{N}} L^2(\mathbb{R}, dx)$,

$$\Psi = \sum_{\mathfrak{C}\in\Gamma} P^{\mathfrak{C}}\Psi \cong (P^{\mathfrak{C}}\Psi)_{\mathfrak{C}\in\Gamma}.$$

In the rest of the document, we will assume this identification and write "=" instead of "≅ " —as such, we will write $\otimes_{j\in\mathbb{N}} L^2(\mathbb{R}, dx) = \oplus_{\mathfrak{C}\in\Gamma}(\otimes_{j\in\mathbb{N}}^{\mathfrak{C}} L^2(\mathbb{R}, dx))$. ⧫

3.4 Results for Later Chapters

Finally, regarding the arbitrary direct sum, we prove some results that we will need later.

Proposition 17. Let I be an arbitrary index set. For each $j \in I$, let $\mathcal{H}_j$ be a Hilbert space of ONB $(f_j^k)_{k\in K(j)} \subset \mathcal{H}_j$, where the cardinality of the index set $K(j)$ is equal to the Hilbert dimension of $\mathcal{H}_j$. Then, defining for each $j_0 \in I, k_0 \in K(j_0)$, $\Phi(j_0, k_0) := (\phi[j_0, k_0]_j)_{j\in I} \in \bigoplus_{j\in I} \mathcal{H}_j$ with

$$\phi[j_0, k_0]_j := \begin{cases} f_{j_0}^{k_0} & \text{if } j = j_0 \\ 0 & \text{if } j \neq j_0 \end{cases}, \tag{3.37}$$

the set $\mathfrak{B} := \Big(\Phi(j_0, k_0)\Big)_{j_0\in I,\ k_0\in K(j_0)}$ is an ONB of $\bigoplus_{j\in I}\mathcal{H}_j$. As such, the Hilbert dimension of $\bigoplus_{j\in I}\mathcal{H}_j$ is $\sum_{j\in I}|K(j)|$. ⧫

Proof: Let there be an arbitrary $G := (g_j)_{j\in I} \in \oplus_{j\in I}^{\mathscr{F}} \mathcal{H}_j$, and let $J := \{j_1, ..., j_m\} \subseteq I$ be the finite set where $g_j \neq 0$. For each $j \in J$, expand $g_j \in \mathcal{H}_j$ in the given ONB of $\mathcal{H}_j$, such that $g_j = \sum_{k\in K(j)} c_k^j f_j^k$ for some coefficients $(c_k^j)_{k\in K(j)} \in \ell^2(K(j),\mathbb{C})$. Since at most countably many coefficients in $(c_k^j)_{k\in K(j)}$ can be non-zero for $\|g_j\|$ to be finite, let $(k_\ell^j)_{\ell\in\mathbb{N}} \subseteq K(j)$ be their indices, i.e., such that $g_j = \sum_{\ell=1}^\infty c_{k_\ell}^j f_j^{k_\ell}$ (we omit the subindex j from k_ℓ^j for clarity). For $j \notin J$, define $c_k^j \equiv 0$. Then, define $G^n := \left(\sum_{\ell=1}^n c_{k_\ell}^j f_j^{k_\ell}\right)_{j\in I}$ (recall that only those entries with $j \in J$ are non-zero).

Let us prove that $\lim_{n\to\infty} \|G^n - G\| = 0$. By definition of expansion in an ONB, $\forall\varepsilon > 0$ $\exists N^\varepsilon(j) \in \mathbb{N}$ such that

$$\left\|\sum_{\ell=1}^n c_{k_\ell}^j f_j^{k_\ell} - g_j\right\|^2 \leq \frac{\varepsilon}{m} \quad \forall n \geq N^\varepsilon(j). \tag{3.38}$$

Take $N_0^\varepsilon := \max\{N^\varepsilon(j_1), ..., N^\varepsilon(j_m)\}$. Then, $\forall\varepsilon > 0$, $\forall n \geq N_0^\varepsilon$

$$\|G^n - G\|^2 = \sum_{j\in I} \left\|\sum_{\ell=1}^n c_{k_\ell}^j f_j^{k_\ell} - g_j\right\|^2 = \sum_{j\in\{j_1,\dots,j_m\}} \left\|\sum_{\ell=1}^n c_{k_\ell}^j f_j^{k_\ell} - g_j\right\|^2 \leq \sum_{j\in\{j_1,\dots,j_m\}} \frac{\varepsilon}{m} = \varepsilon. \tag{3.39}$$

Hence, by definition, $\|G^n - G\|^2 \xrightarrow{n\to\infty} 0$. Therefore, because each G^n is a finite linear combination of elements in $\mathfrak{B}$ (in particular, a combinations of $m \cdot n$ elements), and G was an arbitrary element of $\oplus_{j\in I}^{\mathscr{F}}\mathcal{H}_j$, we proved that $\oplus_{j\in I}^{\mathscr{F}}\mathcal{H}_j \subseteq \overline{\text{span}\mathfrak{B}}$. But by Prop. 14, $\overline{\oplus_{j\in I}^{\mathscr{F}}\mathcal{H}_j} = \bigoplus_{j\in I}\mathcal{H}_j$. Hence, because the closure $\overline{\oplus_{j\in I}^{\mathscr{F}}\mathcal{H}_j}$ is the smallest closed set that contains $\oplus_{j\in I}^{\mathscr{F}}\mathcal{H}_j$, necessarily $\bigoplus_{j\in I}\mathcal{H}_j \subseteq \overline{\text{span}\mathfrak{B}}$. But then, $\overline{\text{span}\mathfrak{B}} = \bigoplus_{j\in I}\mathcal{H}_j$ because $\bigoplus_{j\in I}\mathcal{H}_j$ is the full set in this topological space.

- Lastly, we prove that $\mathfrak{B}$ is an orthonormal set:

$$\left\langle \Phi(j_0,k_0),\ \Phi(\widetilde{j_0},\widetilde{k_0})\right\rangle = \begin{cases} \langle f_{j_0}^{k_0}, f_{j_0}^{k_0}\rangle & \text{if } (j_0,k_0) = (\widetilde{j_0},\widetilde{k_0}) \\ \langle 0,0\rangle \text{ or } \langle f_{j_0}^{k_0}, 0\rangle \text{ or } \langle 0, f_{\widetilde{j_0}}^{\widetilde{k_0}}\rangle & \text{else} \end{cases} = \begin{cases} 1 & \text{if } (j_0,k_0) = (\widetilde{j_0},\widetilde{k_0}) \\ 0 & \text{else} \end{cases}. \tag{3.40}$$

With all, $\mathfrak{B}$ is an orthonormal set with dense span in the Hilbert space $\bigoplus_{j\in I}\mathcal{H}_j$, i.e., it is an ONB. ***o.ε.δ.***

Lemma 16. Given an arbitrary direct sum $\oplus_{j\in I}\mathcal{H}_j$, let $D_j \subseteq \mathcal{H}_j$ be a dense vector subspace for each $j \in I$. Then, both

$$\oplus_{j\in I}^{\mathscr{F}} D_j := \Big\{\ (\psi_j)_{j\in I} \mid \psi_j \in D_j \text{ for finitely many } j \in I \text{ and else } \psi_j = 0\Big\}$$

and $D := \{(\psi_j)_{j\in I} \in \oplus_{j\in I}\mathcal{H}_j \mid \psi_j \in D_j\}$ are *dense* vector subspaces of $\oplus_{j\in I}\mathcal{H}_j$. ♦

Proof: Let there be an arbitrary $F := (f_j)_{j\in I} \in \oplus_{j\in I}^{\mathscr{F}}\mathcal{H}$ such that $f_j \neq 0$ only for j in some finite $J := \{j_1,...,j_m\} \subseteq I$. For each $j \in J$, by density of $D_j \subseteq \mathcal{H}_j$, $\exists (g_j^n)_{n\in\mathbb{N}} \subseteq D_j$ such that $g_j^n \xrightarrow[n\to\infty]{\|\cdot\|_{\mathcal{H}_j}} f_j$, i.e., $\forall\varepsilon > 0$ $\exists N^\varepsilon(j)$ such that, $\left\|g_j^n - f_j\right\|^2 \leq \varepsilon$ $\forall n \geq N^\varepsilon(j)$.

Then, choosing $N_0^\varepsilon(j) := \max_{j\in J}\{N^{\varepsilon/m}(j)\}$,

$$\sum_{\ell=1}^m \left\|g_{j_\ell}^n - f_{j_\ell}\right\|_{\mathcal{H}_j}^2 \leq \sum_{n=1}^m \varepsilon/m = \varepsilon, \qquad \forall n \geq N_0^\varepsilon. \tag{3.41}$$

Now, define $F^n := (g_j^n)_{j\in I}$ (put $g_j^n \equiv 0$ if $j \in I \setminus J$). Then, $\forall \varepsilon > 0$,

$$\|F^n - F\|^2_{\oplus_j \mathcal{H}_j} = \sum_{j\in J} \left\| g_j^n - f_j \right\|^2_{\mathcal{H}_j} \overset{(3.41)}{\leq} \varepsilon \qquad \forall n \geq N_0^{\varepsilon}, \tag{3.42}$$

Therefore, $F^n \xrightarrow[n\to\infty]{\|\cdot\|_{\oplus_j \mathcal{H}_j}} F$. But trivially, $F^n \in \oplus^{\mathscr{F}}_{j\in I} D_j$ so $\oplus^{\mathscr{F}}_{j\in I}\mathcal{H}_j \subseteq \overline{\oplus^{\mathscr{F}}_{j\in I} D_j}$. Then, since by Prop. 14, $\oplus^{\mathscr{F}}_{j\in I}\mathcal{H}_j$ is dense in $\oplus_{j\in I}\mathcal{H}_j$, $\oplus^{\mathscr{F}}_{j\in I} D_j$ must be dense as well. Finally, one trivially has that $\oplus^{\mathscr{F}}_{j\in I} D_j \subseteq \oplus_{j\in I} D_j$, so, the same follows for $\oplus_{j\in I} D_j$. ***o.ε.δ.***

Proposition 18. Given an arbitrary direct sum $\oplus_{j\in I}\mathcal{H}_j$ and for each $j \in I$ a densely defined operator $(A_j, D(A_j))$ acting on $\mathcal{H}_j$, define

$$\oplus_{j\in I} D(A_j) := \left\{ (\psi_j)_{j\in I} \in \bigoplus_{j\in I}\mathcal{H}_j \;\middle|\; \psi_j \in D(A_j) \;\&\; \|(A_j\psi_j)_{j\in I}\|_{\oplus_{j\in I}\mathcal{H}_j} < +\infty \right\}. \tag{3.43}$$

Then, the mapping $(\oplus_{j\in I} A_j)(\psi_j)_{j\in I} := (A_j\psi_j)_{j\in I}$ for $(\psi_j)_{j\in I} \in \oplus_{j\in I} D(A_j)$

- **(i)** makes $(\oplus_{j\in I} A_j,\ \oplus_{j\in I} D(A_j))$ a densely defined linear operator on $\oplus_{j\in I}\mathcal{H}_j$.
- **(ii)** If all $(A_j, D(A_j))$ are *symmetric*, then $(\oplus_{j\in I} A_j,\ \oplus_{j\in I} D(A_j))$ is symmetric.
- **(iii)** If all $(A_j, D(A_j))$ are *closed* operators, then $(\oplus_{j\in I} A_j,\ \oplus_{j\in I} D(A_j))$ is a closed operator.
- **(iv)** If all $(A_j, D(A_j))$ are *self-adjoint*, then $(\oplus_{j\in I} A_j,\ \oplus_{j\in I} D(A_j))$ is self-adjoint. ♦

Proof: **Item (i):** First we check that $\oplus_{j\in I} D(A_j)$ is a vector space. Let $F := (f_j)_{j\in I}, G := (g_j)_{j\in I} \in \oplus_{j\in I} D(A_j)$ and $\alpha, \beta \in \mathbb{C}$. Then,

$$\alpha F + \beta G = (\alpha f_j + \beta g_j)_{j\in I}.$$

Each entry $\alpha f_j + \beta g_j \in D(A_j)$ because $D(A_j)$ are vector spaces. Moreover,

$$\|(\oplus_{j\in I} A_j)(\alpha F + \beta G)\|^2_{\oplus_j\mathcal{H}_j} = \sum_{j\in I} \|A_j(\alpha f_j + \beta g_j)\|^2_{\mathcal{H}_j} \overset{(3.17)}{\leq} 2|\alpha|^2 \sum_{j\in I} \|A_j f_j\|^2 + 2|\beta|^2 \sum_{j\in I} \|A_j g_j\|^2$$

which equals, by definition, $2|\alpha|^2 \|(\oplus_{j\in I} A_j) F\|^2 + 2|\beta|^2 \|(\oplus_{j\in I} A_j) G\|^2$. But this is finite because $F, G \in \oplus_{j\in I} D(A_j)$. Hence, $\alpha F + \beta G \in \oplus_{j\in I} D(A_j)$.

- Now we prove density. Let there be an arbitrary $F := (f_j)_{j\in I} \in \oplus_{j\in I}\mathcal{H}_j$. By density of $\oplus^{\mathscr{F}}_{j\in I}\mathcal{H}_j$ in $\oplus_{j\in I}\mathcal{H}_j$, for $n \in \mathbb{N}$, there exists a sequence, $F^n := (f_j^n)_{j\in I} \in \oplus^{\mathscr{F}}_{j\in I}\mathcal{H}_j$, with some finite $J_n \subseteq I$ such that $f_j^n = 0$ if $j \in I \setminus J_n$, and such that $\|F - F^n\|_{\oplus_j\mathcal{H}_j} \xrightarrow[n\to\infty]{} 0$.
- Now, by density of the $D(A_j)$ in $\mathcal{H}_j$, for each $n \in \mathbb{N}$, $j \in J_n$, there exists a sequence $(\phi_j^{n,k})_{k\in\mathbb{N}} \subseteq D(A_j) : \left\|\phi_j^{n,k} - f_j^n\right\|_{\mathcal{H}_j} \xrightarrow[k\to\infty]{} 0$. That is, $\forall \varepsilon > 0$, $\exists K^{\varepsilon}_{j,n} \in \mathbb{N}$ such that

$$\left\|\phi_j^{n,k} - f_j^n\right\|^2 \leq \varepsilon, \qquad \forall k \geq K^{\varepsilon}_{j,n}. \tag{3.44}$$

Hence, taking $K^{\varepsilon}_{\star,n} := \max_{j\in J_n}\{K^{\varepsilon/|J_n|}_{\star,n}\}$ and defining $\Phi^{n,k} := (\phi_j^{n,k})_{j\in I}$, where $\phi_j^{n,k} := 0$ if $j \notin J_n$,

$$\left\|\Phi^{n,k} - F^n\right\|^2_{\oplus_j\mathcal{H}_j} = \sum_{j\in J_n} \left\|\phi_j^{n,k} - f_j^n\right\|^2 \leq \varepsilon, \qquad \forall k \geq K^{\varepsilon}_{\star,n}. \tag{3.45}$$

- Now, take the diagonal sequence $(\Phi^{n,K_{\star,n}^{1/n^2}})_{n\in\mathbb{N}}$. Each element is in $\oplus_{j\in I}^{\mathscr{F}}\mathcal{H}_j$ because it has finitely many non-zero entries, but since each entry is in $D(A_j)$, in particular each $\Phi^{n,K_{\star,n}^{1/n^2}} \in \oplus_{j\in I}D(A_j)$.[a] Then,

$$\left\|F-\Phi^{n,K_{\star,n}^{1/n^2}}\right\| \leq \|F-F^n\| + \left\|F^n-\Phi^{n,K_{\star,n}^{1/n^2}}\right\| \leq \|F-F^n\| + \frac{1}{n} \xrightarrow[n\to\infty]{} 0.$$

Therefore, $\oplus_{j\in I}^{\mathscr{F}}\mathcal{H}_j \bigcap \oplus_{j\in I}D(A_j)$ is dense in $\oplus_{j\in I}\mathcal{H}_j$ and in particular, $\oplus_{j\in I}D(A_j)$ as well.

Item (ii): Let $F, G \in \oplus_{j\in I}D(A_j)$

$$\langle F,\ (\oplus_{j\in I}A_j)\,G\rangle = \sum_{j\in I}\langle f_j, A_j g_j\rangle \overset{(A_j\text{ sym})}{=} \sum_{j\in I}\langle A_j f_j, g_j\rangle = \langle(\oplus_{j\in I}A_j)\,F,\ G\rangle.$$

Item (iii): By definition, for each $j\in I$, $(A_j, D(A_j))$ is a closed operator when its graph

$$\Gamma(A_j) := \{\ (f_j, A_j f_j) \in \mathcal{H}_j \oplus \mathcal{H}_j \mid f_j \in D(A_j)\ \}$$

is a closed subset of $\mathcal{H}_j \oplus \mathcal{H}_j$ (in the inner product topology). Now, our claim is that this implies the graph

$$\Gamma(\oplus_{j\in I}A_j) := \left\{\ (\Psi,\ \oplus_{j\in I}A_j\Psi) \in (\oplus_{j\in I}\mathcal{H}_j)\oplus(\oplus_{j\in I}\mathcal{H}_j)\ \middle|\ \Psi \in \oplus_{j\in I}D(A_j)\right\}$$

is a closed subset of $(\oplus_{j\in I}\mathcal{H}_j)\oplus(\oplus_{j\in I}\mathcal{H}_j)$.

Consider an arbitrary sequence indexed by $n\in\mathbb{N}$, $(F^n,\ (\oplus_{j\in I}A_j)F^n) \in \Gamma(\oplus_{j\in I}A_j)$ that converges in $\|\cdot\|_{(\oplus_{j\in I}\mathcal{H}_j)\oplus(\oplus_{j\in I}\mathcal{H}_j)}$ somewhere $(F,G)\in(\oplus_{j\in I}\mathcal{H}_j)\oplus(\oplus_{j\in I}\mathcal{H}_j)$. Our claim will be proven if we prove that $F\in\oplus_{j\in I}D(A_j)$ and $G=(\oplus_{j\in I}A_j)F$, i.e., if we prove that the sequence in fact converges in the graph $\Gamma(\oplus_{j\in I}A_j)$. Let us prove it.

- By assumption,

$$\|(F^n,\ (\oplus_{j\in I}A_j)F^n)-(F,G)\|_{(\oplus_{j\in I}\mathcal{H}_j)\oplus(\oplus_{j\in I}\mathcal{H}_j)} \xrightarrow[n\to\infty]{} 0.$$

Denoting the components as $F=(f_j)_{j\in I}$, $F^n=(f_j^n)_{j\in I}$, $G=(g_j)_{j\in I}$, this implies that $\forall\varepsilon>0$, $\exists N^\varepsilon\in\mathbb{N}$ such that $\forall n\geq N^\varepsilon$,

$$\varepsilon \geq \|(F^n,\ (\oplus_{j\in I}A_j)F^n)-(F,G)\|^2_{(\oplus_{j\in I}\mathcal{H}_j)\oplus(\oplus_{j\in I}\mathcal{H}_j)} = \sum_{j\in I}\left\|(f_j^n,\ A_j f_j^n)-(f_j,g_j)\right\|^2_{\mathcal{H}_j\oplus\mathcal{H}_j} \geq$$

$$\geq \left\|(f_j^n,\ A_j f_j^n)-(f_j,g_j)\right\|^2_{\mathcal{H}_j\oplus\mathcal{H}_j}$$

for any particular $j\in I$. This means that $\left\|(f_j^n,\ A_j f_j^n)-(f_j,g_j)\right\|^2_{\mathcal{H}_j\oplus\mathcal{H}_j} \xrightarrow[n\to\infty]{} 0$ for every $j\in I$. But then, since $\Gamma(A_j)\subseteq\mathcal{H}_j\oplus\mathcal{H}_j$ is closed it must be that $f_j\in D(A_j)$ and $g_j=A_jf_j$. By assumption G is a vector in $\oplus_{j\in I}\mathcal{H}_j$, so $\|(A_jf_j)_{j\in I}\| = \|G\| < +\infty$ and therefore, by definition, $F\in\oplus_{j\in I}D(A_j)$. In particular, $(\oplus_{j\in I}A_j)F=(A_jf_j)_{j\in I}=(g_j)_{j\in I}=G$.

Item (iv): Since $(\oplus_{j\in I}A_j,\ \oplus_{j\in I}D(A_j))$ is a densely defined symmetric operator (due to (i) and (ii)), by Theorem 3.92 in (Teufel, 2021), it will be a self-adjoint operator *if and only if* range$(\oplus_{j\in I}A_j \pm i\,Id) = \oplus_{j\in I}\mathcal{H}_j$.

By the same theorem, because each $(A_j, D(A_j))$ is self-adjoint, $\text{range}(A_j \pm i\,Id) = \mathcal{H}_j$. Hence, given an arbitrary $F = (f_j)_{j\in I} \in \oplus_{j\in I}\mathcal{H}_j$, there exist pre-images by $(A_j \pm i)$ for each f_j, $j \in I$, namely, some $g_j \in D(A_j)$ such that $(A_j \pm iId)g_j = f_j$. Now, note that

$$\|(A_j \pm iId)g_j\|^2_{\mathcal{H}_j} = \|A_j g_j\|^2 + \|g_j\|^2 \pm 2\,\text{Re}\{\langle A_j g_j, i g_j\rangle\} = \|A_j g_j\|^2 + \|g_j\|^2 \mp 2\,\text{Im}\{\langle A_j g_j, g_j\rangle\}, \tag{3.46}$$

where the last term is zero because A_j is symmetric.[b] Hence,

$$+\infty \overset{(F\in\oplus_j\mathcal{H}_j)}{>} \|F\|^2_{\oplus_j\mathcal{H}_j} = \sum_{j\in I} \|(A_j \pm i\,Id)g_j\|^2 = \sum_{j\in I} \|A_j g_j\|^2 + \sum_{j\in I} \|g_j\|^2. \tag{3.47}$$

Defining $G := (g_j)_{j\in I}$. this implies that $\|G\| < +\infty$ and $\sum_{j\in I} \|A_j g_j\|^2 < +\infty$, so, $G \in \oplus_{j\in I} D(A_j)$. But then by definition, $(\oplus_{j\in I} A_j \pm i\,Id)G = F$. Since $F \in \oplus_{j\in I}\mathcal{H}_j$ was arbitrary, this proves that $\text{range}(\oplus_{j\in I} A_j \pm i\,Id) = \oplus_{j\in I}\mathcal{H}_j$ and therefore, that $(\oplus_{j\in I} A_j, \oplus_{j\in I} D(A_j))$ is self-adjoint.

o.ε.δ.

---

[a] The sum of the norms after applying A_j in each entry is finite because there are finitely many entries

[b] $\langle A_j g_j, g_j\rangle \overset{(A_j \text{ sym})}{=} \langle g_j, A_j g_j\rangle \overset{(\langle\cdot,\cdot\rangle \text{ conj. sym})}{=} \overline{\langle A_j g_j, g_j\rangle} \implies \langle A_j g_j, g_j\rangle \in \mathbb{R}$.

3.5 The Issue with this Attempt for a Pilot-Wave Theory

Arguably, among the structures we have found, $\bigotimes_{k\in\mathbb{N}} L^2(\mathbb{R}, dx)$ is the one that deserves the most to be called "$L^2(\mathbb{R}^\infty, d^\infty x)$". Yet, it faces the issue of showing no obvious link to the configuration-space $\mathbb{R}^\infty$, when there are two important reasons to desire such a connection.

- Associated to each unit $\Psi \in \otimes_{j\in\mathbb{N}} L^2(\mathbb{R}, dx)$, we would like to have a "Born rule" measure over $\mathbb{R}^\infty$ providing a link to experimental predictions. However, there is no clear way to connect the elements in $\bigotimes_{k\in\mathbb{N}} L^2(\mathbb{R}, dx)$ with measures over $\mathbb{R}^\infty$.
- We would also like to have an ensemble of trajectories on $\mathbb{R}^\infty$ to provide a primitive ontology to the theory. But, such an ensemble of trajectories, according to the pilot-wave archetype should be equivariant with a measure on $\mathbb{R}^\infty$ given by the state vector $\Psi \in \otimes_{j\in\mathbb{N}} L^2(\mathbb{R}, dx)$. And yet, as we just said, such Ψ have no clear connection to measures on $\mathbb{R}^\infty$.

In order to make the issue more explicit, note that the elements of $\bigotimes_{k\in\mathbb{N}} L^2(\mathbb{R}, dx)$ are linear combinations of symbols like $\psi_1 \otimes \psi_2 \otimes \cdots$, representing maps that take as argument things like $(\phi_1, \phi_2, ...) \in \mathscr{C}$. There is nothing about $(x_1, x_2, ...) \in \mathbb{R}^\infty$ in all that. What we would like is something analogous to Lemma 11 to come back to some L^2-space over the whole configuration-space $\mathbb{R}^\infty$. In the next chapter, we will reveal a suggestive way to do this by employing our discoveries of Chapter 2. Interestingly, we will prove in Chapter 5 that the result is what any orthodox quantum physicist would do to recover a wavefunction with "position arguments" (and with it a "Born rule" measure) from an abstract wave-vector, namely, the joint diagonal representation of the position operators.

4

A SUGGESTIVE RELATION BETWEEN THE TWO ATTEMPTS

The issue exposed in §3.5 might be hard to grasp at first because (especially in physics) one is used to the idea that an element $\psi_1 \otimes \cdots \otimes \psi_n \in L^2(\mathbb{R}, dx) \otimes \cdots \otimes L^2(\mathbb{R}, dx)$ "in reality" is also an element of $L^2(\mathbb{R}^n, d^n x)$: just substitute $\otimes$ by product $\cdot$, put each ψ_j a different argument $\psi_1(x_1) \cdots \psi_n(x_n)$, and finally note that the result is a function from $(x_1, ..., x_n) \in \mathbb{R}^n$ to $\mathbb{C}$ (see Lemma 11). Could we do the same for $\psi_1 \otimes \psi_2 \otimes \cdots \in \bigotimes_{k\in\mathbb{N}} L^2(\mathbb{R}, dx)$? The answer is negative precisely because there is no measure $d^\infty x$ to provide the output function "$(x_1, x_2, ...) \mapsto \psi_1(x_1)\psi_2(x_2) \cdots$" a space "$L^2(\mathbb{R}^\infty, d^\infty x)$" where to belong. And yet, if we accept a slight detour it is possible to get functions over $\mathbb{R}^\infty$ also for $\bigotimes_{k\in\mathbb{N}} L^2(\mathbb{R}, dx)$. Recall that for any $d^n\mu \sim d^n x$, $L^2(\mathbb{R}^n, d^n x)$ and $L^2(\mathbb{R}^n, d^n\mu)$ were practically identical spaces for QM (see Prop. 9). The only thing to do in their identification was to divide the wavefunctions $\psi \in L^2(\mathbb{R}^n, d^n x)$ by the Radon-Nikodym derivative ρ —where $d^n\mu = \rho^2\, d^n x$. As such, in order to recover functions over $\mathbb{R}^n$ from $L^2(\mathbb{R}, dx) \otimes \cdots \otimes L^2(\mathbb{R}, dx)$, one could alternatively identify $\psi_1 \otimes \cdots \otimes \psi_n$ with its corresponding function in $L^2(\mathbb{R}^n, d^n\mu)$. For that, the only thing to do would be to change $\psi_1 \otimes \cdots \otimes \psi_n$ by $\psi_1(x_1) \cdots \psi(x_n)$ as before, but now also divide it by $\rho(x_1, ..., x_n)$. It turns out that (bypassing the middle step) this will also be a possibility in $\bigotimes_{k\in\mathbb{N}} L^2(\mathbb{R}, dx)$.

4.1 Some Layers of $\bigotimes\limits_{k\in\mathbb{N}} L^2(\mathbb{R}, dx)$ can be identified with $L^2(\mathbb{R}^\infty, d^\infty\mu)$

We can formulate the idea above in precise words as:

Lemma 17. Given measures $d\mu_j \sim dx$ on $\mathfrak{B}(\mathbb{R})$ for $j \in \{1, ..., n\}$ or equivalently, given $\rho_j^2 : \mathbb{R} \to (0, +\infty]$ such that $d\mu_j = \rho_j^2\, dx$, there *exists a unique unitary isomorphism*

$$\begin{array}{rrcc} & U_{(\rho_1,...,\rho_n)}: \quad \otimes_{k=1}^n L^2(\mathbb{R}, dx) & \longrightarrow & L^2(\mathbb{R}^n,\ \odot_{k=1}^n \rho_j^2\, dx) \\ \text{such that} & \psi_1 \otimes \cdots \otimes \psi_n & \Longleftrightarrow & \left[\ (x_1, .., x_n) \longmapsto \frac{\psi_1(x_1) \cdots \psi_n(x_n)}{\rho_1(x_1) \cdots \rho_n(x_n)}\ \right]. \end{array}$$

◆

Proof: By Lemma 11, there exists a unique unitary $U_{(\cdot\ \leftarrow\otimes)} : L^2(\mathbb{R}, dx) \otimes \cdots \otimes L^2(\mathbb{R}, dx) \longrightarrow L^2(\mathbb{R}^n, d^n x)$ mapping $\psi_1 \otimes \cdots \otimes \psi_n$ to $\psi_1 \cdots \psi_n$. At the same time, by Prop. 9, there exists a unitary $U_{(d^n\mu \leftarrow d^n x)} : L^2(\mathbb{R}^n, d^n x) \to L^2(\mathbb{R}^n, d^n\mu)$ such that $\psi_1 \cdots \psi_n$ is mapped to $\frac{\psi_1 \cdots \psi_n}{\rho_1 \cdots \rho_n}$. Hence, there exists a unique unitary $U_{(\rho_1,...\rho_n)}$ as the one claimed in the present lemma: $U_{(\rho_1,...\rho_n)} = U_{(d^n\mu \leftarrow d^n x)} \circ U_{(\cdot\ \leftarrow\otimes)}$. ***o.ε.δ.***

Now, we could have done the same thing using complex valued $\rho_j : \mathbb{R} \to \mathbb{C}$ as well: as long as $\rho_j(x) \neq 0$ for dx-almost every $x \in \mathbb{R}$, then $|\rho_j|^2(x) > 0$ and thus, by Cor. 4, $|\rho_j|^2 dx_j \sim dx_j$.

On another note, we found that there is a well-behaved infinite product measure whenever the factors are probability measures. In particular, note that if we chose $\rho_j \in L^2(\mathbb{R}, dx) : \|\rho_j\|_{L^2} = 1$, then $|\rho_j|^2 dx_j$ is a probability measure.[1]

With all, in the same spirit as Lemma 17, we can identify some layers of the ITP $\otimes_{j\in\mathbb{N}} L^2(\mathbb{R}, dx)$ (of Chapter 3) with some of the L^2-spaces over $\mathbb{R}^\infty$ (developed in Chapter 2):

Theorem 10. For each $j \in \mathbb{N}$, let $\rho_j \in L^2(\mathbb{R}, dx)$ be such that $\|\rho_j\|_{L^2} = 1$ and $\rho_j(x) \neq 0$ for almost every $x \in \mathbb{R}$ —so that $d\mu_j := |\rho_j|^2 dx$ is a probability measure with $d\mu_j \sim dx$. Then, there *exists a unique unitary isomorphism*

$$\begin{array}{llcl} W_{[\rho]} : & \otimes_{k\in\mathbb{N}}^{[\rho]} L^2(\mathbb{R}, dx) & \longrightarrow & L^2(\mathbb{R}^\infty,\ \odot_{j=1}^{\infty} |\rho_j|^2 dx) \\ \text{such that} & \psi_1 \otimes \cdots \otimes \psi_n \otimes \rho_{n+1} \otimes \cdots & \Longleftrightarrow & \left[\ (x_1, x_2, ..) \longmapsto \dfrac{\psi_1(x_1)\cdots\psi_n(x_n)}{\rho_1(x_1)\cdots\rho_n(x_n)}\ \right]. \end{array}$$

Note that we use $[\rho] \in \Gamma$ to denote the $\approx$-class of equivalence of $\rho := (\rho_j)_{j\in\mathbb{N}}$. ◆

In order to prove this theorem we will need several other prior results. In particular, we will need to develop an explicit understanding of the $L^2(\mathbb{R}^\infty, d^\infty\mu)$ spaces.

4.2 Some Results about $L^2(\mathbb{R}^\infty, d^\infty\mu)$ spaces

Proposition 19. Let (X, Σ) be a measurable space and $d\mu$ a σ-finite measure on it. For any Boolean algebra $\mathcal{A}$ generating Σ (i.e., such that $\sigma(\mathcal{A}) = \Sigma$), the space of *simple functions that only use finite measure sets in* $\mathcal{A}$, namely,

$$S_{\mathcal{A}} := \left\{ \sum_{j=1}^{n} c_j \mathbb{1}_{A_j} \;\middle|\; n \in \mathbb{N},\ c_j \in \mathbb{C},\ A_j \in \mathcal{A},\ d\mu(A_j) < +\infty \right\}, \tag{4.1}$$

is *dense* in $L^p(X, \mathbb{C}, d\mu)$ for all $p \in [1, +\infty)$. ◆

Proof: Fix some $p \in [1, +\infty)$ and an arbitrary $f \in L^p(X, d\mu) \setminus \{0\}$. We will prove that there exists a sequence $(\psi_n)_{n\in\mathbb{N}} \subseteq S_{\mathcal{A}}$ such that $\lim_{n\to\infty} \|f - \psi_n\|_{L^p} = 0$.

By Prop. 6.7 in (Folland, 1999), the simple functions S_Σ

$$\left\{ \sum_{j=1}^{n} c_j \mathbb{1}_{E_j} \;\middle|\; n \in \mathbb{N},\ c_j \in \mathbb{C},\ E_j \in \Sigma,\ d\mu(E_j) < +\infty \right\} \tag{4.2}$$

are dense in $L^p(X, d\mu)$. Hence, there exists a sequence $\phi_n := \sum_{j=1}^{N_n} c_j^n \mathbb{1}_{E_j^n}$ with $\|f - \phi_n\|_{L^p} \xrightarrow[n\to\infty]{} 0$ for some $E_j^n \in \Sigma$, $N_n \in \mathbb{N}$, $c_j^n \in \mathbb{C}$.

By Theorem 13.D in (Halmos, 1950) (or Exercise 1.4.28 in (Tao, 2011)), given a σ-finite measure $d\mu$ and a Boolean algebra $\mathcal{A}$, for every set $E \in \sigma(\mathcal{A})$ with $d\mu(E) < +\infty$ and for every $\varepsilon > 0$, there exists a set $A \in \mathcal{A}$ such that $d\mu(E \Delta A) \leq \varepsilon$ (i.e.,*sets of finite measure can be arbitrarily well approximated —in terms of measure— by sets from the Boolean algebra).* Thus, in our case, $\forall \varepsilon > 0$, $\exists A_j^{\varepsilon,n} \in \mathcal{A}$ with $d\mu(A_j^{\varepsilon,n} \Delta E_j^n) \leq \varepsilon$.

---

[1] Indeed: $|\rho_j|^2 dx(\mathbb{R}) = \int_{x\in\mathbb{R}} |\rho_j|^2(x) dx = \|\rho_j\|_{L^2}^2 = 1$.

Define $\psi_n^\varepsilon := \sum_{j=1}^{N_n} c_j^n \mathbb{1}_{A_j^{\varepsilon,n}}$ (it is trivially in $S_{\mathcal{A}}$). Using in $(\star)$ that if $x \in E_j^n \cap A_j^{\varepsilon,n}$ then $\mathbb{1}_{E_j^n}(x) - \mathbb{1}_{A_j^{\varepsilon,n}}(x) = 0$,

$$\|\phi_n - \psi_n^\varepsilon\|_{L^p}^p = \int_{x\in X} \Big|\sum_{j=1}^{N_n} c_j^n (\mathbb{1}_{E_j^n}(x) - \mathbb{1}_{A_j^{\varepsilon,n}}(x))\Big|^p d\mu \overset{(\star)}{=} \int_{x\in X} \Big|\sum_{j=1}^{N_n} c_j^n (\mathbb{1}_{E_j^n \setminus A_j^{\varepsilon,n}} - \mathbb{1}_{A_j^{\varepsilon,n}\setminus E_j^n})\Big|^p d\mu \le$$

$$\overset{\text{(triangl. ineq)}}{\le} \int_{x\in X} \Bigg[\sum_{j=1}^{N_n} |c_j^n| \underbrace{(\mathbb{1}_{E_j^n \setminus A_j^{\varepsilon,n}} + \mathbb{1}_{A_j^{\varepsilon,n}\setminus E_j^n})}_{\mathbb{1}_{E_j^n \Delta A_j^{\varepsilon,n}}}\Bigg]^p d\mu \overset{\text{(Newton's binomial)}}{=}$$

$$= \int_{x\in X} \sum_{k_1 + \cdots + k_{N_n} = p} \frac{p!}{k_1! \cdots k_{N_n}!} \prod_{j=1}^{N_n} |c_j^n|^{k_j} \Big(\mathbb{1}_{E_j^n \Delta A_j^{\varepsilon,n}}(x)\Big)^{k_j} d\mu =$$

$$= \underbrace{\Big(\sum_{k_1+\cdot+k_{N_n}=p} \frac{p!}{k_1!\cdots k_{N_n}!} \prod_{j=1}^{N_n} |c_j^n|^{k_j}\Big)}_{=:\alpha_{n,p}} \int_{x\in X} \underbrace{\prod_{j=1}^{N_n} \mathbb{1}_{E_j^n \Delta A_j^{\varepsilon,n}}(x)}_{\mathbb{1}_{\cap_{j=1}^{N_n}(E_j^n \Delta A_j^{\varepsilon,n})}(x)} d\mu = \alpha_{n,p} \cdot d\mu\Bigg(\bigcap_{j=1}^{N_n} (E_j^n \Delta A_j^{\varepsilon,n})\Bigg) \le$$

$$\le \alpha_{n,p} \max_{j\in\{1,\dots,N_n\}} d\mu(E_j^n \Delta A_j^{\varepsilon,n}) \le \alpha_{n,p} \cdot \varepsilon.$$

Then, since $\alpha_{n,p} \neq 0$, we can choose $\varepsilon_n := \frac{1}{\alpha_{n,p}} \|f - \phi_n\|_{L^p}^p$ to get:

$$\|f - \psi_n^{\varepsilon_n}\|_{L^p} \le \|f - \phi_n\|_{L^p} + \|\phi_n - \psi_n^{\varepsilon_n}\|_{L^p} \le \|f - \phi_n\|_{L^p} + (\alpha_{n,p}\varepsilon_n)^{1/p} \le 2\|f - \phi_n\|_{L^p}.$$

Therefore, since $\|f - \phi_n\|_{L^p} \xrightarrow[n\to\infty]{} 0$, we get that $\|f - \psi_n^{\varepsilon_n}\|_{L^p} \xrightarrow[n\to\infty]{} 0$.

o.ε.δ.

Lemma 18. Given $\mathbb{R}^\infty$ with the product topology, the set of *finite unions of finite measurable cylinders*

$$C_{meas} := \left\{ \bigcup_{j=1}^{N} B_1^j \times \cdots \times B_{n_j}^j \times \mathbb{R} \times \mathbb{R} \times \cdots \;\middle|\; n_j, N \in \mathbb{N} \text{ and } B_k^j \in \mathfrak{B}(\mathbb{R}) \right\} = \tag{4.3}$$

$$= \left\{ \bigcup_{j=1}^{N} \bigcap_{k=1}^{n_j} \pi_k^{-1}(B_k^j) \;\middle|\; n_j, N \in \mathbb{N},\ \ B_k^j \in \mathfrak{B}(\mathbb{R}) \right\},$$

is a Boolean algebra and it generates $\mathfrak{B}(\mathbb{R}^\infty)$, i.e., $\sigma(C_{meas}) = \mathfrak{B}(\mathbb{R}^\infty)$. ♦

Proof: Let $A, B \in C_{meas}$ arbitrary. Then $A = \bigcup_{j=1}^N \bigcap_{k=1}^{M_j} \pi_k^{-1}(A_k^j)$ and $B = \bigcup_{j=1}^n \bigcap_{k=1}^{m_j} \pi_k^{-1}(B_k^j)$ for some $A_k^j, B_k^j \in \mathfrak{B}(\mathbb{R}), n, N, m_j, M_j \in \mathbb{N}$.

- First, we will prove that C_{meas} is closed under finite union. $A \cup B = \bigcup_{j=1}^{N+n} \bigcap_{k=1}^{s_j} \pi_k^{-1}(C_k^j)$ for $C_k^j := A_k^j$ and $s_j = M_j$ if $j \le N$ while $C_k^j := B_k^{j-N}$ and $s_j = m_{j-N}$ if $j > N$. But if so, by definition, $A \cup B \in C_{meas}$.
- Now, we show that C_{meas} is closed under complement:

$$A^c = \Big(\bigcup_{j=1}^N \bigcap_{k=1}^{M_j} \pi_k^{-1}(A_k^j)\Big)^c \overset{\text{(de Morgan)}}{=} \bigcap_{j=1}^N \Big(\bigcap_{k=1}^{M_j} \pi_k^{-1}(A_k^j)\Big)^c \overset{\text{(de Morgan)}}{=} \bigcap_{j=1}^N \bigcup_{k=1}^{M_j} \Big(\pi_k^{-1}(A_k^j)\Big)^c.$$

Using that pre-images and complements commute, this is equal to $\bigcap_{j=1}^N \bigcup_{k=1}^{M_j} \pi_k^{-1}((A_k^j)^c)$.

But by Theorem 1.4.c in (Willard, 2012) (distributive law for the intersection over the union $X \cap (\cup_j Y_j) = \cup_j (X \cap Y_j)$), this is equal to:

$$\bigcup_{k_1=1}^{M_1} \cdots \bigcup_{k_N=1}^{M_N} \Big[\pi_{k_1}^{-1}((A^1_{k_1})^c) \bigcap \cdots \bigcap \pi_{k_N}^{-1}((A^N_{k_N})^c)\Big]. \tag{4.4}$$

But now, $\pi_{k_j}^{-1}((A^j_{k_j})^c)$ has the shape $C^j_1 \times \cdots \times C^j_{M_j} \times \mathbb{R} \times \cdots$ for some $C^j_k \in \mathfrak{B}(\mathbb{R})$, so each $\pi_{k_1}^{-1}((A^1_{k_1})^c) \bigcap \cdots \bigcap \pi_{k_N}^{-1}((A^1_{k_N})^c)$ in (4.4) has the shape

$$\Big(C^1_1 \times \cdots \times C^1_{M_1} \times \mathbb{R} \times \cdots\Big) \bigcap \cdots \bigcap \Big(C^N_1 \times \cdots \times C^N_{M_N} \times \mathbb{R} \times \cdots\Big) =$$

$$= \Big(C^1_1 \cap \cdots \cap C^N_1\Big) \times \cdots \times \Big(C^1_{M_1} \cap \cdots \cap C^N_{M_N}\Big) \times \mathbb{R} \times \cdots .$$

Each factor $C^1_k \cap \cdots \cap C^N_k$ is again in $\mathfrak{B}(\mathbb{R})$ because σ-algebras are closed under countable intersection. Therefore, (4.4), which equals A^c, is a finite union of finite measurable cylinders. Namely, $A^c \in C_{meas}$.

- With all, C_{meas} is closed under finite unions and complements so it is a Boolean algebra.

- By definition, any element of C_{meas} is a finite union of a finite intersection of elements in the family $\mathfrak{c}_0$ (see Def. 7). Because a σ-algebra is closed under countable intersections and unions, this implies that $\sigma(\mathfrak{c}_0) = \sigma(C_{meas})$. But by definition 7, $\sigma(\mathfrak{c}_0) =: \odot_{k \in \mathbb{N}} \mathfrak{B}(\mathbb{R})$ and we found in Prop. 1 that $\odot_{k \in \mathbb{N}} \mathfrak{B}(\mathbb{R}) = \mathfrak{B}(\mathbb{R}^\infty)$. Hence, $\sigma(C_{meas}) = \mathfrak{B}(\mathbb{R}^\infty)$.

o.ε.δ.

Corollary 12. Given a Borel σ-finite measure $d\mu$ on $\mathfrak{B}(\mathbb{R}^\infty)$, the *simple functions that only use finite measure sets of* C_{meas}, i.e.,

$$S_{C_{meas}} := \left\{ \sum_{j=1}^{N} c_j \mathbb{1}_{E_j} \;\middle|\; N \in \mathbb{N},\ c_j \in \mathbb{C},\ E_j \in C_{meas} \text{ and } d\mu(E_j) < \infty\ \forall j \right\} \tag{4.5}$$

are *dense* in $L^p(\mathbb{R}^\infty, d\mu)$ for all $p \in [1, +\infty)$. ♦

Proof: By Lemma 18 C_{meas} is a Boolean algebra generating $\mathfrak{B}(\mathbb{R}^\infty)$ so the claim follows from Prop. 19. □

Lemma 19. Any $E \in C_{meas}$ can be written as a finite *disjoint* union of measurable cylinders: $E = \bigsqcup_{j=1}^N (A^j_1 \times \cdots \times A^j_{n_j} \times \mathbb{R} \times \mathbb{R} \times \cdots)$ for some $A^j_k \in \mathfrak{B}(\mathbb{R})$ and $N, n_j \in \mathbb{N}$. As a consequence, for all $(x_1, x_2, ..) \in \mathbb{R}^\infty$

$$\mathbb{1}_E(x_1, x_2, ...) = \sum_{j=1}^{N} \mathbb{1}_{A^j_1}(x_1) \cdots \mathbb{1}_{A^j_{n_j}}(x_{n_j}). \tag{4.6}$$

♦

Proof: For an arbitrary set X, consider the sets $A := A_1 \times A_2$ and $B := B_1 \times B_2$ for arbitrary $A_j, B_j \subseteq X$. First, note that $A \setminus B$ is a finite disjoint union of product sets:

$$A \setminus B = (A_1 \times A_2) \setminus (B_1 \times B_2) = (A_1 \times A_2) \setminus (\pi_1^{-1}(B_1) \cap \pi_2(B_2)^{-1}) \overset{\textbf{(de Morg.)}}{=} \tag{4.7}$$

$$= (A_1 \times A_2) \setminus \pi_1^{-1}(B_1) \ \bigcup\ (A_1 \times A_2) \setminus \pi_2^{-1}(B_2) = (A_1 \setminus B_1) \times A_2 \ \bigcup\ A_1 \times (A_2 \setminus B_2) =$$

$$= (A_1 \setminus B_1) \times A_2 \bigsqcup (A_1 \cap B_1) \times (A_2 \setminus B_2). \tag{4.8}$$

- Now, $A \cup B = B \sqcup (A \setminus B)$, so plugging (4.8), $A \cup B$ can also be written as a finite disjoint union of product sets:

$$A \cup B = B_1 \times B_2 \ \bigsqcup \ (A_1 \setminus B_1) \times A_2 \ \bigsqcup \ (A_1 \cap B_1) \times (A_2 \setminus B_2). \tag{4.9}$$

- Next, we can take arbitrary $A := A_1 \times \widetilde{A_2}$, $B := B_1 \times \widetilde{B_2}$ with $\widetilde{A_2} := A_2 \times \cdots \times A_N$ and $\widetilde{B_2} := B_2 \times \cdots \times B_N$, $N \geq 3$ and $A_j, B_j \subseteq X$, and apply (4.9) to get

$$A \cup B = B_1 \times \widetilde{B_2} \ \bigsqcup \ (A_1 \setminus B_1) \times \widetilde{A_2} \ \bigsqcup \ (A_1 \cap B_1) \times (\widetilde{A_2} \setminus \widetilde{B_2}).$$

The first two sets in the r.h.s union are product sets. We now prove that the third one can be written as a finite disjoint union of product sets. For that, we have enough by proving that $\widetilde{A_2} \setminus \widetilde{B_2}$ is writeable as a finite disjoint union of product sets. To do that, beginning with $n := 2$ apply the following algorithm:

1. Define $\widetilde{A_{n+1}} := A_{n+1} \times \cdots \times A_N$ such that $\widetilde{A}_n = A_n \times \widetilde{A_{n+1}}$ and similarly, $\widetilde{B_{n+1}} := B_{n+1} \times \cdots \times B_N$ such that $\widetilde{B}_n = B_n \times \widetilde{B_{n+1}}$.
2. Then, by equation (4.8)
$$\widetilde{A_n} \setminus \widetilde{B_n} = (A_n \setminus B_n) \times \widetilde{A_{n+1}} \quad \bigsqcup \quad (A_n \cap B_n) \times (\widetilde{A_{n+1}} \setminus \widetilde{B_{n+1}}). \tag{4.10}$$
3. The first set in the r.h.s union is already a product set. If $n + 1 = N$, then the second set as well and the algorithm ends there. Else, increase n by one and go back to step 1 to find that the second set in the r.h.s of (4.10) is a finite union of disjoint product sets —which would prove that $\widetilde{A_{n+1}} \setminus \widetilde{B_{n+1}}$ is so.

The algorithm finishes exactly in $N - 2$ iterations, so we have proven our claim.

- Now, given finite cylinders $A := A_1 \times \cdots \times A_N \times \mathbb{R} \times \cdots$, $B := B_1 \times \cdots \times B_N \times \mathbb{R} \times \cdots$ for $A_j, B_j \in \mathfrak{B}(\mathbb{R})$, note that

$$A \cup B = \Big[\Big(A_1 \times \cdots \times A_N \Big) \bigcup \Big(B_1 \times \cdots \times B_N \Big) \Big] \times \mathbb{R} \times \mathbb{R} \times \cdots$$

Hence, by the above item, $A \cup B$ can be expressed as a finite disjoint union of cylinders (with at most N factors different from $\mathbb{R}$). Next, note that in the sets of the resulting disjoint union, each factor is made by taking finitely many $\cap$, $\cup$ and $\setminus$ operations of A_j, B_j, so such factors are still in $\mathfrak{B}(\mathbb{R})$. Finally, because an arbitrary $E \in C_{meas}$ is a *finite* union of elements like A and B, by iterating the argument finitely many times, we can write E as a finite disjoint union of measurable product sets. This proves our claim in the Lemma.

- After that, the claim on the indicator function follows immediately by making its definition explicit. ***o.ε.δ.***

Corollary 13. Given an arbitrary σ-finite Borel measure $d\mu$ on $\mathbb{R}^\infty$, for each $p \in [1, +\infty)$, the space of *L^p integrable products of 1-variable functions,*

$$Y_p := \text{span}\left\{ \Psi \in L^p(\mathbb{R}^\infty, d\mu) \;\middle|\; \begin{array}{c} \Psi(x_1, x_2, ...) = \phi_1(x_1) \cdots \phi_n(x_n) \text{ for some } n \in \mathbb{N} \\ \text{and some measurable } \phi_j : \mathbb{R} \to \mathbb{C} \end{array} \right\} \tag{4.11}$$

is a *dense* subspace of $L^p(\mathbb{R}^\infty, d\mu)$. ♦

Proof: By Lemma 19, any function in $S_{C_{meas}}$ (defined in Corollary 12) is trivially a member of Y_p, i.e., $S_{C_{meas}} \subseteq Y_p$. But by Cor. 12, $\overline{S_{C_{meas}}}^{\|\cdot\|_{L^p}} = L^p(\mathbb{R}^\infty, d\mu)$, so $\overline{Y_p}^{\|\cdot\|_{L^p}} = L^p(\mathbb{R}^\infty, d\mu)$. ***o.ε.δ.***

Corollary 14. Given an infinite product measure $d^\infty\mu$ of Borel probability measures $d\mu_j$ on $\mathbb{R}$, for each $p \in [1, +\infty)$, the space of *L^p integrable products of L^p 1-variable functions*,

$$Z_p := \operatorname{span}\left\{ \Psi \in L^p(\mathbb{R}^\infty, d\mu) \;\middle|\; \begin{array}{c} \Psi(x_1, x_2, ...) = \phi_1(x_1)\cdots\phi_n(x_n) \\ \text{for some } n \in \mathbb{N} \text{ and } \phi_j \in L^p(\mathbb{R}, d\mu_j) \end{array} \right\} \tag{4.12}$$

is a *dense* subspace of $L^p(\mathbb{R}^\infty, d^\infty\mu)$. ♦

Proof: By Lemma 19, any function $\psi \in S_{C_{meas}}$ (with $S_{C_{meas}}$ as in Corollary 12) is such that $\psi(x_1, x_2, ...) = \sum_{j=1}^N c_j \mathbb{1}_{A_1^j}(x_1)\cdots\mathbb{1}_{A_N^j}(x_N)$, with $d^\infty\mu(A_1^j \times \cdots \times A_N^j) < \infty$ for all j (for some $N \in \mathbb{N}$, $c_j \in \mathbb{C}$, $A_k^j \in \mathfrak{B}(\mathbb{R})$). But $d^\infty\mu(A_1^j \times \cdots \times A_N^j) = \prod_{k=1}^N d\mu_k(A_k^j)$. Therefore, $d\mu_k(A_k^j) < +\infty$ and $\left\|\mathbb{1}_{A_k^j}\right\|^p_{L^p(\mathbb{R}, d\mu_j)} = d\mu_j(A_k^j) < +\infty$. Hence, $\psi \in Z_p$ and $S_{C_{meas}} \subseteq Z_p$. Finally, since $\overline{S_{C_{meas}}}^{\|\cdot\|_{L^p}} = L^p(\mathbb{R}^\infty, d^\infty\mu)$, we get that $\overline{Z_p}^{\|\cdot\|_{L^p}} = L^p(\mathbb{R}^\infty, d^\infty\mu)$. ***o.ε.δ.***

Proposition 20. Let $d\mu_j$ be Borel probability measures on $\mathfrak{B}(\mathbb{R})$ and $d^\infty\mu := \odot_{j\in\mathbb{N}} d\mu_j$ be their infinite product measure. Let $f : \mathbb{R}^n \to \mathbb{C}$ be a Borel measurable function for some $n \in \mathbb{N}$. Choose some finite $J := \{j_1, ..., j_n\} \subseteq \mathbb{N}$ and define $F : \mathbb{R}^\infty \to \mathbb{C}$ such that $F(x_1, x_2, ...) := f(x_{j_1}, ..., x_{j_n})$. Then,

$$\int_{x\in\mathbb{R}^\infty} F(x_1, x_2, ...) d^\infty\mu = \int_{x\in\mathbb{R}^n} f(x_{j_1}, ..., x_{j_n}) d^J\mu \tag{4.13}$$

where $d^J\mu := \odot_{j\in J} d\mu_j$. ♦

Proof: We first prove the claim assuming that $f : \mathbb{R}^n \to [0, +\infty]$. By Theorem 2.10 in (Folland, 1999), there exists a sequence of simple functions $s_n : \mathbb{R}^n \to +\infty$, say, $s_n := \sum_{j=1}^{N_n} \alpha_j^n \mathbb{1}_{E_j^n}$ with $N_n \in \mathbb{N}$, $\alpha_j^n \in [0, +\infty)$, $E_j^n \in \mathfrak{B}(\mathbb{R}^n)$ such that $s_n(x) \leq s_{n+1}(x)$ for all $x \in \mathbb{R}^n$ (monotonously increasing) and $\lim_{n\to+\infty} s_n(x) = f(x)$ for almost every $x \in \mathbb{R}^n$. But then, the functions $\widetilde{s_n} := \sum_{j=1}^{N_n} \alpha_j^n \mathbb{1}_{E_j^n \times \prod_{j\in\mathbb{N}\setminus J}\mathbb{R}}$ are a sequence of monotonous increasing (measurable) simple functions over $\mathbb{R}^\infty$, and for almost every $(x_1, x_2, ...) \in \mathbb{R}^\infty$,

$$\lim_{n\to+\infty} \widetilde{s_n}(x_1, x_2, ...) = \lim_{n\to+\infty} s_n(x_{j_1}, ..., x_{j_n}) = f(x_{j_1}, ..., x_{j_n}) = F(x_1, x_2, ...).$$

Hence, applying in $(\star)$ the monotone convergence theorem (2.14 in (Folland, 1999)),

$$\int_{x\in\mathbb{R}^\infty} F(x_1, x_2, ...) d^\infty\mu \overset{(\star)}{=} \lim_{n\to\infty} \int_{x\in\mathbb{R}^\infty} \sum_{j=1}^{N_n} \alpha_j^n \; \mathbb{1}_{E_j^n \times \prod_{j\in\mathbb{N}\setminus J}\mathbb{R}}(x_1, x_2, ...) d^\infty\mu = \tag{4.14}$$

$$= \lim_{n\to\infty} \sum_{j=1}^{N_n} \alpha_j^n \; d^\infty\mu\left(E_j^n \times \prod_{j\in\mathbb{N}\setminus J} \mathbb{R}\right) \overset{(2.20)}{=} \lim_{n\to\infty} \sum_{j=1}^{N_n} \alpha_j^n \; d^J\mu(E_j^n) =$$

$$= \lim_{n\to\infty} \int_{(x_{j_1}, ..., x_{j_n})\in\mathbb{R}^n} \sum_{j=1}^{N_n} \alpha_j^n \; \mathbb{1}_{E_j^n}(x_{j_1}, ..., x_{j_n}) d^J\mu \overset{(\star)}{=} \int_{(x_{j_1}, ..., x_{j_n})\in\mathbb{R}^n} f(x_{j_1}, ..., x_{j_n}) d^J\mu.$$

- In order to prove that the relation holds for $\mathbb{C}$-valued f as well, we just need to split the resulting F into its real and imaginary parts and each of them into their positive and negative parts. Each of the resulting four functions is non-negative (if we take i and the negative signs out) and thus, we can apply the above result to each sub-integral. ***o.ε.δ.***

Lemma 20. In the setting of Corollary 14, let $n \in \mathbb{N}$ and $J := \{j_1, ..., j_n\} \subset \mathbb{N}$ be arbitrary and let $(\phi_j)_{j\in J} \subset L^p(\mathbb{R}, d\mu_j)$ be fixed. Then, for each $k \in \mathbb{N} \setminus J$, the map

$$\begin{array}{ccc} L^p(\mathbb{R}, d\mu_k) & \longrightarrow & L^p(\mathbb{R}^\infty, d^\infty\mu) \\ \psi_k & \longmapsto & \phi_{j_1}\cdots\phi_{j_n}\cdot\psi_k := \Big[(x_1, x_2, ...) \in \mathbb{R}^\infty \longmapsto \phi_{j_1}(x_{j_1})\cdots\phi_{j_n}(x_{j_n})\,\psi_k(x_k)\Big] \end{array}$$

is *continuous.* ♦

Proof: For any $g_k, f_k \in L^p(\mathbb{R}, d\mu_k)$,

$$\Big\|\phi_{j_1}\cdots\phi_{j_N}\cdot g_k - \phi_{j_1}\cdots\phi_{j_N}\cdot f_k\Big\|^p_{L^p(\mathbb{R}^\infty, d^\infty\mu)} = \int_{x\in\mathbb{R}^\infty}\Big|\phi_{j_1}(x_{j_1})\cdots\phi_{j_N}(x_{j_N})\Big|^p\Big|g_k(x_k) - f_k(x_k)\Big|^p d^\infty\mu \overset{\substack{\text{(Prop. 20)}\\ \text{(+ Tonelli)}}}{=}$$

$$= \int_{x_{j_1}\in\mathbb{R}}\Big|\phi_{j_1}(x_{j_1})\Big|^p d\mu_{j_1}\cdots\int_{x_{j_N}\in\mathbb{R}}\Big|\phi_{j_N}(x_{j_N})\Big|^p d\mu_{j_N}\cdot\int_{x_k\in\mathbb{R}}\Big|g_k(x_k) - f_k(x_k)\Big|^p d\mu_k = \prod_{j\in J}\|\phi_j\|^p_{L^p}\cdot\|g_k - f_k\|^p_{L^p}.$$

- Now, let $(g_k^n)_{n\in\mathbb{N}} \subseteq L^p(\mathbb{R}, d\mu_k)$ be a sequence that converges to some $g_k^0 \in L^p(\mathbb{R}, d\mu_k)$ (in $\|\cdot\|_{L^p(\mathbb{R}, d\mu_k)}$-norm). Then, by the above,

$$\Big\|\phi_{j_1}\cdots\phi_{j_N}g_k^n - \phi_{j_1}\cdots\phi_{j_N}\cdot g_k^0\Big\|^p_{L^p(\mathbb{R}^\infty, d^\infty\mu)} = \prod_{j\in J}\|\phi_j\|^p_{L^p}\cdot\Big\|g_k^n - g_k^0\Big\|^p_{L^p}\xrightarrow[n\to+\infty]{} 0.$$

This proves that the map in the lemma is sequentially continuous, but being a map between metric spaces, this implies that it is also continuous. ***o.ε.δ.***

Proposition 21. Let $(d\mu_k)_{k\in\mathbb{N}}$ be an arbitrary sequence of Borel probability measures on $\mathbb{R}$ and for each $k \in \mathbb{N}$, let $(\phi_k^\ell)_{\ell\in\mathbb{N}} \subset L^2(\mathbb{R}, d\mu_k)$ be an ONB of $L^2(\mathbb{R}, d\mu_k)$ such that $\phi_k^1 := 1$. Then, the family

$$\mathscr{O} := \Big\{\ (x_1, x_2, ...) \mapsto \phi_1^{\ell_1}(x_1)\cdots\phi_n^{\ell_n}(x_n)\ \Big|\ n, \ell_1, ..., \ell_n \in \mathbb{N} \text{ and } \ell_n \neq 1\ \Big\} \tag{4.15}$$

is an ONB of $L^2(\mathbb{R}^\infty, d^\infty\mu)$ with $d^\infty\mu := \odot_{j\in\mathbb{N}}d\mu_j$. ♦

Proof: Given that Z_2 is defined as in Corollary 14, for any $\Psi \in Z_2$, $\exists N, M \in \mathbb{N}$, $f_k^j \in L^2(\mathbb{R}, d\mu_k)$ such that $\Psi(x_1, x_2, ...) = \sum_{j=1}^M c_j\ f_1^j(x_1)\cdots f_N^j(x_N)$.[a] Consider the ONB expansions $f_k^j = \sum_{\ell=1}^\infty c_\ell^{(j,k)}\phi_k^\ell$ —where the infinite sum means convergence of partial sums in $L^2(\mathbb{R}, d\mu_k)$-norm. Then, applying N times the continuity of the product proven in Lemma 20, we get that

$$\Psi = \lim_{L_1\to\infty}\cdots\lim_{L_N\to\infty}\sum_{j=1}^M\sum_{\ell_1=1}^{L_1}\cdots\sum_{\ell_N=1}^{L_N}c_j c_{\ell_1}^{(j,1)}\cdots c_{\ell_N}^{(j,N)}\phi_1^{\ell_1}\cdots\phi_N^{\ell_N} = \tag{4.16}$$

$$= \sum_{j=1}^M\sum_{\ell_1=1}^\infty\cdots\sum_{\ell_N=1}^\infty c_j c_{\ell_1}^{(j,1)}\cdots c_{\ell_N}^{(j,N)}\phi_1^{\ell_1}\cdots\phi_N^{\ell_N},$$

where the convergence of the partial sums in each series is in $L^2(\mathbb{R}^\infty, d^\infty\mu)$-norm. Now, we claim that $\forall\varepsilon>0$, there exist $K_1^\varepsilon,...,K_N^\varepsilon\in\mathbb{N}$ such that

$$\Big\|\Psi-\underbrace{\sum_{j=1}^{M}\sum_{\ell_1=1}^{K_1^\varepsilon}\cdots\sum_{\ell_N=1}^{K_N^\varepsilon}c_j c_{\ell_1}^{(j,1)}\cdots c_{\ell_N}^{(j,N)}\phi_1^{\ell_1}\cdots\phi_N^{\ell_N}}_{=:\Xi_\Psi^\varepsilon}\Big\|\le\varepsilon/2. \tag{4.17}$$

To prove it, note that for any $L_1,...,L_N\in\mathbb{N}$, applying the triangle inequality iteratively,

$$\Big\|\Psi-\sum_{j=1}^{M}\sum_{\ell_1=1}^{L_1}\cdots\sum_{\ell_N=1}^{L_N}c_j c_{\ell_1}^{(j,1)}\cdots c_{\ell_N}^{(j,N)}\phi_1^{\ell_1}\cdots\phi_N^{\ell_N}\Big\|\le\Big\|\Psi-\sum_{j=1}^{M}\sum_{\ell_1=1}^{L_1}\sum_{\ell_2=1}^{\infty}\cdots\sum_{\ell_N=1}^{\infty}c_j c_{\ell_1}^{(j,1)}\cdots c_{\ell_N}^{(j,N)}\phi_1^{\ell_1}\cdots\phi_N^{\ell_N}\Big\|+$$

$$+\Big\|\sum_{j=1}^{M}\sum_{\ell_1=1}^{L_1}\sum_{\ell_2=1}^{\infty}\cdots\sum_{\ell_N=1}^{\infty}c_j c_{\ell_1}^{(j,1)}\cdots c_{\ell_N}^{(j,N)}\phi_1^{\ell_1}\cdots\phi_N^{\ell_N}-\sum_{j=1}^{M}\sum_{\ell_1=1}^{L_1}\sum_{\ell_2=1}^{L_2}\sum_{\ell_3=1}^{\infty}\cdots\sum_{\ell_N=1}^{\infty}c_j c_{\ell_1}^{(j,1)}\cdots c_{\ell_N}^{(j,N)}\phi_1^{\ell_1}\cdots\phi_N^{\ell_N}\Big\|+$$

$$+\cdots+\Big\|\sum_{j=1}^{M}\sum_{\ell_1=1}^{L_1}\cdots\sum_{\ell_{N-1}=1}^{L_{N-1}}\sum_{\ell_N=1}^{\infty}c_j c_{\ell_1}^{(j,1)}\cdots c_{\ell_N}^{(j,N)}\phi_1^{\ell_1}\cdots\phi_N^{\ell_N}-\sum_{j=1}^{M}\sum_{\ell_1=1}^{L_1}\cdots\sum_{\ell_N=1}^{L_N}c_j c_{\ell_1}^{(j,1)}\cdots c_{\ell_N}^{(j,N)}\phi_1^{\ell_1}\cdots\phi_N^{\ell_N}\Big\|.$$

Because each iterative series converges, defining $K_j^\varepsilon:=L_j^{\varepsilon/2N}$ —where L_j^δ is the δ-critical index for the convergence of the j-th iterative limit—, each summand in the r.h.s is smaller than $\varepsilon/(2N)$ for every $L_j\ge K_j^\varepsilon$ and every $j\in\{1,...,N\}$. Hence, the whole sum in the r.h.s is $\le\varepsilon/2$.

- But then, given an arbitrary $\Phi\in L^2(\mathbb{R}^\infty,d^\infty\mu)$, since we found that Z_2 is dense in $L^2(\mathbb{R}^\infty,d^\infty\mu)$, $\forall\varepsilon>0$ there exists a $\Psi\in Z_2$ such that $\|\Phi-\Psi\|_{L^2(\mathbb{R}^\infty,d^\infty\mu)}\le\varepsilon/2$. With all,

$$\|\Phi-\Xi_\Psi^\varepsilon\|_{L^2(\mathbb{R}^\infty,d^\infty\mu)}\le\|\Phi-\Psi\|+\|\Psi-\Xi_\Psi^\varepsilon\|\le\varepsilon.$$

Hence, for an arbitrary $\Phi\in L^2(\mathbb{R}^\infty,d^\infty\mu)$, it is possible to find a $\Xi_\Psi^\varepsilon\in\mathrm{span}\mathscr{O}$ that is arbitrarily close to Φ. Namely, $\mathrm{span}\mathscr{O}$ is dense in $L^2(\mathbb{R}^\infty,d^\infty\mu)$.

- Finally, we check that $\mathscr{O}$ is an orthonormal family. Let $\ell_1,...,\ell_N\in\mathbb{N}$ with $\ell_N\neq1$ and $k_1,...,k_M\in\mathbb{N}$ with $k_M\neq1$ for arbitrary $N,M\in\mathbb{N}$ such that $N\le M$. Then,

$$\left\langle\phi_1^{\ell_1}\cdots\phi_N^{\ell_N},\ \phi_1^{k_1}\cdots\phi_M^{k_M}\right\rangle=\int_{x\in\mathbb{R}^\infty}\overline{\phi_1^{\ell_1}}(x_1)\phi_1^{k_1}(x_1)\cdots\overline{\phi_N^{\ell_N}}(x_N)\phi_N^{k_N}(x_N)\phi_{N+1}^{k_{N+1}}(x_{N+1})\cdots\phi_M^{k_M}(x_M)d^\infty\mu=$$

$$\overset{\substack{\text{(Prop. 20)}\\\text{(+ Tonelli)}}}{=}\prod_{j=1}^{N}\langle\phi_j^{\ell_j},\phi_j^{k_j}\rangle\prod_{j=N+1}^{M}\langle0,\phi_j^{k_j}\rangle=\begin{cases}0 & \text{if } N\neq M\\ 0 & \text{if } \ell_j\neq k_j \text{ for some } j\\ 1 & \text{if } N=M \text{ and } (\ell_1,...,\ell_N)=(k_1,...,k_N).\end{cases}$$

- All in all, $\mathscr{O}$ is an orthonormal family whose span is dense in $L^2(\mathbb{R}^\infty,d^\infty\mu)$, namely, an ONB.

o.ε.δ.

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[a] A priori, it would be $\sum_{j=1}^{M}c_j\ f_1^j(x_1)\cdots f_{N_j}^j(x_N)$ with $N_j\in\mathbb{N}$, but since the function $g(x_j)\equiv1$ is in $L^2(\mathbb{R},d\mu_j)$ for all j, we can "fill" each summand with enough g's.

The Proof of Theorem 10

Proof **(of Theorem 10):** **(i) Proof of Existence and Uniqueness:** Define

$$V := \operatorname{span}\Big\{\phi_1 \otimes \phi_2 \otimes \cdots \mid \exists N \in \mathbb{N} : \ \phi_j = \rho_j \quad \forall j \geq N\Big\}.$$

It is a dense vector subspace of $\otimes_{k\in\mathbb{N}}^{[\rho]} L^2(\mathbb{R}, dx)$ by Theorem 9.

Denote $d^\infty\mu := \odot_{j\in\mathbb{N}}|\rho_j|^2 dx$ and define:

$$\begin{array}{llcl} W : & V & \longrightarrow & L^2(\mathbb{R}^\infty, d^\infty\mu) \\ & \sum\limits_{j=1}^{N} c_j\phi_1^j \otimes \cdots \otimes \phi_{M_j} \otimes \rho_{M_{j+1}} \otimes \cdots & \longmapsto & \Big[(x_1, x_2, \ldots) \longmapsto \sum\limits_{j=1}^{N} c_j \frac{\phi_1^j(x_1)\cdots\phi_{M_j}^j(x_{M_j})}{\rho_1(x_1)\cdots\rho_{M_j}(x_{M_j})}\Big]. \end{array}$$

When there is no risk of confusion we will denote the class of functions in the r.h.s by $\sum_{j=1}^N c_j \frac{\phi_1^j\cdots\phi_{M_j}^j}{\rho_1\cdots\rho_{M_j}}$. Now, W is linear by definition, but it is also isometric as we proceed to prove.

$$\Bigg\| W\Bigg(\sum_{j=1}^{N} c_j\phi_1^j \otimes \cdots \otimes \phi_{M_j} \otimes \rho_{M_{j+1}} \otimes \cdots\Bigg)\Bigg\|^2_{L^2(\mathbb{R}^\infty, d^\infty\mu)} = \Bigg\|\sum_{j=1}^{N} c_j \frac{\phi_1^j\cdots\phi_{M_j}^j}{\rho_1\cdots\rho_{M_j}}\Bigg\|^2_{L^2(\mathbb{R}^\infty, d^\infty\mu)} =$$

$$= \sum_{j=1}^{N}\sum_{i=1}^{N} \bar{c}_j c_i \Big\langle \frac{\phi_1^j\cdots\phi_{M_j}^j}{\rho_1\cdots\rho_{M_j}}, \frac{\phi_1^i\cdots\phi_{M_i}^i}{\rho_1\cdots\rho_{M_i}}\Big\rangle_{L^2(\mathbb{R}^\infty, d^\infty\mu)} = (\star) \tag{4.18}$$

For $M_j < M_i$ we have that:

$$\Big\langle \frac{\phi_1^j\cdots\phi_{M_j}^j}{\rho_1\cdots\rho_{M_j}}, \frac{\phi_1^i\cdots\phi_{M_i}^i}{\rho_1\cdots\rho_{M_i}}\Big\rangle_{L^2(\mathbb{R}^\infty, d^\infty\mu)} =$$

$$= \int_{x\in\mathbb{R}^\infty} \frac{\overline{\phi_1^j(x_1)}\phi_1^i(x_1)\cdots\overline{\phi_{M_j}^j(x_{M_j})}\phi_{M_j}^i(x_{M_j}) \quad \phi_{M_{j+1}}^i(x_{M_{j+1}})\cdots\phi_{M_i}^i(x_{M_i})}{|\rho_1|^2(x_1)\cdots|\rho_{M_j}|^2(x_{M_j}) \qquad \rho_{M_{j+1}}(x_{M_{j+1}})\cdots\rho_{M_i}(x_{M_i})} d^\infty\mu \overset{\left(\substack{\text{Prop. } 20, \\ \text{\& Fubini [a]}}\right)}{=}$$

$$= \prod_{\ell=1}^{M_j}\int_{x_\ell\in\mathbb{R}} \frac{\overline{\phi_\ell^j(x_\ell)}\phi_\ell^i(x_\ell)}{|\rho_\ell|^2(x_\ell)} d\mu_\ell \cdot \prod_{\ell=M_{j+1}}^{M_i}\int_{x_\ell\in\mathbb{R}} \frac{\phi_\ell^i(x_\ell)}{\rho_\ell(x_\ell)} d\mu_\ell \overset{(d\mu_\ell = |\rho_\ell|^2 dx)}{=}$$

$$= \prod_{\ell=1}^{M_j}\int_{x_\ell\in\mathbb{R}} \overline{\phi_\ell^j(x_\ell)}\phi_\ell^i(x_\ell)dx_\ell \cdot \prod_{\ell=M_{j+1}}^{M_i}\int_{x_\ell\in\mathbb{R}} \phi_\ell^i(x_\ell)\overline{\rho_\ell(x_\ell)}dx_\ell = \prod_{\ell=1}^{M_j}\langle\phi_\ell^j, \phi_\ell^i\rangle \cdot \prod_{\ell=M_{j+1}}^{M_i}\langle\rho_\ell, \ \phi_\ell^i\rangle.$$

The $M_j \geq M_i$ cases follow similarly. Going back to (4.18), these leave:

$$(\star) = \sum_{j=1}^{N}\sum_{i=1}^{N} \bar{c}_j c_i \prod_{\ell=1}^{M_j}\langle\phi_\ell^j, \phi_\ell^i\rangle \cdot \prod_{\ell=M_{j+1}}^{M_i}\langle\rho_\ell, \ \phi_\ell^i\rangle \overset{(\ \langle\rho_\ell,\rho_\ell\rangle=1\ )}{=}$$

$$= \sum_{j=1}^{N}\sum_{i=1}^{N} \overline{c_j}c_j \Big\langle(\phi_1^j \otimes \cdots \otimes \phi_{M_j}^j \otimes \rho_{M_j+1} \otimes \cdots), \ (\phi_1^i \otimes \cdots \otimes \phi_{M_i}^i \otimes \rho_{M_i+1} \otimes \cdots)\Big\rangle_{\otimes_k L^2(\mathbb{R}, dx)} =$$

$$= \Bigg\|\sum_{j=1}^{N} c_j\Big(\phi_1^j \otimes \cdots \otimes \phi_{M_j}^j \otimes \rho_{M_j+1} \otimes \cdots\Big)\Bigg\|^2_{\otimes_k L^2(\mathbb{R}, dx)}.$$

Therefore, W is indeed an isometry.

- As a consequence, W is a bounded operator of norm 1. But then, since it maps from a dense subspace to a Banach space, by Theorem 8, there exists a unique extension of W to a map $\widetilde{W} : \bigotimes_{k\in\mathbb{N}} L^2(\mathbb{R}, dx) \to L^2(\mathbb{R}^\infty, d^\infty\mu)$ and it has operator norm 1.

(ii) Proof of $\widetilde{W}$ being a Unitary Isomorphism: We prove first that $\widetilde{W}$ is an isometry. Given an arbitrary $\Psi \in \bigotimes_{k\in\mathbb{N}} L^2(\mathbb{R}, dx)$, by density, there exists a sequence $(\Psi^n)_{n\in\mathbb{N}} \subseteq V$ with $\Psi_n \xrightarrow[n\to\infty]{} \Psi$. But then, by Theorem 8, $\lim_{n\to\infty} W(\Psi^n)$ exists and $\widetilde{W}(\Psi) = \lim\limits_{n\to\infty} W(\Psi^n)$. This implies:

$$\left\|\widetilde{W}(\Psi)\right\| = \left\|\lim_{n\to\infty} W(\Psi^n)\right\| \overset{(\|\cdot\| \text{ is continuous})}{=} \lim_{n\to\infty} \|W(\Psi^n)\| \overset{(W \text{ isom})}{=} \lim_{n\to\infty} \|\Psi^n\| = \|\Psi\|. \tag{4.19}$$

Therefore, $\widetilde{W}$ is an isometry and it is injective by definiteness of the norm.

- Finally, we prove surjectivity. Let $\psi \in L^2(\mathbb{R}^\infty, d^\infty\mu)$. By Prop. 21,[b] there exist some $c_n \in \mathbb{C}$, $\phi_j^k \in L^2(\mathbb{R}, d\mu_k)$ and some sequence $\{(k_1^n, ..., k_{M_n}^n)\}_{n\in\mathbb{N}}$ such that the partial sums $R_N := \sum_{n=1}^N c_n\, \phi_1^{k_1^n} \cdots \phi_{M_n}^{k_{M_n}^n}$ converge to Ψ. Now, the sequence of partial sums $S_N := \sum_{n=1}^N c_n(\phi_1^{k_1^n}\rho_1) \otimes \cdots \otimes (\phi_{M_n}^{k_{M_n}^n}\rho_n) \otimes \rho_{n+1} \otimes \cdots$ in $\otimes_{k\in\mathbb{N}}^{[\rho]} L^2(\mathbb{R}, dx)$ satisfies $WS_N = R_N$, so it is Cauchy because

$$\|S_N - S_M\|_{\otimes_k L^2(\mathbb{R},dx)} \overset{(\widetilde{W} \text{ isometry})}{=} \left\|\widetilde{W}S_N - \widetilde{W}S_M\right\|_{L^2(\mathbb{R}^\infty, d^\infty\mu)} = \|R_N - R_M\|_{L^2(\mathbb{R}^\infty, d^\infty\mu)},$$

which converges uniformly in N, M because $(R_N)_{N\in\mathbb{N}}$ is convergent and therefore Cauchy. But $\otimes_{k\in\mathbb{N}}^{[\rho]} L^2(\mathbb{R}, dx)$ is a closed subspace of $\otimes_{k\in\mathbb{N}} L^2(\mathbb{R}, dx)$ and the latter is Cauchy complete. Hence, $(S_N)_{N\in\mathbb{N}}$ converges to some $\Phi \in \otimes_{k\in\mathbb{N}}^{[\rho]} L^2(\mathbb{R}, dx)$. But then,

$$\widetilde{W}(\Phi) = \widetilde{W}(\lim_{N\to\infty} S_N) \overset{(\widetilde{W} \text{ is continuous})}{=} \lim_{N\to\infty} \widetilde{W}(S_N) = \lim_{n\to\infty} R_N = \Psi. \tag{4.20}$$

o.ε.δ.

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[a] After using Proposition 20 to reduce the $\mathbb{R}^\infty$ integral to an integral over $\mathbb{R}^{M_i}$, we can apply Fubini's theorem (2.37 in (Folland, 1999)) because putting absolute values to the integrands (to check if they are jointly in L^1), one can apply Tonelli's theorem and every resulting integral is trivially finite.

[b] This was the exact reason to develop all the results of section 4.2.

4.3 *Every* Layer Can be Identified with some $L^2(\mathbb{R}^\infty, d^\infty\mu)$

Now, after proving that Theorem 10 holds indeed, the idea would be to make such an identification for all layers $\mathfrak{C} \in \Gamma$. However, this would require that all layers possess a generator $\rho_1 \otimes \rho_2 \otimes \cdots$ such that $\rho(x) \neq 0$ almost everywhere. Is this possible? Indeed:

Proposition 22. For each class $\mathfrak{C} \in \Gamma$ there exists an elementary tensor product $\rho_1 \otimes \rho_2 \otimes \cdots \in \otimes_{j\in\mathbb{N}}^{\mathfrak{C}} L^2(\mathbb{R}, dx)$ such that

- $\|\rho_j\|_{L^2(\mathbb{R},dx)} = 1$. Hence,
    1. $d\mu_j := |\rho_j|^2 dx$ is a probability measure and
    2. $\rho_1 \otimes \rho_2 \otimes \cdots$ is a generator of the layer $\mathfrak{C}$ (see Theorem 9).
- $\rho_j(x) \neq 0$ for almost every $x \in \mathbb{R}$. Hence,
    3. $|\rho_j|^2(x) > 0$ for almost every $x \in \mathbb{R}$, and thus, $d\mu_j \sim dx$ (by Cor. 4).

♦

Proof: By Thm. 9, for any fixed $\mathfrak{C} \in \Gamma$, $\exists (f_j)_{j\in\mathbb{N}} \in \mathfrak{C}$ with $\|f_j\| = 1$. Hence, to prove the proposition we just need to build a sequence $(\rho_j)_{j\in\mathbb{N}} \subset L^2(\mathbb{R}, dx)$ such that

(a) $\|\rho_j\|_{L^2(\mathbb{R},dx)} = 1$ (implying it is a $\mathscr{C}$-sequence with $\rho_1 \otimes \rho_2 \otimes \cdots \neq 0$).

(b) $\rho_j(x) \neq 0$ for almost every $x \in \mathbb{R}$.

(c) $\sum_{j\in\mathbb{N}} |\langle \rho_j,\ f_j\rangle - 1|$ exists (such that $(\rho_j)_j \approx (f_j)_j$, and hence $\otimes_j \rho_j$ is also a generator of $\mathfrak{C}$'s layer).

- Take any $h \in L^2(\mathbb{R}, dx)$ with $h(x) \neq 0$ for almost every $x \in \mathbb{R}$, e.g., $h(x) := e^{-x^2}$. Then define for each $j \in \mathbb{N}$

$$h_j(x) := \begin{cases} h(x) & \text{if } x \in f_j^{-1}(\{0\}) \\ 0 & \text{else} \end{cases}. \tag{4.21}$$

They satisfy $\|h_j\|_{L^2(\mathbb{R},dx)} = \left\| h \mathbb{1}_{f_j^{-1}(\{0\})} \right\|_{L^2(\mathbb{R},dx)} \leq \|h\| < +\infty$. Hence, $h_j \in L^2(\mathbb{R}, dx)$. Moreover,

$$\langle f_j, h_j \rangle = \int_{x\in\mathbb{R}} f_j(x) h(x) \mathbb{1}_{f_j^{-1}(\{0\})}(x) dx = \int_{x\in f_j^{-1}(\{0\})} 0 \cdot h(x) dx + \int_{x\in\mathbb{R}\setminus f_j^{-1}(\{0\})} f_j(x) \cdot 0 dx = 0.$$

Thus, $f_j \perp h_j$ for all $j \in \mathbb{N}$. Now, for some $\delta_j, \varepsilon_j > 0$ to be fixed later, define for each $j \in \mathbb{N}$

$$\rho_j := \delta_j \Big(f_j + \varepsilon_j \frac{h_j}{\|h_j\|} \Big). \tag{4.22}$$

By definition, each $\rho_j(x) \neq 0$ almost everywhere in $x \in \mathbb{R}$, so they satisfy (b). Next, note that

$$\|\rho_j\|^2_{L^2(\mathbb{R},dx)} = \delta_j^2 \left\| f_j + \varepsilon_j \frac{h_j}{\|h_j\|} \right\|^2_{L^2} = \delta_j^2 \Big(\cancelto{1}{\|f_j\|^2} + \varepsilon_j^2 \frac{\|h_j\|^2}{\|h_j\|^2} + 2 \frac{\varepsilon_j}{\|h_j\|} \operatorname{Re}\Big\{ \cancelto{0}{\langle f_j, h_j \rangle} \Big\} \Big) = \delta_j^2 (1 + \varepsilon_j^2).$$

If we fix $\delta_j := 1/\sqrt{1 + \varepsilon_j^2}$ we get that $\|\rho_j\| = 1$, making them satisfy (a). Finally, note that:

$$\langle \rho_k, f_k \rangle = \Big\langle \delta_k \Big(f_k + \varepsilon_k \frac{h_k}{\|h_k\|} \Big), f_k \Big\rangle = \delta_k \langle f_k, f_k \rangle + \frac{\delta_k \varepsilon_k}{\|h_k\|} \cancelto{0}{\langle h_k, f_k \rangle} = \delta_k \|f_k\|^2 = \delta_k = \frac{1}{\sqrt{1 + \varepsilon_k^2}} \Longrightarrow$$

$$\Longrightarrow \sum_{k\in\mathbb{N}} |\langle \rho_k, f_k \rangle - 1| = \sum_{k=1}^{\infty} \Big| \frac{1}{\sqrt{1 + \varepsilon_k^2}} - 1 \Big|.$$

If we choose ε_k such that this sum converges, then (c) will be satisfied. For instance, take $\varepsilon_{k-1}^2 := [k^4/(k^2 - 1)^2] - 1$ with $k \in \mathbb{N} \setminus \{1\}$. It satisfies $\varepsilon_j > 0$ for all $j \in \mathbb{N}$ and

$$\frac{1}{\sqrt{\varepsilon_j^2 + 1}} = \frac{((j+1)^2 - 1)}{(j+1)^2} = 1 - \frac{1}{(j+1)^2} \Longrightarrow \sum_{j=1}^{\infty} |\langle \rho_j, f_j \rangle - 1| = \sum_{j=1}^{\infty} \frac{1}{(\ell + 1)^2} < +\infty.$$

o.ε.δ.

Theorem 10 and Proposition 22 together prove that *for any* $\mathfrak{C} \in \Gamma$, there exists some generator $\bigotimes_{j\in\mathbb{N}} \rho_j^{\mathfrak{C}}$ of the layer $\otimes_{k\in\mathbb{N}}^{\mathfrak{C}} L^2(\mathbb{R}, dx)$ giving a probability measure $d^\infty \mu_{\mathfrak{C}} := \odot_{j=1}^{\infty} |\rho_j^{\mathfrak{C}}|^2 dx$ such that $L^2(\mathbb{R}^\infty, d^\infty \mu_{\mathfrak{C}}) \cong \otimes_{k\in\mathbb{N}}^{\mathfrak{C}} L^2(\mathbb{R}, dx)$.

But, in principle, there could exist multiple such generators of a same layer $\mathfrak{C} \in \Gamma$, say, $(\rho_j^{\mathfrak{C}})_{j\in\mathbb{N}}$ and $(\eta_j^{\mathfrak{C}})_{j\in\mathbb{N}}$, with measures $d^\infty\mu_{\mathfrak{C}} := \odot_{j\in\mathbb{N}}|\rho_j^{\mathfrak{C}}|^2dx$ and $d^\infty\nu_{\mathfrak{C}} := \odot_{j\in\mathbb{N}}|\eta_j^{\mathfrak{C}}|^2dx$. After we choose one of them, say, $(\rho_j^{\mathfrak{C}})_{j\in\mathbb{N}}$, there is an obvious identification $\otimes_{k\in\mathbb{N}}^{\mathfrak{C}}L^2(\mathbb{R},dx) \cong L^2(\mathbb{R}^\infty, d^\infty\mu_{\mathfrak{C}})$ (in the sense of Theorem 10). But, what if $L^2(\mathbb{R}^\infty, d^\infty\mu_{\mathfrak{C}})$ and $L^2(\mathbb{R}^\infty, d^\infty\nu_{\mathfrak{C}})$ are *inequivalent* spaces (in the sense of not allowing equivalent QM as in Theorem 5)? That is, what if $d^\infty\mu_{\mathfrak{C}} \not\sim d^\infty\nu_{\mathfrak{C}}$? If so, we could not use the identification to give any generator-independent notion of Born rule to $\bigotimes_{k\in\mathbb{N}} L^2(\mathbb{R},dx)$. Miraculously, this never happens:

Theorem 11. For any fixed $\mathfrak{C} \in \Gamma$, given an arbitrary pair of generators $(\rho_1 \otimes \rho_2 \otimes \cdots)$, $(\eta_1 \otimes \eta_2 \otimes \cdots) \in \otimes_{k\in\mathbb{N}}^{\mathfrak{C}}L^2(\mathbb{R},dx)$ such that $\rho_j(x), \eta_j(x) \neq 0$ for dx-almost every $x \in \mathbb{R}$, their infinite measures $d^\infty\mu := \odot_{j\in\mathbb{N}}|\rho_j|^2dx_j$ and $d^\infty\nu := \odot_{k\in\mathbb{N}}|\eta_j|^2dx_j$ are always *mutually absolutely continuous,* i.e., $d^\infty\mu \sim d^\infty\nu$. In particular, taking $W_{[\rho]}$ as in Theorem 10, the Radon-Nikodym derivative is given by:

$$d^\infty\nu = \Big|W_{[\rho]}(\eta_1 \otimes \eta_2 \otimes \cdots)\Big|^2 d^\infty\mu. \quad \text{Moreover,} \quad W_{[\rho]}(\eta_1 \otimes \eta_2 \otimes \cdots) = \overset{L^2(\mathbb{R}^\infty,d^\infty\mu)}{\lim_{n\to\infty}} \; \frac{\eta_1\cdots\eta_n}{\rho_1\cdots\rho_n}.$$

♦

We said “miraculously” because the result that allows us to prove the main claim ($d^\infty\mu \sim d^\infty\nu$) seems almost exactly designed for us to prove it:[2]

Theorem 12 (*Kakutani's Theorem*)**.** Let $\{\Omega_n\}_{n\in\mathbb{N}}$ be topological spaces and for each $n \in \mathbb{N}$ let $d\mu_n$, $d\nu_n$ be Borel probability measures on Ω_n with $d\mu_n \sim d\nu_n$. Let $\varphi_n : \Omega_n \to (0, +\infty]$ be such that $d\nu_n = \varphi_n d\mu_n$.[a] Consider the infinite product measures $d^\infty\mu := \odot_{j\in\mathbb{N}}\, d\mu_j$ and $d^\infty\nu := \odot_{j\in\mathbb{N}}\, d\nu_j$ on $\odot_{j\in\mathbb{N}}\mathfrak{B}(\Omega_j)$.

Then, defining

$$\mathscr{K} := \prod_{j\in\mathbb{N}} \left(\int_{\omega_j\in\Omega_j} \sqrt{\varphi_n(\omega_j)} d\mu_j \right), \tag{4.23}$$

- either: $\Big(\mathscr{K} > 0\Big) \iff \Big(d^\infty\mu \sim d^\infty\nu\Big)$
- or: $\Big(\mathscr{K} = 0\Big) \iff \Big(d^\infty\mu \perp d^\infty\nu\Big)$.

Proof: It is the main result proven in (Kakutani, 1948). □

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[a] Recall that because $d\nu_n$ is a probability measure, it must be that $\varphi_n \in L^1(\Omega_n, d\mu_n)$.

Corollary 15. For each $j \in \mathbb{N}$ let there be some $\rho_j, \eta_j \in L^2(\mathbb{R}, dx)$ with $\rho_j(x), \eta_j(x) \neq 0$ for dx-almost every $x \in \mathbb{R}$ and $\|\rho_j\|, \|\eta_j\| = 1$. Define $d^\infty\mu := \odot_{j\in\mathbb{N}}\, |\rho_j|^2dx_j$ and $d^\infty\nu := \odot_{j\in\mathbb{N}}\, |\eta_j|^2dx_j$. Then,

- either: $\left(d^\infty\mu \sim d^\infty\nu\right) \iff \left(\prod_{k\in\mathbb{N}}\langle\, |\eta_j|, |\rho_j|\, \rangle > 0\right) \iff \left(\sum_{k\in\mathbb{N}}\Big|\langle\, |\eta_j|,\ |\rho_j|\, \rangle - 1\Big| \text{ exists}\right)$
- or: $\left(d^\infty\mu \perp d^\infty\nu\right) \iff \left(\prod_{k\in\mathbb{N}}\langle\, |\eta_j|, |\rho_j|\, \rangle = 0\right) \iff \left(\sum_{k\in\mathbb{N}}\Big|\langle\, |\eta_j|,\ |\rho_j|\, \rangle - 1\Big| \text{ does not exist}\right)$

♦

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[2] In fact, it is not at all a miracle because, as we explain in the historical remarks of Chapter 7, von Neumann was also involved in Kakutani's theorem.

Proof: • (Leftmost $\Leftrightarrow$'s :) First, note that $\forall B \in \mathfrak{B}(\mathbb{R})$

$$\frac{|\eta_j|^2}{|\rho_j|^2} d\mu_j(B) = \int_{x \in B} \frac{|\eta_j|^2(x)}{|\rho_j|^2(x)} d\mu_j \overset{(d\mu_j = |\rho_j|^2 dx)}{=} \int_{x \in \mathbb{R}} |\eta_j|^2(x) dx = d\nu_j(B).$$

Hence, $\frac{|\eta_j|^2}{|\rho_j|^2} d\mu_j = d\nu_j$. Then, Kakutani's $\mathscr{K}$ integral reads:

$$\mathscr{K} = \prod_{j \in \mathbb{N}} \int_{x_j \in \mathbb{R}} \frac{|\eta_j|(x_j)}{|\rho_j|(x_j)} d\mu_j = \prod_{j \in \mathbb{N}} \int_{x_j \in \mathbb{R}} |\eta_j|(x_j)|\rho_j|(x_j) dx_j = \prod_{j \in \mathbb{N}} \langle |\eta_j|,\ |\rho_j| \rangle. \tag{4.24}$$

This product cannot diverge nor fail to converge: η_j, ρ_j are unit vectors, so by the Cauchy-Schwarz inequality, $\langle |\eta_j|.\ |\rho_j| \rangle \in [0, 1]$ —hence, the product is either a finite positive number or it is zero.[a] More rigorously, the product $\mathscr{K}$ exists by Prop. 44.(i). But then, Kakutani's theorem (Thm. 12) immediately gives what we wanted to prove.

- (Rightmost $\Leftrightarrow$'s :) By Prop. 44.(ii),

$$\prod_{j \in \mathbb{N}} \langle |\eta_j|,\ |\rho_j| \rangle \neq 0 \Longleftrightarrow \left(\begin{array}{c} \langle |\eta_j|, |\rho_j| \rangle \neq 0 \ \forall j \in \mathbb{N} \\ \text{and } \sum_{j \in \mathbb{N}} \Big| \langle |\eta_j|,\ |\rho_j| \rangle - 1 \Big| \text{ exists} \end{array} \right) \overset{(\star)}{\Longleftrightarrow} \left(\sum_{j \in \mathbb{N}} \Big| \langle |\eta_j|,\ |\rho_j| \rangle - 1 \Big| \text{ exists} \right).$$

$(\star)$: By hypothesis, $|\eta_j|, |\rho_j|$ are almost everywhere > 0 so $\langle |\eta_j|,\ |\rho_j| \rangle$ cannot be zero. Finally, because the product can only converge to > 0 or to 0 (i.e., it must exist), the dichotomy is:

$$\text{either: } \left(\prod_{j \in \mathbb{N}} \langle |\eta_j|,\ |\rho_j| \rangle > 0 \right) \Longleftrightarrow \left(\sum_{j \in \mathbb{N}} \Big| \langle |\eta_j|,\ |\rho_j| \rangle - 1 \Big| \text{ exists} \right)$$

$$\text{or its negation: } \left(\prod_{j \in \mathbb{N}} \langle |\eta_j|,\ |\rho_j| \rangle = 0 \right) \Longleftrightarrow \left(\sum_{j \in \mathbb{N}} \Big| \langle |\eta_j|,\ |\rho_j| \rangle - 1 \Big| \text{ not exists} \right).$$

o.ε.δ.

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[a] Intuitively, as we keep multiplying new factors, the partial products will either decrease (if the new factor is in $(0, 1)$), keep constant (if the new factor is 1) or will decay immediately to 0 (if it is 0).

The relation that discerns between $d^\infty\mu \sim d^\infty\nu$ and $d^\infty\mu \perp d^\infty\nu$ (namely, the existence of $\sum_{j \in \mathbb{N}} |\langle\ |f_j|,\ |g_j| \rangle - 1|$) looks almost exactly like the $\approx$-equivalence relation (namely, the existence of $\sum_{j \in \mathbb{N}} |\langle f_j, g_j \rangle - 1|$). Could they be the same condition? The answer turns out to be negative.

Proposition 23. Given $\rho_1 \otimes \rho_2 \otimes \cdots$, $\eta_1 \otimes \eta_2 \otimes \cdots \in \otimes_{k \in \mathbb{N}} L^2(\mathbb{R}, dx)$ such that $\|\rho_j\|, \|\eta_j\| = 1$ and $\rho_j(x), \eta_j(x) \neq 0$ for almost every $x \in \mathbb{R}$,

$$\left((\eta_j)_{j \in \mathbb{N}} \approx (\rho_j)_{j \in I} \overset{\text{(by def.)}}{\Longleftrightarrow} \sum_{j \in \mathbb{N}} \Big| \langle\ \eta_j,\ \rho_j\ \rangle - 1 \Big| \ \text{ exists} \right) \quad \begin{array}{c} \Longrightarrow \\ \not\Longleftarrow \end{array} \quad \left(\sum_{j \in \mathbb{N}} \Big| \langle\ |\eta_j|,\ |\rho_j|\ \rangle - 1 \Big| \ \text{ exists} \right).$$

♦

Proof: Let us prove ($\Rightarrow$). First,

$$|\langle \eta_j, \rho_j \rangle| = \Big| \int_{x \in \mathbb{R}} \eta_j(x) \rho_j(x) dx \Big| \leq \int_{x \in \mathbb{R}} |\eta_j|(x) |\rho_j|(x) dx = \langle |\eta_j|, |\rho_j| \rangle \left(\overset{\text{(Cau.Sch.)}}{\leq} \|\eta_j\| \|\rho_j\| = 1 \right).$$

This implies that the distance from $|\langle \eta_j, \rho_j \rangle|$ to 1 must be bigger than that from $\langle |\eta_j|, |\rho_j| \rangle$ to

1, i.e., that $|\langle|\eta_j|,|\rho_j|\rangle - 1| \le |\,|\langle\eta_j,\,\rho_j\rangle| - 1|$.

Second, by the reverse triangle inequality, for any $z \in \mathbb{C}$, $||z|-1| \le |z-1|$, so $|\,|\langle\eta_j,\rho_j\rangle|-1\,| \le |\langle\eta_j,\rho_j\rangle - 1|$ for all $j \in \mathbb{N}$. With all,

$$\big|\ \langle|\eta_j|,|\rho_j|\rangle - 1\ \big| \overset{\left(\substack{\text{we used}\\ \|\eta_j\|,\|\rho_j\|\le 1}\right)}{\le} \big|\,|\langle\eta_j,\rho_j\rangle| - 1\ \big| \overset{\text{(always)}}{\le} \big|\ \langle\eta_j,\rho_j\rangle - 1\ \big|. \tag{4.25}$$

Hence,

$$\sum_{j\in\mathbb{N}} |\langle|\eta_j|,|\rho_j|\rangle - 1| \ \le \sum_{j\in\mathbb{N}} |\langle\eta_j,\rho_j\rangle| - 1\ | \le \sum_{j\in\mathbb{N}} |\langle\eta_j,\rho_j\rangle - 1|. \tag{4.26}$$

Therefore, if $\sum_{j\in\mathbb{N}} |\langle\eta_j,\rho_j\rangle - 1|$ exists, then $\sum_{j\in\mathbb{N}} |\langle|\eta_j|,|\rho_j|\rangle - 1|$ exists.

- We now prove ($\not\Longleftarrow$) by giving a counterexample. For any $(\rho_j)_{j\in\mathbb{N}}$ as in the hypothesis, $\eta_j := -\rho_j$ is a possible η_j within the hypothesis since $\|\eta_j\| = 1$, $|\eta_j|^2(x) = |\rho_j|^2(x) > 0$ almost everywhere, and,

$$\sum_{j=1}^{\infty}\Big|\ \langle|\rho_j|,\ |\eta_j|\rangle - 1\Big| = \sum_{j=1}^{\infty}\Big|\ \langle|\rho_j|,\ |\rho_j|\rangle - 1\Big| = \sum_{j=1}^{\infty}\Big|\|\rho_j\|^2 - 1\Big| = 0. \tag{4.27}$$

So, the sum exists.[a] However, $\sum_{j=1}^{\infty} |\langle\rho_j,\eta_j\rangle - 1| = \sum_{j=1}^{\infty} 2$, the limit of which does not exist in $\mathbb{R}$.[b]

o.ε.δ.

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[a] In particular, by Prop. 15, the measures $d^\infty\mu$ and $d^\infty\nu$ are mut. a.c.

[b] In particular, this means that $(\rho_j)_{j\in\mathbb{N}} \not\approx (\eta_j)_{j\in\mathbb{N}}$.

The reader will agree that even with the negative answer of Prop. 23, all this seems too coincidental. In the next section we will parenthetically explore this in slightly greater detail.

Now, with the above results, we have enough to prove the main claim of Theorem 11: that the measures $d^\infty\mu$ and $d^\infty\nu$ of a same layer are always mut. a.c. However, for the technicality about the particular shape of the Radon-Nikodym derivative, we need two additional (very technical) prior results.

Proposition 24. Given a family of Hilbert spaces $(\mathcal{H}_j)_{j\in I}$, and a fixed $\approx$-equivalence class $\mathfrak{C} \in \Gamma$, let there be arbitrary $(f_j)_{j\in I}, (\rho_j)_{j\in I} \in \mathfrak{C}$ with $\|\rho_j\| = 1\ \forall j \in I$ and $\bigotimes_{j\in I} f_j \neq 0$. Then, for every $\varepsilon > 0$ there is a finite $K^\varepsilon \subseteq I$ such that for all finite $J \supseteq K^\varepsilon$, with $J \subseteq I$,

$$\left\|\bigotimes_{j\in I} f_j - \bigotimes_{j\in J} f_j \otimes \bigotimes_{j\in I\setminus J} \rho_j\right\|_{\otimes_j\mathcal{H}_j} \le \varepsilon \quad . \tag{4.28}$$

In particular, $\bigotimes_{j\in J} f_j \otimes \bigotimes_{j\in I\setminus J} \rho_j \in \mathfrak{C}$.

(*"We can approximate any elementary tensor product in $\mathfrak{C}$ by finite changes in $\bigotimes_{j\in I}\rho_j$"*).

For instance, if $I = \mathbb{N}$,

$$\Big\|(f_1 \otimes f_2 \otimes \cdots)\ -\ (f_1 \otimes \cdots \otimes f_n \otimes \rho_{n+1} \otimes \rho_{n+2} \otimes \cdots)\Big\|_{\otimes_j\mathcal{H}_j} \xrightarrow[n\to\infty]{} 0 \tag{4.29}$$

♦

Proof: The proof's idea is taken from the proof of Lemma 4.1.2 in (von Neumann, 1939).

- First, note that because $(f_j)_{j\in I} \approx (\rho_j)_{j\in I}$, by definition, $\sum_{j\in I} |\langle f_j,\rho_j\rangle - 1|$ exists. But then, by definition, for all $\delta \in (0,1)$ there exists a finite $I^\delta \subseteq I$ such that for all finite $J \subseteq I$, with $I^\delta \subseteq J$,

$$\Big|\sum_{j\in I}|\langle f_j,\rho_j\rangle - 1| - \sum_{j\in J}|\langle f_j,\rho_j\rangle - 1|\Big| \le \delta \Longrightarrow \sum_{j\in I\setminus J}|\langle f_j,\rho_j\rangle - 1| \le \delta. \tag{4.30}$$

Consequently, for all finite $\{k_1, ..., k_n\} \subseteq I \setminus J$,

$$|\prod_{\ell=1}^{n}\langle f_{k_\ell},\rho_{k_\ell}\rangle - 1| \overset{(\star)}{\le} \exp\Big(\sum_{\ell=1}^{n}|\langle f_{k_\ell},\rho_{k_\ell}\rangle - 1|\Big) - 1 \le \exp\Big(\sum_{j\in I\setminus J}|\langle f_j,\rho_j\rangle - 1|\Big) - 1 \overset{(4.30)}{\le} e^\delta - 1 \le e\delta.$$

In $(\star)$ we used that $|z_1\cdots z_n - 1| \le \exp(|z_1-1|+\cdots+|z_n-1|)-1$ for all $z_j\in\mathbb{C}$. This proves that $|\prod_{j\in I\setminus J}\langle f_j,\rho_j\rangle - 1| \le e\delta$.

Now,

$$\Big\|\bigotimes_{j\in I} f_j - \bigotimes_{j\in J} f_j \otimes \bigotimes_{j\in I\setminus J}\rho_j\Big\|^2 = \Big\|\bigotimes_{j\in I} f_j\Big\|^2 + \Big\|\bigotimes_{j\in J} f_j \otimes \bigotimes_{j\in I\setminus J}\rho_j\Big\|^2 - 2\,\mathrm{Re}\Big\{\Big\langle \bigotimes_{j\in I} f_j,\ \bigotimes_{j\in J} f_j \otimes \bigotimes_{j\in I\setminus J}\rho_j\Big\rangle\Big\} =$$

$$= \prod_{j\in I}\|f_j\|^2 + \prod_{j\in J}\|f_j\|^2 \cancelto{1}{\prod_{j\in I\setminus J}\|\rho_j\|^2} - 2\,\mathrm{Re}\Big\{\prod_{j\in J}\|f_j\|^2 \prod_{j\in I\setminus J}\langle f_j,\rho_j\rangle\Big\} =$$

$$= \prod_{j\in J}\|f_j\|^2\Big(\prod_{j\in I\setminus J}\|f_j\|^2 + 1 - 2\,\mathrm{Re}\prod_{j\in I\setminus J}\langle f_j,\rho_j\rangle\Big) \overset{(\star\star)}{\le} C_f\Big(\prod_{j\in I\setminus J}\|f_j\|^2 + 1 - 2\,\mathrm{Re}\prod_{j\in I\setminus J}\langle f_j,\rho_j\rangle\Big) \overset{(\star\star\star)}{\le}$$

$$\le C_f\Big(1 + \lambda + 1 - 2\,\mathrm{Re}\prod_{j\in I\setminus J}\langle f_j,\rho_j\rangle\Big) = C_f\lambda + C_f 2\,\mathrm{Re}\Big(1 - \prod_{j\in I\setminus J}\langle f_j,\rho_j\rangle\Big) \le$$

$$\le C_f\lambda + C_f 2\cdot\Big|1 - \prod_{j\in I\setminus J}\langle f_j,\rho_j\rangle\Big| \le C_f\lambda + C_f 2e\delta.$$

In $(\star\star)$ we used that since $\prod_{j\in I}\|f_j\|^2$ exists by hypothesis,[a] all the finite partial products must be bounded by some $C_f \ge 0$.[b] On the other hand, in $(\star\star\star)$ we used that since $\prod_{j\in I}\|f_j\| \ne 0$ (because $\bigotimes_{j\in I} f_j \ne 0$), then,[a] $\prod_{j\in I}\|f_j\|^2 \ne 0$, and thus,[c] for any $\lambda > 0$ there exists some finite $D^\lambda \subseteq I$ such that $\prod_{j\in I\setminus J}\|f_j\|^2 \le 1+\lambda$ for all finite $J \supseteq D^\lambda$. With all, the inequality we found:

$$\Big\|\bigotimes_{j\in I} f_j - \bigotimes_{j\in J} f_j \otimes \bigotimes_{j\in I\setminus J}\rho_j\Big\|^2 \le C_f\lambda + C_f 2e\delta \tag{4.31}$$

holds for any finite $J \subseteq I$ for which $I^\delta \cup D^\lambda \subseteq J$. Finally, for a given $\varepsilon > 0$, the choices $\lambda_\varepsilon = \frac{\varepsilon}{2C_f}$ and $\delta_\varepsilon = \frac{\varepsilon}{4eC_f}$, together with $K^\varepsilon := I^{\delta_\varepsilon} \cup D^{\lambda_\varepsilon}$, prove the main claim of the lemma.

- For the claim on the sequence, the limit is immediately given by choosing for each $\varepsilon > 0$, $N^\varepsilon := \max\{I^{\delta_\varepsilon} \cup D^{\lambda_\varepsilon}\}$. ***o.ε.δ.***

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[a] By Prop. 50 and Prop. 44, if $\|f_j\| \ne 0$ for all f_j (e.g., if $\otimes_{j\in I} f_j \ne 0$), then ($\prod_{j\in I}\|f_j\|$ exists and $\ne 0$) $\Longleftrightarrow$ ($\prod_{j\in I}\|f_j\|^2$ exists and $\ne 0$).

[b] By definition, given any fixed $\varepsilon_0 > 0$, there exists $Y^{\varepsilon_0} \subseteq I$ finite s.th. all finite super-sets $R \subseteq I$ satisfy $|\prod_{j\in I}\|f_j\|^2 - \prod_{j\in R}\|f_j\|^2| \le \varepsilon_0$. Hence, all $\prod_{j\in R}\|f_j\|^2$ are below $\prod_{j\in I}\|f_j\|^2 + \varepsilon_0$. The remaining finite products to be checked are those that can be taken using only elements in Y^{ε_0}. But there is a finite number of possible combinations we can take there. Hence, there exists a finite upper bound C_f.

[c] If $\prod_{j\in I}\|f_j\|^2 \ne 0$, then the finite partial products have a lower bound $m > 0$. The proof is essentially the same as that of the upper bound in [a]. But then, taking $\varepsilon_0 = m\lambda$, by convergence of $\prod_{j\in I}\|f_j\|^2$, for all finite $J \supseteq Y^{m\lambda} =: D^\lambda$ we get that $|\prod_{j\in I}\|f_j\|^2 - \prod_{j\in J}\|f_j\|^2| \le m\lambda \Longrightarrow |\prod_{j\in I\setminus J}\|f_j\|^2 - 1| \le m\lambda/\prod_{j\in J}\|f_j\|^2 \le m\lambda/m = \lambda$.

Lemma 21. Let there be a fixed $\mathfrak{C} \in \Gamma$ and a pair of generators $(\rho_1 \otimes \rho_2 \otimes \cdots)$, $(\eta_1 \otimes \eta_2 \otimes \cdots) \in \otimes^{\mathfrak{C}}_{k\in\mathbb{N}} L^2(\mathbb{R}, dx)$ such that $\rho_j(x), \eta_j(x) \neq 0$ for almost every $x \in \mathbb{R}$. Denote their associated infinite measures $d^\infty\mu := \odot_{j\in\mathbb{N}} |\rho_j|^2 dx_j$ and $d^\infty\nu := \odot_{k\in\mathbb{N}} |\eta_j|^2 dx$ and let $W_{[\rho]}$, $W_{[\eta]}$ be the unitaries of Theorem 10 associated to them. Then, for any $f_1, ..., f_n \in L^2(\mathbb{R}, dx)$, defining $F := f_1 \otimes \cdots \otimes f_n \otimes \eta_{n+1} \otimes \eta_{n+2} \otimes \cdots$,

$$\mathcal{M}_{W_{[\rho]}(\otimes_{j\in\mathbb{N}}\eta_j)} \circ W_{[\eta]}(F) = W_{[\rho]}(F), \tag{4.32}$$

where $(\mathcal{M}_{W_{[\rho]}(\otimes_{j\in\mathbb{N}}\eta_j)}\Psi)(x) := \Big(W_{[\rho]}(\otimes_{j\in\mathbb{N}}\eta_j)\Big)(x) \cdot \Psi(x)$ for almost every $x \in \mathbb{R}^\infty$. ♦

Proof: On the one hand, by Prop. 24,

$$F_N := (f_1 \otimes \cdots \otimes f_n \otimes \eta_{n+1} \otimes \cdots \otimes \eta_N \otimes \rho_{N+1} \otimes \rho_{N+2} \cdots) \xrightarrow[N\to\infty]{\otimes_{k\in\mathbb{N}} L^2(\mathbb{R},dx)} F.$$

Therefore,

$$W_{[\rho]}(F) = W_{[\rho]}(\overset{\otimes_j}{\lim_{N\to\infty}} F_N) \overset{(W_{[\rho]}\text{ conts.})}{=} \overset{L^2}{\lim_{N\to\infty}} W_{[\rho]}(F_N) \overset{(\text{by def})}{=} \overset{L^2}{\lim_{N\to\infty}} \underbrace{\frac{f_1 \cdots f_n \cdot \eta_{n+1} \cdots \eta_N}{\rho_1 \cdots \rho_N}}_{=:\Phi_N}. \tag{4.33}$$

By Theorem 3.12 in (Rudin, 1987) this implies that there is a subsequence of $(\Phi_N)_{N\in\mathbb{N}}$, say, $(\Phi_{N_k})_{k\in\mathbb{N}}$, that converges point-wise almost everywhere in $x \in \mathbb{R}^\infty$ to $W_{[\rho]}(F)$. That is, for almost every $x \in \mathbb{R}^\infty$,

$$W_{[\rho]}(F)(x) = \overset{\mathbb{C}}{\lim_{k\to\infty}} \frac{f_1(x_1) \cdots f_n(x_n) \cdot \eta_{n+1}(x_{n+1}) \cdots \eta_{N_k}(x_{N_k})}{\rho_1(x_1) \cdots \rho_N(x_{N_k})}. \tag{4.34}$$

- On the other hand,

$$\left\| W_{[\rho]}(\eta_1 \otimes \eta_2 \otimes \cdots) - \frac{\eta_1 \cdots \eta_{N_k}}{\rho_1 \cdots \rho_{N_k}} \right\|_{L^2(\mathbb{R}^\infty, d\mu_\rho)} \overset{(W^{-1}_{[\rho]}\text{ isometry})}{=} \left\| (\eta_1 \otimes \eta_2 \otimes \cdots) - W^{-1}_{[\rho]}\Big(\frac{\eta_1 \cdots \eta_{N_k}}{\rho_1 \cdots \rho_{N_k}}\Big) \right\|_{\otimes_j L^2(\mathbb{R})} = \tag{4.35}$$

$$= \|(\eta_1 \otimes \eta_2 \otimes \cdots) - (\eta_1 \otimes \cdots \otimes \eta_{N_k} \otimes \rho_{N_k+1} \otimes \cdots)\| \xrightarrow[k\to\infty]{(\text{by Prop. } 24)} 0.$$

Therefore, $W_{[\rho]}(\eta_1 \otimes \eta_2 \otimes \cdots) = \lim^{L^2(\mathbb{R}^\infty, d^\infty\mu)}_{k\to\infty} \frac{\eta_1 \cdots \eta_{N_k}}{\rho_1 \cdots \rho_{N_k}}$. As above, this implies point-wise convergence almost everywhere for some subsequence $(\frac{\eta_1 \cdots \eta_{N_{k_\ell}}}{\rho_1 \cdots \rho_{N_{k_\ell}}})_{\ell\in\mathbb{N}}$ to $W_{[\rho]}(\eta_1 \otimes \eta_2 \otimes \cdots)$. Using this in $(\star)$, we get that for almost every $x \in \mathbb{R}^\infty$,

$$(\mathcal{M}_{W_{[\rho]}(\otimes_{j\in\mathbb{N}}\eta_j)} \circ W_{[\eta]}F)(x) = W_{[\rho]}(\otimes_{j\in\mathbb{N}}\eta_j)(x) \cdot (W_{[\eta]}F)(x) \overset{(\star)}{=}$$

$$= \left(\overset{\mathbb{C}}{\lim_{\ell\to\infty}} \frac{\eta_1(x_1) \cdots \eta_{N_{k_\ell}}(x_{N_{k_\ell}})}{\rho_1(x_1) \cdots \rho_{N_{k_\ell}}(x_{N_{k_\ell}})} \right) \cdot \frac{f_1(x_1) \cdots f_n(x_n)}{\eta_1(x_1) \cdots \eta_n(x_n)} \overset{(\text{eventually } N_{k_\ell} > n)}{=}$$

$$= \overset{\mathbb{C}}{\lim_{\ell\to\infty}} \frac{f_1(x_1) \cdots f_n(x_n) \cdot \eta_{n+1}(x_{n+1}) \cdots \eta_{N_{k_\ell}}(x_{N_{k_\ell}})}{\rho_1(x_1) \cdots \rho_{N_{k_\ell}}(x_{N_{k_\ell}})}.$$

But every subsequence of a convergent sequence must converge to the same value, so this must equal (4.34). Hence, $(\mathcal{M}_{W_{[\rho]}(\otimes_{j\in\mathbb{N}}\eta_j)} \circ W_{[\eta]}F)(x) = W_{[\rho]}(F)(x)$ almost-everywhere in $x \in \mathbb{R}^\infty$, which is to say that $(\mathcal{M}_{W_{[\rho]}(\otimes_{j\in\mathbb{N}}\eta_j)} \circ W_{[\eta]}F) = W_{[\rho]}(F)$. ***o.ε.δ.***

The Proof of Theorem 11

Proof **of Theorem 11:** By hypothesis, both, $\eta_1 \otimes \eta_2 \otimes \cdots$ and $\rho_1 \otimes \rho_2 \otimes \cdots$ belong to the same layer of the ITP, so, $(\eta_j)_{j\in\mathbb{N}} \approx (\rho_j)_{j\in\mathbb{N}}$ and by definition, $\sum_{j\in\mathbb{N}} |\langle \eta_j,\ \rho_j\rangle - 1|$ exists. But by Prop. 23, this implies that $\sum_{j\in\mathbb{N}} |\langle\ |\eta_j|,\ |\rho_j|\ \rangle - 1|$ exists, which holds by Cor. 15, *if and only if* $d^\infty\mu \sim d^\infty\nu$.

- For the claim on the limit, repeat exactly (4.35) with $(n)_{n\in\mathbb{N}}$ in place of $(N_k)_{k\in\mathbb{N}}$.
- Finally, to prove the Radon-Nikodym derivative claim, note that for any $n \in \mathbb{N}$ and arbitrary $B_1, ...B_n \in \mathfrak{B}(\mathbb{R})$, given $B := \prod_{j=1}^n B_j \times \prod_{j\in I\setminus\{1,\dots,n\}} \mathbb{R}$,

$$\int_B \Big|W_{[\rho]}(\eta_1 \otimes \eta_2 \otimes \cdots)\Big|^2 d^\infty\mu_{\mathfrak{C}} = \Big\|W_{[\rho]}(\eta_1 \otimes \eta_2 \otimes \cdots)\mathbb{1}_B\Big\|^2_{L^2} \overset{\left(\substack{\mathbb{1}_B(x)=\mathbb{1}_{B_1}(x_1)\cdots\mathbb{1}_{B_n}(x_n)) \\ +\text{ by def. of } W_{[\eta]}}\right)}{=}$$

$$= \left\|W_{[\rho]}(\eta_1 \otimes \eta_2 \otimes \cdots) \cdot W_{[\eta]}\Bigg(\underbrace{(\mathbb{1}_{B_1}\eta_1) \otimes \cdots \otimes (\mathbb{1}_{B_1}\eta_n) \otimes \eta_{n+1} \otimes \cdots}_{=:F}\Bigg)\right\|^2 \overset{(\text{ Lem. } 21)}{=} \Big\|W_{[\rho]}(F)\Big\|^2 =$$

(Use that by Lemma 24, $F_N := (\mathbb{1}_{B_1}\eta_1 \otimes \cdots \otimes \mathbb{1}_{B_n}\eta_n \otimes \eta_{n+1} \cdots \otimes \rho_N \otimes \cdots) \xrightarrow[N\to\infty]{\otimes_{k\in\mathbb{N}} L^2(\mathbb{R},dx)} F$)

$$\overset{(W_{[\rho]}\text{ conts.})}{=} \lim_{N\to\infty} \Big\|W_{[\rho]}(F_N)\Big\|^2 \overset{(\text{by def})}{=} \lim_{N\to\infty} \left\|\mathbb{1}_{B_1}\cdots\mathbb{1}_{B_n}\frac{\eta_1\cdots\eta_N}{\rho_1\cdots\rho_N}\right\|^2 =$$

$$= \lim_{N\to\infty} \int_{x\in\mathbb{R}^\infty} \frac{\mathbb{1}_1(x_1)|\eta_1(x_1)|^2\cdots\mathbb{1}_{B_n}(x_n)|\eta_n(x_n)|^2|\eta_{n+1}(x_{n+1})|^2\cdots|\eta_N(x_N)|^2}{|\rho_1(x_1)|^2\cdots|\rho_N(x_N)|^2} d^\infty\mu =$$

$$\overset{\left(\substack{\text{Prop. } 20 \\ +\text{ Tonelli}}\right)}{=} \lim_{N\to\infty} \left(\prod_{j=1}^n \int_{x_j\in B_j} \frac{|\eta_j|^2(x_j)}{|\rho_j|^2(x_j)} \underbrace{d\mu_j}_{|\rho_j|^2(x_j)dx_j} \prod_{j=n+1}^N \int_{x_j\in\mathbb{R}} \frac{|\eta_j|^2(x_j)}{|\rho_j|^2(x_j)} \underbrace{d\mu_j}_{|\rho_j|^2(x_j)dx_j}\right) \overset{\|\eta_j\|^2=1}{=}$$

$$= \lim_{N\to\infty} \prod_{j=1}^n \underbrace{|\eta_j|^2 dx_j}_{d\nu_j}(B_j) = \prod_{j=1}^n d\nu_j(B_j) \overset{(\text{prod. meas.})}{=} d^\infty\nu(B). \tag{4.36}$$

Now, let C_{meas} be the Boolean algebra of Lemma 19. Due to Lemma 19, any $B \in C_{meas}$ is such that $B = \sqcup_{\ell=1}^M B^\ell := \sqcup_{\ell=1}^M B_1^\ell \times \cdots \times C_n^\ell \times \mathbb{R} \times \cdots$ for some $B_k^\ell \in \mathfrak{B}(\mathbb{R})$. Then, because $\Big|W_{[\rho]}(\eta_1 \otimes \eta_2 \otimes \cdots)\Big|^2 d^\infty\mu$ is a measure by Lemma 6, finite additivity gives:

$$\Big(\Big|W_{[\rho]}(\eta_1\otimes\eta_2\otimes\cdots)\Big|^2 d^\infty\mu\Big)(B) = \sum_{\ell=1}^M \Big(\Big|W_{[\rho]}(\eta_1\otimes\eta_2\otimes\cdots)\Big|^2 d^\infty\mu\Big)(B^\ell) \overset{(4.36)}{=} \sum_{\ell=1}^M d^\infty\nu(B^\ell) = d^\infty\nu(B). \tag{4.37}$$

Finally, by Theorem 1 two σ-finite measures that coincide in a Boolean algebra must be equal. Therefore, $\Big|W_{[\rho]}(\eta_1 \otimes \eta_2 \otimes \cdots)\Big|^2 d^\infty\mu = d^\infty\nu$. ***o.ε.δ.***

Parenthesis: On an even Coarser Dissection of the ITP

The condition in Corollary 15 (the existence of $\sum_{j\in\mathbb{N}} |\langle|\rho_j|, |\eta_j|\rangle - 1|$) is so similar to the condition for $\approx$-equivalence (the existence of $\sum_{j\in\mathbb{N}} |\langle\rho_j, \eta_j\rangle - 1|$), that one expects that if $(\rho_j)_{j\in\mathbb{N}}, (\eta_j)_{j\in\mathbb{N}}$ are in the same $\approx$-class their associated measures satisfy $d^\infty\mu_\rho \sim d^\infty\mu_\eta$. And indeed this is the case as proven in Theorem 11. However, one would also expect naively that if $(\rho_j)_{j\in\mathbb{N}}, (\eta_j)_{j\in\mathbb{N}}$ are in different $\approx$-classes, by Cor. 15 the associated measures satisfy $d^\infty\mu_\rho \perp d^\infty\mu_\eta$. This would be the case indeed if the $(\Leftarrow)$ implication in Prop. 23 held. However, we proved that this is *not* the case, and the counterexample we used to prove it was precisely a pair of generators $(\rho_j)_{j\in\mathbb{N}}, (\eta_j)_{j\in\mathbb{N}}$ of *different* layers with $d^\infty\mu_\rho \sim d^\infty\mu_\eta$.

Now, there is a second equivalence relation for ITPs, introduced in §A.2.2: the *quasi-equivalence* relation $\overset{q}{\approx}$. Go for a second to Proposition 57 to see its definition. In particular, note that $(\rho_j)_{j\in\mathbb{N}} \approx (\eta_j)_{j\in\mathbb{N}} \Longrightarrow (\rho_j)_{j\in\mathbb{N}} \overset{q}{\approx} (\eta_j)_{j\in\mathbb{N}}$.

With this "coarser"/weaker relation, we can state the following analogue of Theorem 11, where yet still, we see that $\overset{q}{\approx}$ is not the precise condition to identify mutual singularity of the generated measures:

Proposition 25. Given $\rho_1\otimes\rho_2\otimes\cdots,\ \eta_1\otimes\eta_2\otimes\cdots \in \otimes_{k\in\mathbb{N}}L^2(\mathbb{R},dx)$ such that $\|\rho_j\|, \|\eta_j\| = 1$ and $\rho_j(x), \eta_j(x) \neq 0$ for almost every $x\in\mathbb{R}$,

$$\left((\eta_j)_{j\in\mathbb{N}} \overset{q}{\approx} (\rho_j)_{j\in I} \overset{\text{(by def.)}}{\Longleftrightarrow} \sum_{j\in\mathbb{N}} \Big| |\langle\, \eta_j,\ \rho_j\, \rangle| - 1 \Big| \ \text{ exists} \right) \ \substack{\Longrightarrow \\ \not\Longleftarrow} \ \left(\Big| \langle\, |\eta_j|,\ |\rho_j|\, \rangle - 1 \Big| \ \text{ exists} \right).$$

$$\text{Hence,} \qquad \left((\rho_j)_{j\in\mathbb{N}} \overset{q}{\approx} (\eta_j)_{j\in\mathbb{N}} \right) \ \substack{\Longrightarrow \\ \not\Longleftarrow} \ \left(d^\infty\mu_\rho \sim d^\infty\mu_\eta \right). \tag{4.38}$$

♦

Proof: To prove $(\Rightarrow)$ we use (4.26) of Prop. 23:

$$(\rho_j)_{j\in\mathbb{N}} \overset{q}{\approx} (\eta_j)_{j\in\mathbb{N}} \overset{\text{(by def)}}{\Longleftrightarrow} \sum_{j\in\mathbb{N}} \big|\, |\langle \eta_j, \rho_j\rangle| - 1 \big| \text{ exists} \overset{(4.26)}{\Longrightarrow} \sum_{j\in\mathbb{N}} \big|\, \langle |\eta_j|,\ |\rho_j|\rangle - 1 \big| \text{ exists} \overset{\text{Cor. 15}}{\Longleftrightarrow} d^\infty\mu_\eta \sim d^\infty\mu_\rho.$$

- To prove $(\not\Leftarrow)$, let us provide a counterexample. Let $\rho \in L^2(\mathbb{R},dx)$ be the normalized standard Gaussian so that $\|\rho\|_{L^2} = 1$. Define $\eta := \mathbb{1}_{(-\infty,0)}\cdot\rho - \mathbb{1}_{[0,+\infty)}\cdot\rho$. Then, trivially, $\langle\rho,\eta\rangle = 0$ but $\langle|\rho|,|\eta|\rangle = \|\rho\|^2 = 1$. Hence,

$$\sum_{j\in\mathbb{N}} \big|\, |\langle\rho,\eta\rangle| - 1 \big| = \sum_{j\in\mathbb{N}} 1 \quad \text{(does not exist)} \quad \text{but} \quad \sum_{j\in\mathbb{N}} \big|\, \langle|\rho|,|\eta|\rangle - 1 \big| = 0 \text{ (exists)}.$$

Therefore, $(\rho_j)_{j\in\mathbb{N}} \overset{q}{\not\approx} (\eta_j)_{j\in\mathbb{N}}$ but $d^\infty\mu_\rho \sim d^\infty\mu_\eta$. ***o.ε.δ.***

With all, the situation for the $(\eta_j)_j, (\rho_j)_j$ as in Prop 25 is the following one:

$$\begin{array}{ccccc}
\approx & & \overset{q}{\approx} & & \overset{k}{\approx}\,? \\
\left(\displaystyle\sum_{j\in\mathbb{N}} \Big| \langle\eta_j,\ \rho_j\rangle - 1 \Big| \text{ exists} \right) & \substack{\Longrightarrow \\ \not\Longleftarrow} & \left(\displaystyle\sum_{j\in\mathbb{N}} \Big| |\langle\eta_j,\ \rho_j\rangle| - 1 \Big| \text{ exists} \right) & \substack{\Longrightarrow \\ \not\Longleftarrow} & \left(\displaystyle\sum_{j\in\mathbb{N}} \Big| \langle|\eta_j|,\ |\rho_j|\rangle - 1 \Big| \text{ exists} \right) \\
\Updownarrow & & \Updownarrow & & \Updownarrow \\
\otimes^{[\eta]}_{k\in\mathbb{N}} L^2(\mathbb{R},dx) = \otimes^{[\rho]}_{k\in\mathbb{N}} L^2(\mathbb{R},dx) & \Big| & \exists\ (\theta_j)_{j\in\mathbb{R}} \subset [-\pi,\pi):\ (\eta_j)_j \approx (e^{i\theta_j}\rho_j)_j & \Big| & \odot_{j\in\mathbb{N}}|\eta_j|^2 dx_j \sim \odot_{j\in\mathbb{N}}|\rho_j|^2 dx_j \\
\text{and else} & \Big| & \text{and else} & \Big| & \text{and else} \\
\otimes^{[\eta]}_{k\in\mathbb{N}} L^2(\mathbb{R},dx) \perp \otimes^{[\rho]}_{k\in\mathbb{N}} L^2(\mathbb{R},dx) & \Big| & \substack{\text{their quasi-layers are independent degrees} \\ \text{of freedom for } \mathcal{L}^q(\otimes_{j\in\mathbb{N}}L^2(\mathbb{R},dx))} & \Big| & \odot_{j\in\mathbb{N}}|\eta_j|^2 dx_j \perp \odot_{j\in\mathbb{N}}|\rho_j|^2 dx_j
\end{array}$$

Clearly, this suggests a third kind of equivalence relation. Note that unlike the other two, this third kind would only be definable for tensor products of number-valued function-spaces like $L^2(\mathbb{R},dx)$.

Proposition 26. Let I be arbitrary and $(X_j, \Sigma_j, d\mu_j)_{j\in I}$ be measure spaces. Consider the ITP $\bigotimes_{j\in I} L^2(X_j, d\mu_j)$ with its corresponding notion of $\mathscr{C}_0$-sequences (Def. 36). Then, for $(f_j)_{j\in I}, (g_j)_{j\in I} \in \mathscr{C}_0$, the relation

$$\left((f_j)_{j\in I} \overset{k}{\approx} (g_j)_{j\in I} \right) \quad :\Longleftrightarrow \quad \left(\sum_{j\in I} \Big| \langle |f_j|,\ |g_j| \rangle - 1 \Big| \text{ exists} \right) \tag{4.39}$$

is an *equivalence relation*. We call it the *Kakutani equivalence* or *k-equivalence*. In particular,

$$\left((f_j)_{j\in I} \overset{k}{\approx} (g_j)_{j\in I} \right) \quad \Longleftrightarrow \quad \left((|f_j|)_{j\in I} \approx (|g_j|)_{j\in I} \right).$$

♦

Proof: **Reflexive :** $(f_j)_{j\in I} \in \mathscr{C}_0$ implies by definition that $\sum_{j\in I} |\|f_j\| - 1|$ exists. But by Prop. 50, this holds *if and only if* $\sum_{j\in I} \Big|\|f_j\|^2 - 1\Big|$ exists. Hence, by definition, $(f_j)_{j\in I} \overset{k}{\approx} (f_j)_{j\in I}$.

Symmetric: Note that

$$\langle |f_j|, |g_j| \rangle \overset{\text{(conj.sym.)}}{=} \overline{\langle |g_j|, |f_j| \rangle} = \overline{\int_{x\in X} \overline{|g_j|(x)}\, |f_j|(x) d\mu} \overset{\text{(all real)}}{=} \int_{x\in X} |g_j|(x)\, |f_j|(x) d\mu = \langle |g_j|, |f_j| \rangle. \tag{4.40}$$

Hence, $\sum_{j\in I} \Big| \langle |f_j|,\ |g_j| \rangle - 1 \Big| = \sum_{j\in I} \Big| \langle |g_j|,\ |f_j| \rangle - 1 \Big|$, and thus, $(f_j)_{j\in I} \overset{k}{\approx} (g_j)_{j\in I}$ *if and only if* $(g_j)_{j\in I} \overset{k}{\approx} (f_j)_{j\in I}$.

Transitive : First, note that

$$\left((|f_j|)_{j\in I} \approx (|g_j|)_{j\in I} \right) \overset{\text{(by def)}}{\Longleftrightarrow} \left(\sum_{j\in I} \Big| \langle |f_j|,\ |g_j| \rangle - 1 \Big| \text{ exists} \right). \tag{4.41}$$

Hence, $(f_j)_{j\in I} \overset{k}{\approx} (g_j)_{j\in I}$ *if and only if* $(|f_j|)_{j\in I} \approx (|g_j|)_{j\in I}$. But we know that $\approx$ is transitive, so $\overset{k}{\approx}$ must be transitive. ***o.ε.δ.***

Proposition 27. In the situation of Proposition 26, for any $(f_j)_{j\in I}, (g_j)_{j\in I} \in \mathscr{C}_0$,

$$(f_j)_{j\in I} \overset{k}{\approx} (g_j)_{j\in I} \quad \Longrightarrow \quad \left(\begin{array}{c} \exists \theta_j : X_j \to [-\pi, \pi) \ \text{ measurable :} \\ (f_j)_{j\in I} \approx (e^{i\theta_j}\, g_j)_{j\in I} \end{array} \right). \tag{4.42}$$

The reverse implication also holds when $\|f_j\|, \|g_j\| \leq 1$ for all $j \in \mathbb{N}$. (Possibly, it holds for any $(f_j)_{j\in I}, (g_j)_{j\in I} \in \mathscr{C}_0$.)

♦

Proof: $(\Rightarrow)$: For any function $h \in L^2(X_j, d\mu_j)$ one can define the map $\arg(h) : X \to [-\pi, \pi)$,

$$\arg(h)(x) := \begin{cases} \text{atan2}(h(x)) = \text{Im}(log(h(x)) & \text{if } h(x) \neq 0 \\ 0 & \text{if } h(x) = 0 \end{cases}. \tag{4.43}$$

It is the composition of two measurable maps, so it is itself measurable too. Now, let $z, w \in \mathbb{C} \setminus \{0\}$. Given their polar forms $z = |z| e^{i\alpha}$, $w = |w| e^{i\beta}$, we get that $|z||w| = z\, w\, e^{-i(\alpha+\beta)}$. But $\overline{z} = |z| e^{-i\alpha}$ so, $|z||w| = |z| e^{i\alpha}\, w\, e^{-i(\alpha+\beta)} = \overline{z}\, w\, e^{i(\alpha-\beta)}$. We can do this point-wise too: let $\alpha_j := \arg(f_j)$ and $\beta_j := \arg(g_j)$, then, for $x \in X_j$,

$$|f_j(x)|\, |g_j(x)| = \overline{f_j(x)}\, g_j(x)\, e^{i(\alpha_j(x) - \beta_j(x))}. \tag{4.44}$$

As such,

$$\langle |f_j|,\ |g_j|\rangle = \int_{x\in X} |f_j(x)|\,|g_j(x)| d\mu_j \overset{(4.44)}{=} \int_{x\in X_j} \overline{f_j(x)}\, g_j(x)\, e^{i(\alpha_j(x)-\beta_j(x))} d\mu_j = \langle f_j,\ e^{i(\alpha_j-\beta_j)} g_j\rangle.$$

Therefore,

$$\sum_{j\in I} \Big| \langle |f_j|,\ |g_j|\rangle - 1 \Big| = \sum_{j\in I} \Big| \langle f_j,\ e^{i(\alpha_j-\beta_j)} g_j\rangle - 1 \Big|. \tag{4.45}$$

Taking $\theta_j := \alpha_j - \beta_j$, the statement to be proven holds.

- ($\Leftarrow$): It holds if $\|f_j\|, \|g_j\| \leq 1$ because:

$$\Big((f_j) \approx (e^{i\theta_j}\, g_j) \Big) \Longleftrightarrow \Big(\sum_{j\in I} \Big| \langle f_j,\ e^{i\theta_j} g_j\rangle - 1 \Big| \text{ exists} \Big) \overset{(4.26)}{\Longrightarrow}$$

$$\Longrightarrow \Big(\sum_{j\in I} \Big| \langle |f_j|,\ \cancelto{1}{|e^{i\theta_j}|} |g_j|\rangle - 1 \Big| \text{ exists} \Big) \Longleftrightarrow (f_j) \overset{k}{\approx} (g_j).$$

o.ε.δ.

In Theorem 5 we found that two pilot-wave theories over the same configuration-space but on L^2-spaces with different measures yield experimentally and ontologically identical quantum theories *if and only if* the measures are mut. a.c. We can now add to the discussion that if we employ individual layers of the ITP for QM, each class of $\overset{k}{\approx}$-related generators would provide an "equivalent" $L^2(\mathbb{R}^\infty, d^\infty \mu_\rho)$ space independently of the chosen generator. Interestingly, we found that the isomorphism between such equivalent pilot-wave theories was constrained to be a unitary map exchanging the Radon-Nikodym derivative times an arbitrary phase choice per configuration —which we called "gauge phase". Proposition 27 suggests that something similar is going on for $L^2(\mathbb{R}^\infty, d^\infty \mu)$ spaces.

We leave the full clarification of the $\overset{k}{\approx}$-equivalence relation's significance as future work.

4.4 Solve the Impasse in both Approaches by Merging Them!

Let us assemble all the obtained results to clarify the picture. First of all, we found in Chapter 3 that $\bigotimes_{k\in\mathbb{N}} L^2(\mathbb{R}, dx)$ —which has a unique construction and hence it is the obvious candidate for "$L^2(\mathbb{R}^\infty, d^\infty x)$"— can be "sectorized" into some "layers" indexed by $\mathfrak{C} \in \Gamma$ as

$$\bigotimes_{k\in\mathbb{N}} L^2(\mathbb{R}, dx) = \bigoplus_{\mathfrak{C}\in\Gamma} \Big[\otimes^{\mathfrak{C}}_{k\in\mathbb{N}} L^2(\mathbb{R}, dx) \Big]. \tag{4.46}$$

Every such layer was generated by finite "perturbations" and linear combinations of a fixed tailed tensor product of unit vectors $\rho_1^{\mathfrak{C}} \otimes \rho_2^{\mathfrak{C}} \otimes \cdots \in \bigotimes_{k\in\mathbb{N}} L^2(\mathbb{R}, dx)$, i.e.,

$$\bigoplus_{\mathfrak{C}\in\Gamma} [\otimes^{\mathfrak{C}}_{k\in\mathbb{N}} L^2(\mathbb{R}, dx)] = \overline{\operatorname{span}\Big\{ \psi_1 \otimes \cdots \otimes \psi_n \otimes \rho^{\mathfrak{C}}_{n+1} \otimes \cdots \ \Big|\ n\in\mathbb{N},\ \psi_j \in L^2(\mathbb{R}, dx) \Big\}}. \tag{4.47}$$

We observed that the inconvenience of such abstract tensor spaces is that there is no longer an explicit reference to $x \in \mathbb{R}^\infty$. But, we also realized that one faces the same issue with finite tensor products. In the finite case, what one does to make $\mathbb{R}^n$ manifest is to find an isomorphism $\bigotimes_{k=1}^n L^2(\mathbb{R}, dx) \cong L^2(\mathbb{R}^n, d^n x)$. Inspired by that, almost with the same unitary map $W_{\mathfrak{C}}$, we found (via Theorem 10 and Proposition 22) a way to identify each $\mathfrak{C}$-layer with an $L^2(\mathbb{R}^\infty, d^\infty\mu_{\mathfrak{C}})$ space. In particular, we avoided the non-existence of an infinite Lebesgue measure by employing $d^\infty\mu_{\mathfrak{C}} := \odot_{j\in\mathbb{N}}|\rho_j^{\mathfrak{C}}|^2 dx_j$, a probability measure naturally emerging from the generator of the layer, $(\rho_j^{\mathfrak{C}})_{j\in\mathbb{N}}$, in a way that every finite "n-truncation" or "cut-off" yields a measure in $\mathbb{R}^n$ equivalent to the Lebesgue measure, $d^n\mu_{\mathfrak{C}} := \odot_{j=1}^n|\rho_j^{\mathfrak{C}}|^2 dx_j \sim d^n x$. With all, we found a way to make manifest the configuration-space $\mathbb{R}^\infty$ in each $\bigotimes_{k\in\mathbb{N}}^{\mathfrak{C}} L^2(\mathbb{R}, dx)$ layer. Namely, the abstract tensor products of each layer became actual functions over $(x_1, x_2, ...) \in \mathbb{R}^\infty$.[3] We can extend this to any $\Psi \in \otimes_{j\in\mathbb{N}} L^2(\mathbb{R}, dx)$ as follows: after a choice of generator $\rho_1^{\mathfrak{C}} \otimes \rho_2^{\mathfrak{C}} \otimes \cdots$ per layer $\mathfrak{C} \in \Gamma$, with the unitary $W_{\mathfrak{C}}$ of Theorem 10, we can build a big unitary identification,

$$
\begin{array}{rcl}
\mathscr{W} : \bigotimes_{k\in\mathbb{N}} L^2(\mathbb{R}, dx) & = \bigoplus\limits_{\mathfrak{C}\in\Gamma} \left(\otimes_{k\in\mathbb{N}}^{\mathfrak{C}} L^2(\mathbb{R}, dx) \right) \xrightarrow{\mathscr{W} = \oplus_{\mathfrak{C}\in\Gamma} W_{\mathfrak{C}}} & \bigoplus\limits_{\mathfrak{C}\in\Gamma} L^2(\mathbb{R}^\infty, d^\infty\mu_{\rho_{\mathfrak{C}}}) \\
\Psi & = (\Psi_{\mathfrak{C}})_{\mathfrak{C}\in\,\Gamma} \qquad\quad \longmapsto & (\widetilde{\Psi_{\mathfrak{C}}})_{\mathfrak{C}\in\Gamma}
\end{array} .
$$

Then, considering $\bigotimes_{k\in\mathbb{N}} L^2(\mathbb{R}, dx) \cong \bigoplus\limits_{\mathfrak{C}\in\Gamma} L^2(\mathbb{R}^\infty, d^\infty\mu_{\rho_{\mathfrak{C}}})$, one could denote $\Psi \cong (\widetilde{\Psi_{\mathfrak{C}}})_{\mathfrak{C}\in\Gamma}$. Namely, we could understand each vector $\Psi \in \bigotimes_{k\in\mathbb{N}} L^2(\mathbb{R}, dx)$ as a tuple of infinitely many wavefunctions over $\mathbb{R}^\infty$. Using this, one can think of an obvious Born rule-measure over $\mathbb{R}^\infty$ for each $\Psi \in \bigotimes_{k\in\mathbb{N}} L^2(\mathbb{R}, dx)$. The way to do it is exactly the same as when one has a spinor valued wavefunction, which, after choosing a spin basis, is also representable as a tuple of functions. Namely, for $\Psi \in \bigotimes_{k\in\mathbb{N}} L^2(\mathbb{R}, dx)$, one could define, for any $B \in \mathfrak{B}(\mathbb{R}^\infty)$,

$$
\text{“}\mathbb{P}\begin{pmatrix} \text{The system is found in} \\ x \in B \text{ if described by } \Psi \end{pmatrix}\text{” := “}\int_{x\in B} |\Psi|^2(x) d^\infty x\text{” := } \sum_{\mathfrak{C}\in\Gamma} \int_{x\in B} |\widetilde{\Psi_{\mathfrak{C}}}|^2(x)\; d^\infty\mu_{\rho_{\mathfrak{C}}}, \tag{4.48}
$$

which also yields an associated equivariance law for families of homeomorphisms $Q_t : \mathbb{R}^\infty \to \mathbb{R}^\infty$. Because by Lemma 14, at most countably many $\widetilde{\Psi_{\mathfrak{C}}} \neq 0$, the sum $\sum_{\mathfrak{C}\in\Gamma}$ is well-defined in the usual sense.

And yet, there is a potential flaw in this construction. In the identification $\bigotimes_{k\in\mathbb{N}} L^2(\mathbb{R}, dx) \cong \bigoplus\limits_{\mathfrak{C}\in\Gamma} L^2(\mathbb{R}^\infty, d^\infty\mu_{\rho_{\mathfrak{C}}})$ we made a choice: we had to choose a generator $\bigotimes_{k\in\mathbb{N}} \rho_k^{\mathfrak{C}}$ per each $\mathfrak{C} \in \Gamma$. What if different choices lead to different Born rules? We would like that this choice is merely a "choice of coordinate chart" that makes computations easier but nothing else —precisely as in the case of spinors, where one uses a particular choice of basis to talk about a Born rule and trajectories, but any such choice gives the same result. "Luckily", we found in Prop. 22 that for generators of the same layer, the given $d^\infty\mu_{\mathfrak{C}}$ are mut. a.c. That, together with Theorem 5 is all we need to prove that this Born rule is indeed "independent of the coordinate chart" $\mathfrak{R} := (\otimes_{j\in\mathbb{N}}\rho_j^{\mathfrak{C}})_{\mathfrak{C}\in\Gamma}$.

---

[3] Recall that in Chapter 2, we found an issue with the $L^2(\mathbb{R}^\infty, d^\infty\mu)$ spaces: there were too many possible inequivalent choices of infinite measure $d^\infty\mu$ —some not mut. a.c. to each other although they were equivalent in every "n-cut-off". The solution in the end turns out to be that they were not too many, but *just the right number*: their non-uniqueness is just the right thing to match the multiple layers appearing in the unique limit of tensor products.

Theorem 13. We call a choice of generator $\rho_1^{\mathfrak{C}} \otimes \rho_2^{\mathfrak{C}} \otimes \cdots \bigotimes_{k\in\mathbb{N}} \in L^2(\mathbb{R}, dx)$ for each class $\mathfrak{C} \in \Gamma$, such that $\rho(x) \neq 0$ for dx-almost every $x \in \mathbb{R}$, a *wavefunction-representation (WR) basis* $\mathfrak{R} := (\bigotimes_{k\in\mathbb{N}} \rho_k^{\mathfrak{C}})_{\mathfrak{C}\in\Gamma}$. The associated (Borel probability) *measures* $d^\infty\mu_{\mathfrak{C}} := \odot_{j=1}^{\infty}|\rho_j^{\mathfrak{C}}|^2 dx_j$, $\mathfrak{C} \in \Gamma$, are called the *WR background measures.*

(i) Any such $\mathfrak{R}$ has an associated identification

$$L^2(\mathbb{R}, dx) \otimes L^2(\mathbb{R}, dx) \otimes \cdots \overset{\mathfrak{R}}{\cong} \bigoplus_{\mathfrak{C}\in\Gamma} L^2(\mathbb{R}^\infty,\ d^\infty\mu_{\mathfrak{C}}) \tag{4.49}$$

given by the block-diagonal unitary isomorphism

$$\begin{array}{lcccc} \mathscr{W}_{\mathfrak{R}} : \bigotimes_{k\in\mathbb{N}} L^2(\mathbb{R}, dx) & = \bigoplus_{\mathfrak{C}\in\Gamma} \Big(\otimes_{k\in\mathbb{N}}^{\mathfrak{C}} L^2(\mathbb{R}, dx)\Big) & \xrightarrow{\mathscr{W}_{\mathfrak{R}} = \oplus_{\mathfrak{C}\in\Gamma} W_{[\rho^{\mathfrak{C}}]}} & \bigoplus_{\mathfrak{C}\in\Gamma} L^2(\mathbb{R}^\infty, d^\infty\mu_{\rho_{\mathfrak{C}}}) \\ \Psi & = (\Psi_{\mathfrak{C}})_{\mathfrak{C}\in\ \Gamma} & \longmapsto & (\widetilde{\Psi_{\mathfrak{C}}})_{\mathfrak{C}\in\Gamma} \end{array} \tag{4.50}$$

where $W_{[\rho^{\mathfrak{C}}]}$ is the unitary of Theorem 10 associated to $\bigotimes_{k\in\mathbb{N}} \rho_k^{\mathfrak{C}}$. We call this identification the *wavefunction representation of* $\otimes_{j\in\mathbb{N}} L^2(\mathbb{R}, dx)$ (in $\mathfrak{R}$ basis).

(ii) Given a different choice of WR basis $\tilde{\mathfrak{R}} := (\bigotimes_{k\in\mathbb{N}} \eta_k^{\mathfrak{C}})_{\mathfrak{C}\in\Gamma}$, one has that

$$W_{[\rho^{\mathfrak{C}}]} = (U_{\rho^{\mathfrak{C}}\leftarrow\eta^{\mathfrak{C}}} \circ W_{[\eta^{\mathfrak{C}}]}), \tag{4.51}$$

where $U_{\rho^{\mathfrak{C}}\leftarrow\eta^{\mathfrak{C}}}$ is the *change of layer-WR*,

$$\begin{array}{lllc} U_{\rho^{\mathfrak{C}}\leftarrow\eta^{\mathfrak{C}}} : & L^2(\mathbb{R}^\infty, d^\infty\mu_{\eta^{\mathfrak{C}}}) & \longrightarrow & L^2(\mathbb{R}^\infty, d^\infty\mu_{\rho^{\mathfrak{C}}}) \\ & g & \longmapsto & \Big[\ x \in \mathbb{R}^\infty \ \longmapsto \ W_{[\rho^{\mathfrak{C}}]}(\otimes_{k\in\mathbb{N}}\eta_k^{\mathfrak{C}})(x) \cdot g(x)\Big] \end{array} . \tag{4.52}$$

It is the multiplication operator that unitarily exchanges the WR background measures by multiplying the input function with the (square root) of their Radon-Nikodym derivative (see Theorems 5 and 11). Then, the WR of $\mathfrak{R}$ and $\tilde{\mathfrak{R}}$ are related by

$$\mathscr{W}_{\mathfrak{R}} = \Big(\bigoplus_{\mathfrak{C}\in\Gamma} U_{\rho^{\mathfrak{C}}\leftarrow\eta^{\mathfrak{C}}}\Big) \circ \mathscr{W}_{\tilde{\mathfrak{R}}}, \tag{4.53}$$

and $\bigoplus_{\mathfrak{C}\in\Gamma} U_{\rho^{\mathfrak{C}}\leftarrow\eta^{\mathfrak{C}}}$ is called the *change of WR.*

(iii) For each $\Psi = (\Psi^{\mathfrak{C}})_{\mathfrak{C}\in\Gamma} \in \bigotimes_{k\in\mathbb{N}} L^2(\mathbb{R}, dx)$ the map $\mathbb{P}^\Psi : \mathfrak{B}(\mathbb{R}^\infty) \to [0, +\infty]$ given by

$$\mathbb{P}^\Psi(B) := \sum_{\mathfrak{C}\in\Gamma} \int_{x\in B} |W_{[\rho^{\mathfrak{C}}]}\Psi^{\mathfrak{C}}|^2(x)\ d^\infty\mu_{\rho^{\mathfrak{C}}} \qquad \text{for } B \in \mathfrak{B}(\mathbb{R}^\infty). \tag{4.54}$$

is a Borel finite measure on $\mathbb{R}^\infty$ and it is *independent of the choice of WR basis* $\mathfrak{R}$. We denote "$\int_{x\in B} |\Psi|^2(x)d^\infty x$" $:= \mathbb{P}^\Psi(B)$ and we call $\mathbb{P}^\Psi$ *the Born rule of* Ψ. In particular, note that

$$\|\Psi\|^2_{\otimes_{k\in\mathbb{N}} L^2(\mathbb{R},dx)} = \mathbb{P}^\Psi(\mathbb{R}^\infty) = \sum_{\mathfrak{C}\in\Gamma} \left\| W_{\mathfrak{C}}\Psi^{\mathfrak{C}} \right\|^2_{L^2(\mathbb{R}^\infty, d\mu_{\mathfrak{C}})} = \|\mathscr{W}_{\mathfrak{R}}\Psi\|^2_{\oplus_{\mathfrak{C}\in\Gamma} L^2(\mathbb{R}^\infty, d\mu_{\mathfrak{C}})}. \tag{4.55}$$

Consequently, for unit Ψ, $\mathbb{P}^\Psi$ is a *probability measure.*

(iv) Given $t \in \mathbb{R} \mapsto \Psi_t \in \bigotimes_{k\in\mathbb{N}} L^2(\mathbb{R}, dx)$, and a family of homeomorphisms $Q_t : \mathbb{R}^\infty \to \mathbb{R}^\infty$, $t \in \mathbb{R}$, with $Q_0 = Id$, we say that Q_t *is equivariant with* Ψ_t when $\mathbb{P}^{\Psi_0} \circ Q_0 = \mathbb{P}^{\Psi_t} \circ Q_t$, i.e., when

$$\text{“}\int_{x\in B} |\Psi_0|^2(x) d^\infty x\text{ ”} = \text{“}\int_{x\in Q_t(B)} |\Psi_t|^2(x) d^\infty x\text{ ”} \qquad \forall B \in \mathfrak{B}(\mathbb{R}^\infty),\ \forall t \in \mathbb{R} \tag{4.56}$$

i.e., when *for one (and hence any) WR basis* $\mathfrak{R}$,

$$\sum_{\mathfrak{C}\in\Gamma}\int_{x\in B}|W_{\mathfrak{C}}\Psi_0^{\mathfrak{C}}|^2(x)\,d^\infty\mu_{\rho^{\mathfrak{C}}}=\sum_{\mathfrak{C}\in\Gamma}\int_{x\in Q_t(B)}|W_{\mathfrak{C}}\Psi_t^{\mathfrak{C}}|^2(x)\,d^\infty\mu_{\rho^{\mathfrak{C}}}\qquad\forall B\in\mathfrak{B}(\mathbb{R}^\infty),\ \forall t\in\mathbb{R}.$$

◆

Proof: • First of all, a choice of generator as the one required for each class $\mathfrak{C}\in\Gamma$ is possible by Prop. 22, so there exists at least one WR basis $\mathfrak{R}$.

Item (i): By Theorem 10, there exists a unitary $W_{[\rho^{\mathfrak{C}}]}$ with the required structure for all $\bigotimes_{k\in\mathbb{N}}\rho_k^{\mathfrak{C}}\in\mathfrak{R}$, such that one can define $\mathscr{W}_{\mathfrak{R}}:=\oplus_{\mathfrak{C}\in\Gamma}W_{[\rho^{\mathfrak{C}}]}$ satisfying the required structure. Such a direct sum of unitaries is an isometry since,

$$\|\mathscr{W}_{\mathfrak{R}}\Psi\|^2=\left\|(\oplus_{\mathfrak{C}\in\Gamma}W_{\mathfrak{C}})(\Psi^{\mathfrak{C}})_{\mathfrak{C}\in\Gamma}\right\|^2=\sum_{\mathfrak{C}\in\Gamma}\left\|W_{\mathfrak{C}}\Psi^{\mathfrak{C}}\right\|^2=\sum_{\mathfrak{C}\in\Gamma}\left\|\Psi^{\mathfrak{C}}\right\|^2=\left\|(\Psi^{\mathfrak{C}})_{\mathfrak{C}\in\Gamma}\right\|^2=\|\Psi\|^2.\tag{4.57}$$

It is trivially still linear and bijective. Hence, $\mathscr{W}_{\mathfrak{R}}$ is a unitary isomorphism.

Item (ii): By Theorem 11, given the two WR bases $\mathfrak{R},\tilde{\mathfrak{R}}$, for each $\mathfrak{C}\in\Gamma$ it holds that

$$d^\infty\mu_{\rho^{\mathfrak{C}}}\sim d^\infty\mu_{\eta^{\mathfrak{C}}}\quad\text{with}\quad d^\infty\mu_{\eta^{\mathfrak{C}}}=|W_{[\rho^{\mathfrak{C}}]}(\otimes_{k\in\mathbb{N}}\eta_k^{\mathfrak{C}})|^2\,d^\infty\mu_{\rho^{\mathfrak{C}}},\quad\text{i.e.,}\quad\frac{d\mu_{\eta^{\mathfrak{C}}}}{d\mu_{\rho^{\mathfrak{C}}}}=|W_{[\rho^{\mathfrak{C}}]}(\otimes_{k\in\mathbb{N}}\eta_k^{\mathfrak{C}})|^2.\tag{4.58}$$

- But then, Theorem 5 (taking $d\nu$ to be $d\mu_{\eta^{\mathfrak{C}}}$ and $d\mu$ to be $d\mu_{\rho^{\mathfrak{C}}}$, such that $\frac{1}{\rho}$ in Thm. 5 is now $\frac{d\mu_{\eta^{\mathfrak{C}}}}{d\mu_{\rho^{\mathfrak{C}}}}$), there exists a unitary $U_{\rho\leftarrow\eta}:L^2(\mathbb{R}^\infty,d^\infty\mu_{\eta^{\mathfrak{C}}})\to L^2(\mathbb{R}^\infty,d^\infty\mu_{\rho^{\mathfrak{C}}})$ such that

$$|\phi^{\mathfrak{C}}|^2d^\infty\mu_{\eta^{\mathfrak{C}}}=|U_{\rho\leftarrow\eta}\ (\phi^{\mathfrak{C}})|^2d^\infty\mu_{\rho^{\mathfrak{C}}},\qquad\forall\phi^{\mathfrak{C}}\in L^2(\mathbb{R}^\infty,d^\infty\mu_{\eta^{\mathfrak{C}}}).\tag{4.59}$$

In particular, the theorem allows to choose $U_{\rho\leftarrow\eta}(\phi^{\mathfrak{C}})=\sqrt{\frac{d\mu_{\eta^{\mathfrak{C}}}}{d\mu_{\rho^{\mathfrak{C}}}}}\cdot e^{i\theta}\cdot\phi^{\mathfrak{C}}$ for any phase $\theta:\mathbb{R}^\infty\to[-\pi,\pi)$ (as long as it is measurable). We fix $e^{i\theta}$ to be the phase of $W_{[\rho^{\mathfrak{C}}]}(\otimes_{k\in\mathbb{N}}\eta_k^{\mathfrak{C}})$, namely

$$e^{i\theta(x)}:=\frac{W_{[\rho^{\mathfrak{C}}]}(\otimes_{k\in\mathbb{N}}\eta_k^{\mathfrak{C}})(x)}{|W_{[\rho^{\mathfrak{C}}]}(\otimes_{k\in\mathbb{N}}\eta_k^{\mathfrak{C}})|(x)}\qquad\text{for }x\in\mathbb{R}^\infty.\tag{4.60}$$

As such, $U_{\rho\leftarrow\eta}(\phi^{\mathfrak{C}})=W_{[\rho^{\mathfrak{C}}]}(\otimes_{k\in\mathbb{N}}\eta_k^{\mathfrak{C}})\cdot\ \phi^{\mathfrak{C}}$ for all $\phi^{\mathfrak{C}}\in L^2(\mathbb{R}^\infty,d^\infty\mu_{\eta^{\mathfrak{C}}})$.

- Putting everything together, we have that $U_{\rho\leftarrow\eta}\circ W_{[\eta^{\mathfrak{C}}]}:\otimes_{k\in\mathbb{N}}L^2(\mathbb{R},dx)\longrightarrow L^2(\mathbb{R}^\infty,d^\infty\mu_{\rho^{\mathfrak{C}}})$ is a unitary operator because it is the composition of two unitary operators. Now, for any $f_1,\ldots,f_n\in L^2(\mathbb{R},dx)$, define $F:=(f_1\otimes\cdots\otimes f_n\otimes\rho_{n+1}^{\mathfrak{C}}\otimes\cdots)$. By Prop. 24,

$$F_N:=(f_1\otimes\cdots\otimes f_n\otimes\rho_{n+1}^{\mathfrak{C}}\otimes\cdots\otimes\rho_N^{\mathfrak{C}}\otimes\eta_{N+1}^{\mathfrak{C}}\otimes\cdots)\xrightarrow[N\to\infty]{\|\cdot\|_{\otimes_kL^2(\mathbb{R},dx)}}F.$$

But then, as the culmination of the technical Lemma 21,

$$(U_{\rho\leftarrow\eta}\circ W_{[\eta^{\mathfrak{C}}]})(F)\overset{(U_{\rho\leftarrow\eta}\circ W_{[\eta^{\mathfrak{C}}]}\text{ conts})}{=}\lim_{N\to\infty}\left[(U_{\rho\leftarrow\eta}\circ W_{[\eta^{\mathfrak{C}}]})(F_N)\right]\overset{(\text{Lem. }21)}{=}$$

$$=\lim_{N\to\infty}\left[W_{[\rho^{\mathfrak{C}}]}(F_N)\right]\overset{(W_{[\rho]}\text{ conts})}{=}W_{[\rho^{\mathfrak{C}}]}(F)$$

Now, we proved in Theorem 10 that any unitary coinciding with $W_{[\rho^{\mathfrak{C}}]}$ on vectors like F equals $W_{[\rho^{\mathfrak{C}}]}$ everywhere. Therefore:

$$(U_{\rho\leftarrow\eta}\circ W_{[\eta^{\mathfrak{C}}]})=W_{[\rho^{\mathfrak{C}}]}.\tag{4.61}$$

Item (iii): Let us first prove that $\mathbb{P}^\Psi$ is a measure for each $\Psi \in \bigotimes_{k\in\mathbb{N}} L^2(\mathbb{R}, dx)$. It is well-defined because for each $B \in \mathfrak{B}(\mathbb{R}^\infty)$

$$\begin{aligned}\mathbb{P}^\Psi(B) = \sum_{\mathfrak{C}\in\Gamma} \int_{x\in B} |W_\mathfrak{C}\Psi^\mathfrak{C}|^2(x) d^\infty\mu \leq \sum_{\mathfrak{C}\in\Gamma} \int_{x\in\mathbb{R}^\infty} |W_\mathfrak{C}\Psi|^2 d^\infty\mu = \sum_{\mathfrak{C}\in\Gamma} \left\| W_\mathfrak{C}\Psi^\mathfrak{C} \right\|^2_{L^2(\mathbb{R}^\infty, d^\infty\mu_\rho)} \overset{\text{(by def)}}{=} \\ = \|\mathscr{W}_\mathfrak{R}\Psi\|^2_{\oplus_\mathfrak{C} L^2(\mathbb{R}^\infty, d^\infty\mu^\mathfrak{C}_\rho)} \overset{(\mathscr{W}_\mathfrak{R}\text{ is unitary})}{=} \|\Psi\|^2 < +\infty.\end{aligned} \tag{4.62}$$

The first inequality is trivially an equality if $B = \mathbb{R}^\infty$, so this also proves (4.55).

- Now, $\mathbb{P}^\Psi(\emptyset) = 0$ trivially because for all $\mathfrak{C} \in \Gamma$, $d^\infty\mu_{\rho^\mathfrak{C}}(\emptyset) = 0$. Next, let $(B_n)_{n\in\mathbb{N}}$ be a sequence of disjoint sets in $\mathfrak{B}(\mathbb{R}^\infty)$. Then,

$$\mathbb{P}^\Psi\Big(\bigsqcup_{n\in\mathbb{N}} B_n\Big) = \sum_{\mathfrak{C}\in\Gamma} \int_{x\in\bigsqcup_{n\in\mathbb{N}} B_n} |W_\mathfrak{C}\Psi^\mathfrak{C}|^2(x)\ d^\infty\mu_{\rho^\mathfrak{C}} \overset{(\star)}{=} \sum_{\mathfrak{C}\in\Gamma} \Big(\lim_{N\to\infty} \underbrace{\Big[\sum_{n=1}^N \int_{x\in B_n} |W_\mathfrak{C}\Psi^\mathfrak{C}|^2(x)\ d^\infty\mu_{\rho^\mathfrak{C}} \Big]}_{=:\varphi_N(\mathfrak{C})} \Big) \overset{(\star\star)}{=}$$

$$= \lim_{N\to\infty} \Big(\sum_{\mathfrak{C}\in\Gamma} \underbrace{\Big[\sum_{n=1}^N \int_{x\in B_n} |W_\mathfrak{C}\Psi^\mathfrak{C}|^2(x)\ d^\infty\mu_{\rho^\mathfrak{C}} \Big]}_{\varphi_N(\mathfrak{C})} \Big) \overset{\text{(finite sum)}}{=} \lim_{N\to\infty} \sum_{n=1}^N \Big(\sum_{\mathfrak{C}\in\Gamma} \Big[\int_{x\in B_n} |W_\mathfrak{C}\Psi^\mathfrak{C}|^2(x)\ d^\infty\mu_{\rho^\mathfrak{C}} \Big] \Big),$$

which by definition, equals $\lim_{N\to\infty} \sum_{n=1}^N \mathbb{P}^\Psi(B_n)$. Hence, if $(\star), (\star\star)$ hold, this proves that $\mathbb{P}^\Psi$ is σ-additive, which finishes the proof that it is a measure on $\mathfrak{B}(\mathbb{R}^\infty)$.

Claim $(\star)$ holds because by Lemma 6, $|W_\mathfrak{C}\Psi^\mathfrak{C}|^2\ d^\infty\mu_{\rho^\mathfrak{C}}$ is a measure, so in particular it is σ-additive. About claim $(\star\star)$: by Prop. 16.(iii), at most countably many $\mathfrak{C}$ summands are non-zero, say $J := (\mathfrak{C}_k)_{k\in\mathbb{N}} \subseteq \Gamma$. As such,[a] we can substitute $\sum_{\mathfrak{C}\in\Gamma} \mapsto \sum_{k=1}^\infty$ if we switch $\mathfrak{C} \mapsto \mathfrak{C}_k$. But then, writing the series as an integral in the counting measure $d\nu$, the r.h.s of $(\star\star)$ is

$$\lim_{N\to\infty} \int_{k\in\mathbb{N}} \varphi_N(\mathfrak{C}_k)\ d\nu. \tag{4.63}$$

In particular, $\varphi_N(\mathfrak{C}) \leq \varphi_{N+1}(\mathfrak{C})$ for all $N, \mathfrak{C}$ (it is an increasing sequence of functions). Moreover, the point-wise limit $\lim_{N\to\infty} \varphi_N(\mathfrak{C})$ exists by $(\star)$. Hence, by the Monotone Convergence Theorem, (4.63) equals $\int_{k\in N} (\lim_{N\to\infty}\ \varphi_N(\mathfrak{C}_k)\ ) d\nu$, which is exactly the l.h.s of $(\star\star)$.

- Finally, we prove that $\mathbb{P}^\Psi$ is independent of the choice of WR basis $\mathfrak{R}, \tilde{\mathfrak{R}}$ employed to compute it. Given $\Psi = (\Psi^\mathfrak{C})_{\mathfrak{C}\in\Gamma}$,

$$\left| W_{[\eta^\mathfrak{C}]}\Psi^\mathfrak{C} \right|^2 d^\infty\mu_{\eta^\mathfrak{C}} \overset{(4.59)}{=} \left| U_{\rho^\mathfrak{C}\leftarrow\eta^\mathfrak{C}} W_{[\eta^\mathfrak{C}]}\Psi^\mathfrak{C} \right|^2 d^\infty\mu_{\rho^\mathfrak{C}} \overset{(4.61)}{=} \left| W_{[\rho^\mathfrak{C}]}\Psi^\mathfrak{C} \right|^2 d^\infty\mu_{\rho^\mathfrak{C}}.$$

Hence,

$$\sum_{\mathfrak{C}\in\Gamma} \int_{x\in B} \left| W_{[\eta^\mathfrak{C}]}\Psi^\mathfrak{C} \right|^2 d^\infty\mu_{\eta^\mathfrak{C}} = \sum_{\mathfrak{C}\in\Gamma} \int_{x\in B} \left| W_{[\rho^\mathfrak{C}]}\Psi^\mathfrak{C} \right|^2 d^\infty\mu_{\rho^\mathfrak{C}} \qquad \forall B \in \mathfrak{B}(\mathbb{R}^\infty), \tag{4.64}$$

and no matter if we use $\mathfrak{R}$ or $\tilde{\mathfrak{R}}$ to compute $\mathbb{P}^\Psi$, the results agree.

(Note that we will find an alternative way to prove WR-independence of $\mathbb{P}^\Psi$ as a byproduct of next chapter, with no need to use the change of WR explicitly as in the present proof.)

Item (iv): Follows trivially from (4.64). ***o.ε.δ.***

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[a] Use that by Proposition 43, von Neumann's notion of arbitrary sum matches the usual notion of countable sum —in the absolute convergence sense.

5

“$L^2(\mathbb{R}^\infty,\ d^\infty x)$” IS THE “POSITION REPRESENTATION” OF $L^2(\mathbb{R}, dx) \otimes L^2(\mathbb{R}, dx) \otimes \cdots$

Although the decomposition we found in Theorem 13 smoothly matched the narrative and indeed provided an invariant arena for pilot-wave theories over $\mathbb{R}^\infty$, the reader might feel that it has a certain *ad hoc* quality. That is, the reader might be unconvinced by the heuristic arguments that lead us to Theorem 13 and (very fairly) ask the author: “why this type of decomposition, and Born rule and not any other?” In this chapter we provide a third and most elaborate attempt to make sense of “$L^2(\mathbb{R}^\infty, d^\infty x)$” in the context of QM, and it will turn out to coincide with the decomposition of Theorem 13. This will vindicate our structure as *the* arena for Schrödinger picture configuration-space QM over $\mathbb{R}^\infty$.

5.1 The Diagonal Representation Space of an “Observable”

With the advent of abstract Hilbert spaces in the age of von Neumann’s axiomatization of QM during the 1930’s, many physicists might have (fairly) needed to understand where in those abstract Hilbert spaces and operator algebras was there the position configuration-space. Namely, where “the functions with $(x_1, ..., x_n)$ arguments” and their associated $|\psi|^2(x_1, ..., x_n)$-Born rule went. The reconciliation, as it is still explained in physics today, comes from the spectral theorem (its “diagonalization” or “multiplication operator” version). The game is the following one. Given an abstract Hilbert space $\mathcal{H}$, one can take a self-adjoint operator $(\hat{\mathcal{A}}, D(\mathcal{A}))$ of spectrum $\sigma(\hat{\mathcal{A}}) \subseteq \mathbb{R}$, said to model an “observable quantity” A with the possible “recordings of its quantum measurements” being in $\sigma(\hat{\mathcal{A}})$. Then, the spectral theorem provides a unitary operator $U_{\hat{\mathcal{A}}} : \mathcal{H} \longrightarrow L^2(\mathbb{R}, d\mu)$ such that each abstract $\Psi \in \mathcal{H}$ is mapped to a “function” $U_{\hat{\mathcal{A}}}\Psi =: \psi$ with $a \in \sigma(A) \longmapsto \psi(a)$ —often denoted by physicists as $\Psi =$ “ $\int_{\sigma(\hat{\mathcal{A}})} \psi(a)\, |a\rangle\, da$”. For example, if $\hat{\mathcal{A}}$ is the “Hamiltonian operator” $\hat{H}$, modeling the “energy observable”, then $\psi(E) = (U_{\hat{H}}\Psi)(E)$ with $E \in \sigma(H)$, is said to be Ψ but represented in the “energy basis”, or in *energy representation* —in symbols, $\Psi =$ “ $\int_{\sigma(\hat{H})} \psi(E)\, |E\rangle\, dE$”. If it is the “momentum operator” $\hat{p}$, modeling the “momentum observable”, then this leads to $p \mapsto \psi(p)$: the quantum state in *momentum representation* —in symbols, $\Psi =$ “ $\int_{\sigma(\hat{p})} \psi(p)\, |p\rangle\, dp$”. And if it is the “position operator” $\hat{x}$, then $\psi(x)$ is called the *position representation* of the quantum state-vector Ψ —symbolically, $\Psi =$ “ $\int_{\sigma(\hat{x})} \psi(x)\, |x\rangle\, dx$”.

The feature defining the different $U_{\hat{\mathcal{A}}}$ and the reason to call the arrival space "$\mathcal{A}$-representation" or "diagonalization space for $\hat{\mathcal{A}}$" is that $\hat{\mathcal{A}}\Psi$ is mapped to the function ψ multiplied by its argument, $a \mapsto a\,\psi(a)$, which physicists represent in symbols as $\hat{\mathcal{A}} =$ "$\int_{\sigma(\hat{\mathcal{A}})} a\,|a\rangle\langle a|\,da$" $\implies$ $\hat{\mathcal{A}}\Psi =$ "$\int_{\sigma(\hat{\mathcal{A}})} a\,\psi(a)\;|a\rangle da$".

Now, in order to fully recover usual Schrödinger QM over $\mathbb{R}^n$, one requires an additional rule in the game: the so-called "joint diagonalization". Given a family of "commuting" self-adjoint operators $\mathcal{A}_1, ..., \mathcal{A}_n$, they can be "simultaneously diagonalized". That is, the joint spectral theorem gives a unitary $U : \mathcal{H} \longrightarrow L^2(\mathbb{R}^n, d^n\mu)$, identifying each $\Psi \in \mathcal{H}$ with a function $\psi(a_1, ..., a_n)$ such that the action of $\mathcal{A}_j$ corresponds to multiplication by the j-th variable: $a_j\,\psi(a_1, ..., a_n)$. In the symbols employed by physicists, $\Psi =$ "$\int \psi(a_1, ..., a_n)|a_1, ..., a_n\rangle d^n a$" and $\mathcal{A}_j =$ "$\int a_j\;|a_1, ..., a_n\rangle\langle a_1, ..., a_n|\,d^n a$". For instance, given the n operators $\hat{q}_1, ..., \hat{q}_n$ that represent the n degrees of freedom of a quantum system (say, the $3N$ "position observables" of N quantum particles), the abstract vector $\Psi \in \mathcal{H}$ can be represented by a "function" $(q_1, ..., q_n) \mapsto \psi(q_1, ..., q_n)$ such that $\hat{q}_k\Psi$ corresponds to multiplication by the argument: $q_k\psi(q_1, ..., q_n)$. Lastly, physicists explain that this function is what one used to employ in the original Schrödinger theory, namely, that $\psi(x_1, ..., x_n)$ —the *configuration or joint position representation* of the quantum state $\Psi \in \mathcal{H}$— gives back the original Born rule of configuration space as $|\psi|^2 d^n x$.

On another note, in von Neumann's axiomatization, the state space for the composite of two quantum systems is the tensor product of their respective Hilbert spaces, say, $\mathcal{H}_1 \otimes \mathcal{H}_2$. As such, one can take a self-adjoint operator from $\mathcal{H}_1$, say A_1, "lift it" to the composite with an identity $A_1 \otimes Id$, and do the same for A_2 from $\mathcal{H}_2$: $Id \otimes A_2$. Since they commute trivially, heuristically, one can jointly "diagonalize them" by the above, providing A_1-representation to system 1 and A_2-representation to system 2, namely, making abstract vectors $\Psi \in \mathcal{H}_1 \otimes \mathcal{H}_2$ be represented by functions $(a_1, a_2) \in \sigma(A_1) \times \sigma(A_2) \to \psi(a_1, a_2)$.

Now, if $\otimes_{k\in\mathbb{N}} L^2(\mathbb{R}, dx)$ is really an abstract Hilbert space, why not do the same? One could take the position operator of $L^2(\mathbb{R}, dx)$, call it $\hat{q}$ and for each $k \in \mathbb{N}$, lift it to the infinite tensor product by putting identities in the other factors as: $\widehat{q_k} = Id \otimes \cdots \otimes \hat{q} \otimes Id \otimes \cdots$. Since they all commute trivially, one could give them a joint diagonalization map $U : \bigotimes_{k\in\mathbb{N}} L^2(\mathbb{R}, dx) \to L^2(\mathbb{R}^\infty, d\mu)$. If this were possible, each abstract $\Psi \in \otimes_{k\in\mathbb{N}} L^2(\mathbb{R}, dx)$ would be represented by a function $(q_1, q_2, ...) \mapsto \psi(q_1, q_2, ...)$ for which $\hat{q}_k$ acts as multiplication by the argument q_k —in physicist notation $\Psi =$ "$\int_{\mathbb{R}^\infty} \psi(q_1, q_2, ...)\,|q_1, q_2, ...\rangle d^\infty q$" and $\widehat{q_k} =$ "$\int_{\mathbb{R}^\infty} q_k\,|q_1, q_2, ...\rangle\langle q_1, q_2, ...|d^\infty q$". What we prove in this chapter is that if one follows this idea with mathematical precision, it yields *exactly* the decomposition of Theorem 13!

There is a very important detail that we omitted above to avoid the confusion of the physicist readers. The reason why it could confuse a physicist in particular is that (to the author's knowledge) this detail is hardly ever mentioned in a physics lecture or textbook on QM.[1] The omitted detail is that what the diagonalization theorem truly says (even back in von Neumann's days) is: "...there is a unitary $U : \mathcal{H} \longrightarrow \oplus_{\mathfrak{N}\in\mathfrak{X}} L^2(\mathbb{R}, d\mu_{\mathfrak{N}})$...", so not only one arrival $L^2(\mathbb{R}, d\mu)$ space but a direct sum of $L^2(\mathbb{R}, d\mu_{\mathfrak{N}})$ indexed by some $\mathfrak{N} \in \mathfrak{X}$. Moreover, it says that each of those L^2-spaces is "generated" by some vector $\Psi^{\mathfrak{N}} \in \mathcal{H}$ —via what is called the *functional calculus*. A family $(\Psi^{\mathfrak{N}})_{\mathfrak{N}\in\mathfrak{X}}$ of such generators is called a *spectral basis* or *diagonal representation*

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[1] And this perhaps explains why none of the (unfortunately few) physicists that research the Schrödinger picture of QFT (e.g. Jackiw (1995)) ever talk about "tuples of infinitely many wavefunctionals" to describe scalar quantum fields —as we will.

basis. Likewise, in the case of joint diagonalization for n operators, the theorem says "...there is a unitary $U : \mathcal{H} \longrightarrow \oplus_{\mathfrak{N}\in\mathfrak{X}} L^2(\mathbb{R}^n, d^n\mu_{\mathfrak{N}})$...". In any of the cases, only if the spectrum of the self-adjoint operator(s) is, so-called, *simple* is there a chance to have an isomorphism to a *single* $L^2(\mathbb{R}, d\mu)$ (or $L^2(R^n, d^n\mu)$) space. In the case of the position operators in $\mathbb{R}^n$ the spectrum is simple for instance. But this is an exceptional case. And thus, the emergence of all the $L^2(\mathbb{R}^\infty, d^\infty\mu_{\mathfrak{C}})$ layers from Theorem 13 would not be that mysterious from a mathematically honest point of view.

With all, perhaps the most important moral of this thesis is that sometimes mathematical rigor does pay off. Without ever going to the rigorous detail, meaning, staying in mere heuristic, symbolic and formal manipulations, a mathematical technicality like the one being suggested in this document could hardly ever be discovered.[2] What is more, even backed by formal manipulations suggesting it, who would claim in their five senses, that the wavefunctional of a scalar field is *a "spinor" of uncountably many entries?* Only the apodictic force of a theorem could render such a wild claim worthy of debate.

5.2 A Joint Diagonalization Theorem for Arbitrarily Many Strongly Commuting Self-Adjoint Operators

In order to make the "joint position representation" strategy projected above rigorous, we first need a joint diagonalization spectral theorem *for arbitrarily many self-adjoint operators in non-separable Hilbert spaces.* Although possibly known, we could not find any reference explicitly proving such a theorem, so we will build it step by step in this section. It will be considerably technical, so the reader may skip this section on a first read.

5.2.1 A PVM and its Functional Calculus on an Arbitrary Measurable space

In order to deal with PVMs over $\mathbb{R}^\infty$ we require a slight generalization of the usual notion of PVM —defined over $\mathbb{R}$ or $\mathbb{R}^n$. Fortunately, Schmüdgen (2012) defined them with sufficient generality to spare us several lengthy proofs.

Definition 17. Let Σ be a σ-algebra (resp., a Boolean algebra) on a set X and let $\mathcal{H}$ be an arbitrary Hilbert space. A *projector valued measure* or PVM (resp., a projector valued *pre*-measure or PVpreM) is a mapping $P : \Sigma \to \mathcal{L}(\mathcal{H})$ satisfying all the following:

- **(i)** $P(B)$ is an *orthogonal projector* for all $B \in \Sigma$, i.e., it is an *idempotent* ($P^2(B) = P(B)$) and *self-adjoint bounded* operator.
- **(ii)** $P(X) = Id$.
- **(iii)** If $(B_j)_{j\in\mathbb{N}} \subseteq \Sigma$ is a disjoint sequence such that $\sqcup_{j\in\mathbb{N}} B_j \in \Sigma$, then

$$\lim_{N\to\infty} \sum_{j=1}^{N} P(B_j)\psi = P\Big(\bigcup_{j\in\mathbb{N}} B_j\Big)\psi \qquad \forall\psi \in \mathcal{H}. \tag{5.1}$$

When we wish to specify all the dependencies of a PVM/PVpreM, we denote it by $(\mathcal{H}, (X, \Sigma), P)$.
♦

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[2] Although, as explained in the chapter on historical remarks, one could have guessed something similar from "Haag's theorem".

Lemma 22. Given conditions (i) and (ii) in Definition 17 are satisfied for some $(\mathcal{H},(X,\Sigma),P)$,

$$\Big(\text{(iii) holds}\Big) \iff \left(\text{the mapping} \quad \begin{array}{rccc} d\mu_\psi: & \Sigma & \longrightarrow & \mathbb{R} \\ & B & \longmapsto & \langle\psi,\ P(B)\psi\rangle \end{array} \quad \text{is } \sigma\text{-additive } \forall\psi\in\mathcal{H}.\right)$$

If so, for a PVpreM $d\mu_\psi$ is a *pre-measure* on the Boolean algebra Σ, while for a PVM, $d\mu_\psi$ is a *measure* on the σ-algebra Σ. ♦

Proof: See Lemma 4.4 in Schmüdgen (2012). For the last claim: note that taking $B_j=\emptyset$ in (iii) for all $j\in\mathbb{N}$, one gets that necessarily $P(\emptyset)=0$. As such, $d\mu_\psi(\emptyset)=0$, which was the missing check. □

Corollary 16. Given a PVM $(\mathcal{H},(X,\Sigma),P)$, for each $\psi\in\mathcal{H}$ the map

$$d\mu_\psi(B) := \langle\psi,\ P(B)\psi\rangle = \|P(B)\psi\|^2 \tag{5.2}$$

with $B\in\Sigma$ defines a finite measure on (X,Σ) such that $d\mu_\psi(X)=\|\psi\|^2$. We call it *the (scalar) spectral measure of ψ.*

For each pair $\psi,\phi\in\mathcal{H}$, $d\mu_{\phi,\psi}(B):=\langle\phi,P(B)\psi\rangle$ defines a *complex measure* on (X,Σ) because it is the following linear combination of measures:

$$d\mu_{\phi,\psi}(B) = \frac{1}{4}\Big(d\mu_{\phi+\psi}(B) - d\mu_{\phi-\psi}(B) + i\,d\mu_{\phi-i\psi}(B) - i\,d\mu_{\phi+i\psi}(B)\Big). \tag{5.3}$$

♦

Proof: Let $B\in\Sigma$. Then:

$$d\mu_\psi(B) = \langle\psi,\ P(B)\psi\rangle \overset{\substack{P(B)^2=P(B)\\ P(B)=P(B)^*}}{=} \|P(B)\psi\|^2 \quad\Longrightarrow\quad d\mu_\psi(X) = \|P(X)\psi\|^2 = \|\psi\|^2 < +\infty.$$

As such, by Lemma 22, $d\mu_\psi$ is a finite measure. On the other hand, expanding $\langle\phi,P(B)\psi\rangle$ with the polarization identity yields exactly 5.3. ***o.ε.δ.***

Definition 18. Given a measurable space (X,Σ), denote by $\mathcal{M}_b(X,\Sigma)$ the vector space

$$\mathcal{M}_b(X,\Sigma) := \left\{ f: X\to\mathbb{C} \;\middle|\; f \text{ is measurable and bounded: } \|f\|_\infty := \sup_{x\in X}|f(x)| < +\infty \right\}.$$

We will assume it is equipped with the $\|\cdot\|_\infty$ norm —with which, it is a Banach space.[a] ♦

Proposition 28. Given a PVM $(\mathcal{H},(X,\Sigma),P)$, for each $f\in\mathcal{M}_b(X,\Sigma)$ there *exists a unique bounded operator* $\Phi(f)\in\mathcal{L}(\mathcal{H})$ such that

$$\langle\phi,\ \Phi(f)\ \psi\rangle = \int_{x\in X} f(x)\ d\mu_{\phi,\psi} \quad \forall\phi,\psi\in\mathcal{H}. \tag{5.4}$$

The mapping $\Phi:(\mathcal{M}_b,\|\cdot\|_\infty)\to(\mathcal{L}(\mathcal{H}),\|\cdot\|_{op})$ is a bounded linear operator and we call it the *bounded functional calculus of the PVM P.* One also denotes $\int_{x\in X} f(x)\ dP := \Phi(f)$. ♦

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[a] 2.1.12.A of (Abraham et al., 2001) proves that $\mathcal{M}_b(X,\Sigma)$ would be a Banach space if we removed the measurability condition. Now, linear combinations of measurable functions are measurable and by Prop. 2.7 in (Folland, 1999), point-wise limits —and hence in particular $\|\cdot\|_\infty$ norm limits— of measurable functions are measurable. Thus, $\mathcal{M}_b(X,\Sigma)$ is a closed subspace of a Banach space and hence, it is itself Banach.

Proof: See Theorem 37.1 in (Halmos, 1957). A sketch: one checks that because f is bounded, the integral $\varphi(\phi,\psi) := \int_{x\in X} f(x)\, d\mu_{\phi,\psi}$ for $\phi,\psi\in\mathcal{H}$ is well-defined and a sesquilinear form. But then, $|\varphi(\phi,\psi)| \leq \|f\|_\infty \|\psi\|\|\phi\|$. Hence, φ is a bounded sesquilinear form and by Theorem 22.1 in (Halmos, 1957), there exists a unique bounded operator $\Phi(f)\in\mathcal{H}$ such that $\varphi(\phi,\psi) = \langle\phi,\Phi(f)\psi\rangle$. Moreover, it has the same bound as the form, so $\|\Phi(f)\|_{op} \leq \|f\|_\infty$. ***o.ε.δ.***

Note that any simple function $\sum_{j=1}^N c_j \mathbb{1}_{E_j}$ (with $N\in\mathbb{N},\ E_j\in\Sigma$ and $c_j\in\mathbb{C}$), is trivially a member of $\mathcal{M}_b(X)$. In particular,

$$\Phi\left(\sum_{j=1}^N c_j \mathbb{1}_{E_j}\right) \overset{\text{(linear)}}{=} \sum_{j=1}^N c_j \Phi(\mathbb{1}_{E_j}) \overset{\{\ \langle\psi,\Phi(\mathbb{1}_{E_j})\varphi\rangle=\langle\psi,P(E_j)\varphi\rangle\ \forall\psi,\varphi\in\mathcal{H}\ \}}{=} \sum_{j=1}^N c_j P(E_j). \tag{5.5}$$

With that, now for an arbitrary $f\in\mathcal{M}_b(X,\Sigma)$, we can get a slightly more explicit shape for $\Phi(f)$. By Theorem 2.10 in (Folland, 1999), for an arbitrary $f\in\mathcal{M}_b(X,\Sigma)$, there exists a sequence of simple functions (s_n) uniformly approximating f, i.e.: $s_n \xrightarrow{\|\cdot\|_\infty} f$. As such, using that Φ is continuous, one gets that $\Phi(f) = \lim_{n\to\infty}^{\|\cdot\|_{op}} \Phi(s_n)$, where each $\Phi(s_n)$ is a finite linear combination of projection operators as in (5.5). This is why, one can consider that a functional calculus is a "device" to approximate operators by sums of scaled projection operators.[a]

One can extend Φ to allow the input of arbitrary measurable functions, but only at the cost of having to deal with *unbounded* operators.

Proposition 29. Given a PVM $(\mathcal{H},(X,\Sigma),P)$ and an arbitrary measurable $f: X\to\mathbb{C}$, the set

$$D_f := \left\{\psi\in\mathcal{H} \;\middle|\; \int_{x\in X} |f(x)|^2\, d\mu_\psi < +\infty\right\} = \left\{\psi\in\mathcal{H} \;\middle|\; f\in L^2(X,d\mu_\psi)\right\} \tag{5.6}$$

is a dense vector subspace of $\mathcal{H}$. Define the "cut-offs" $f_n := f\cdot\mathbb{1}_{\Omega_n}$ with $\Omega_n := |f|^{-1}([0,n))$ for $n\in\mathbb{N}$. Since $f_n\in\mathcal{M}_b$, one can formally define for each $\psi\in D_f$

$$\Phi(f)\psi := \lim_{n\to\infty} \Phi(f_n)\psi. \tag{5.7}$$

It turns out that all such limits exist and $(\Phi(f),\ D_f)$ defines a densely defined linear operator. In particular, the action of $\Phi(f)$ is independent of the chosen sequence $(f_n)_{n\in\mathbb{N}}$ in the following sense. Given any other sequence $(g_k)_{k\in\mathbb{N}}\subseteq\mathcal{M}_b(X)$ such that (i) $g_n(x)\xrightarrow{n\to\infty} f(x)$ for all $x\in X$ and (ii) $\sup_{n\in\mathbb{N}}\|g_n\|_{L^2(X,d\mu_\psi)} < +\infty$, then, $\lim_{n\to\infty}\Phi(g_n)\psi = \Phi(f)\psi$.

We call the resulting map Φ *the functional calculus* of the PVM P and we still denote $\int_{x\in X} f(x)dP := \Phi(f)$. Note that if $f\in\mathcal{M}_b(X)$ then $D_f = \mathcal{H}$ and $\Phi(f)$ is the same as the one defined in Prop. 28. ♦

Proof: See Theorem 4.13 in (Schmüdgen, 2012). □

Proposition 30. Let $(\mathcal{H},P,(X,\Sigma))$ be a PVM and let $f,g: X\to\mathbb{C}$ be arbitrary measurable functions. Then,

- **(i)** $\|\Phi(f)\psi\|^2 = \int_{x\in X}|f(x)|^2 d\mu_\psi = \|f\|^2_{L^2(X,d\mu_\psi)}$ for all $\psi\in D_f$.
- **(ii)** $\langle\Phi(g)\psi,\Phi(f)\phi\rangle = \int_{x\in X}\overline{g(x)}f(x)d\mu_{\psi,\phi}$ for all $\psi,\phi\in D_f$.

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[a] This also shows that we are talking about a generalized notion of Lebesgue integration where the measure is operator valued, thereby explaining the notation $\int_{x\in X} f(x)dP := \Phi(f)$ —the integrals over that measure yield operators instead of numbers.

(iii) $\Phi(f)^* = \Phi(\overline{f}\,)$ and $D(\Phi(f)^*) = D_f$. Hence, $(\Phi(f), D_f)$ is always a closed *normal operator.*[a] In particular, it is a *self-adjoint* operator whenever f is real-valued.

(iv) $\alpha\Phi(f) + \beta\Phi(g) \ \subseteq \Phi(\alpha f + \beta g)$ and $D(\alpha\Phi(f) + \beta\Phi(g)) = D_{|f|+|g|} = D_f \cap D_g$.

(v) $\Phi(f) \circ \Phi(g) \subseteq \Phi(f \cdot g)$ with $D(\Phi(f) \circ \Phi(g)) = D_f \cap D_{f \cdot g}$. ♦

Proof: In (Schmüdgen, 2012) see Prop. 4.15 for items (i) and (ii) and Thm. 4.16 for the rest. ☐

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[a] A *normal operator* is an unbounded operator $(A, D(A))$ such that $D(A) = D(A^*)$ and $\|A\psi\| = \|A^*\psi\|$ for any $\psi \in D(A)$.

5.2.2 The Arbitrary Product of Commuting PVMs

Theorem 14. Let $\mathcal{H}$ be an arbitrary Hilbert space and let $(X_j)_{j=1}^n$ be $n \in \mathbb{N}$ locally compact, second countable and Hausdorff topological spaces. If $(\mathcal{H},\ (X_j, \mathfrak{B}(X_j)),\ P_j)_{j=1}^n$ is a *commuting family of PVMs* on $\mathcal{H}$, i.e., if they satisfy

$$P_j(B_j) \circ P_k(B_k) = P_k(B_k) \circ P_j(B_j) \qquad \forall k, j \in \{1, ..., n\},\ \forall B_j \in \mathfrak{B}(X_j),\ B_k \in \mathfrak{B}(X_k), \tag{5.8}$$

then,

(i) there *exists a unique* PVM $(\mathcal{H},\ (\prod_{j=1}^n X_j,\ \mathfrak{B}(\prod_{j=1}^n X_j)),\ P)$ such that for all $B_j \in \mathfrak{B}(X_j)$

$$P\Big(B_1 \times \cdots \times B_n\Big) = P_1(B_1) \circ \cdots \circ P_n(B_n). \tag{5.9}$$

We call P the *joint-* or *product- PVM of the commuting family* $(P_j)_{j=1}^n$ and denote it by $\odot_{j=1}^n P_j \equiv P_1 \odot \cdots \odot P_n$.

- Product PVMs satisfy the *consistency property* that for any $J \subset \{1, ..., n\}$,

$$(\odot_{j=1}^n P_j)\Bigg(E_J \times \prod_{j \in \{1,...,n\} \setminus J} X_j\Bigg) = (\odot_{j \in J} P_j)(E_J), \qquad \forall E_J \in \mathfrak{B}\Big(\prod_{j \in J} X_j\Big). \tag{5.10}$$

(ii) Denote the functional calculus of P and P_j respectively, by Φ and Φ_j.[3] If $f(x_1, ..., x_n) := g_k(x_k)$ for some $g_k : X_k \to \mathbb{C}$ measurable, then $\Phi(f) = \Phi_k(g_k)$ (with equal domains). In the integral notation this reads:

$$\int_{(x_1,...,x_n) \in \prod_{j=1}^n X_j} g_k(x_k)\ d(P_1 \odot \cdots \odot P_n) = \int_{x_k \in X_k} g_k(x_k)\ dP_k. \tag{5.11}$$

♦

Proof: For everything in **item (i)** except the consistency property, see Theorem 4.10 in (Schmüdgen, 2012). To prove the consistency property, given a fixed $\{j_1, ..., j_m\} =: J \subset \{1, ..., n\}$, define $Q(E_J) := (\odot_{j=1}^n P_j)\Big(E_J \times \prod_{j \in \{1,...,n\} \setminus J} X_j\Big)$ for each $E_J \in \mathfrak{B}(\prod_{j \in J} X_j)$. Then, $(\mathcal{H},\ (\prod_{j \in J} X_j,\ \mathfrak{B}(\prod_{j \in J} X_j)),\ Q)$ is trivially a PVM and by (5.9),

$$Q(B_{j_1} \times \cdots \times B_{j_m}) = P_{j_1}(B_{j_1}) \cdots P_{j_n}(B_{j_n}) \cdot \prod_{j \in \{1,...,n\} \neq J} \cancelto{Id}{P_j(X_j)} = P_{j_1}(B_{j_1}) \cdots P_{j_n}(B_{j_n}).$$

for all $B_{j_k} \in \mathfrak{B}(X_{j_k})$. Hence, Q coincides with $\odot_{j \in J} P_j$ on product sets. But then, by uniqueness, it must be that $Q = \odot_{j \in J} P_j$ everywhere.

---

[3] Note that Φ takes measurable functions $f : \prod_{j=1}^n X_j \to \mathbb{C}$ and outputs unbounded operators in $\mathcal{H}$.

- For **item (ii),** we proceed as Schmüdgen (2012) does in his Lemma 5.22 but now for general X_j. Let $f = g_k$ as in the statement. Then, by definition of the unbounded functional calculus, for $i \in \mathbb{N}$ and $f_i = f\mathbb{1}_{\Omega_i} \in \mathcal{M}_b(\prod_{j=1}^n X_j)$ with $\Omega_i := |f|^{-1}([0,i))$, it holds that $\Phi(f) = \lim_{i\to\infty}\Phi(f_i)$ strongly on D_f. Now, because f only depends on x_k, then, $\Omega_i = \widetilde{\Omega}_i \times \prod_{\ell=1,\neq k}^n X_\ell$ for $\widetilde{\Omega}_i := |g_k|^{-1}([0,i))$ and $f_i(x_1,...,x_n) = g_k(x_k)\mathbb{1}_{\Omega_i}(x_1,...,x_n) = g_k(x_k)\mathbb{1}_{\widetilde{\Omega}_i}(x_k)$. By Theorem 2.10 in (Folland, 1999), for each $i \in \mathbb{N}$, $(g\mathbb{1}_{\widetilde{\Omega}_i}) \in \mathcal{M}_b(X_k)$ can be expanded in a sequence of simple functions $(s_i^\ell)_{\ell\in\mathbb{N}} \in \mathcal{M}_b(X_k)$ such that $s_i^\ell \xrightarrow[\ell\to\infty]{\|\cdot\|_\infty} g_k\mathbb{1}_{\widetilde{\Omega}_i}$, say $s_i^\ell(x_k) = \sum_{j=1}^{N_i^\ell} c_{j,i}^\ell \mathbb{1}_{E_{j,i}^\ell}(x_k)$ (for some $N_i^\ell \in \mathbb{N}$, $c_{j,i}^\ell \in \mathbb{C}$, $E_{j,i}^\ell \in \mathfrak{B}(X_k)$). But then, considering s_i^ℓ as a map $\mathbb{R}^\infty \to \mathbb{C}$ that does not depend on x_j for $j \neq k$, abusing notation, one trivialllly gets that $s_i^\ell \xrightarrow[\ell\to\infty]{\|\cdot\|_\infty} f_i$. Altogether, for each $\psi \in D_f$,

$$\Phi(f)\psi = \overset{\mathcal{H}}{\lim_{i\to\infty}} \Phi(f_i)\psi \overset{(\Phi(\cdot)\text{ conts.})}{=} \overset{\mathcal{H}}{\lim_{i\to\infty}}\ \overset{\mathcal{L}(\mathcal{H})}{\lim_{\ell\to\infty}} \Phi(s_i^\ell)\psi = \overset{\mathcal{H}}{\lim_{i\to\infty}}\ \overset{\mathcal{L}(\mathcal{H})}{\lim_{i\ell\to\infty}} \sum_{j=1}^{N_i^\ell} c_{j,i}^\ell (P_1 \odot \cdots \odot P_i)\Big(E_{j,i}^\ell \times \prod_{r=1,\neq k}^{n} X_r\Big) =$$

$$\overset{(5.10)}{=} \overset{\mathcal{H}}{\lim_{i\to\infty}}\ \overset{\mathcal{L}(\mathcal{H})}{\lim_{\ell\to\infty}} \sum_{j=1}^{N_i^\ell} c_{j,i}^\ell\ P_k(E_{j,i}^\ell) = \overset{\mathcal{H}}{\lim_{i\to\infty}}\ \overset{\mathcal{L}(\mathcal{H})}{\lim_{\ell\to\infty}} \Phi_k(s_i^\ell)\psi \overset{(\Phi(\cdot)\text{ conts.})}{=} \overset{\mathcal{H}}{\lim_{i\to\infty}} \Phi_k(g_k\mathbb{1}_{\widetilde{\Omega}_i})\psi = \Phi_k(g_k)\psi.$$

In particular, this also proves the equality of the domains. ***o.ε.δ.***

Proposition 31. Let $(\mathcal{H},\ (X,\Sigma_0),\ P_0)$, be a PVpreM (i.e., Σ_0 is a Boolean algebra). Then, there *exists a unique* PVM $(\mathcal{H},\ (X,\sigma(\Sigma_0)),\ P)$ extending P_0 to $\sigma(\Sigma_0)$, i.e., such that $P(E_0) = P_0(E_0)$ for all $E \in \Sigma_0$. ♦

Proof: The existence result is Lemma 4.9 in (Schmüdgen, 2012). To prove uniqueness, note that for each $\psi \in \mathcal{H}$, $d\mu_\psi^0 : \Sigma_0 \to \mathbb{R}$; $d\mu_\psi^0(E_0) := \langle\psi,\ P_0(E_0)\psi\rangle$ is a pre-measure on Σ_0 by Lemma 22. Moreover, it is σ-finite because by Cor. 16 it is finite: $d\mu_\psi^0(X) = \|\psi\|^2$. Hence, by Theorem 1, there exists a unique measure extending $d\mu_\psi^0$ to $\sigma(\Sigma_0)$. Now, suppose that $(\mathcal{H},(X,\sigma(\Sigma_0)),P)$ and $(\mathcal{H},(X,\sigma(\Sigma_0)),Q)$ are both PVMs extending P_0. Then, their associated measures $d\mu_\psi(E) := \langle\psi, P(E)\psi\rangle$ and $d\nu_\psi(E) := \langle\psi, Q(E)\psi\rangle$ for $E \in \sigma(\Sigma_0)$ are extensions of $d\mu_\psi^0$, because, $d\mu_\psi(E_0) = \langle\psi,\ P(E_0)\psi\rangle = \langle\psi,\ P_0(E_0)\psi\rangle = d\mu_\psi^0(E_0)$ if $E \in \Sigma_0$ (likewise for $d\nu_\psi$). Hence, $d\nu_\psi = d\mu_\psi\ \forall\psi \in \mathcal{H}$. With that, by Cor. 16 it must be that $d\nu_{\psi,\varphi} = d\mu_{\psi,\varphi}$ for all $\psi,\varphi \in \mathcal{H}$. But, by definition this means that $\langle\psi, Q(E)\varphi\rangle = \langle\psi, P(E)\varphi\rangle\ \forall E \in \sigma(\Sigma_0)$ and $\forall\psi,\varphi \in \mathcal{H}$. Therefore, $Q(E) \equiv P(E)$. ***o.ε.δ.***

Theorem 15. Let $\mathcal{H}$ be an arbitrary Hilbert space, I an arbitrary index set and let $(X_j)_{j\in I}$ be a family of locally compact, second countable and Hausdorff topological spaces. Denote $X^\infty := \prod_{j\in I} X_j$ and equip it with the product σ-algebra $\odot_{j\in I}\mathfrak{B}(X_j)$.

If $(\mathcal{H},\ (X_j,\mathfrak{B}(X_j)),\ P_j)_{j\in I}$ is a *commuting family of PVMs* on $\mathcal{H}$, i.e., if they satisfy

$$P_j(B_j)\circ P_k(B_k) = P_k(B_k)\circ P_j(B_j) \qquad \forall k,j \in \{1,...,n\},\ \forall B_j \in \mathfrak{B}(X_j),\ B_k \in \mathfrak{B}(X_k), \tag{5.12}$$

then,

(i) There *exists a unique* PVM $(\mathcal{H},\ (X^\infty,\ \odot_{j\in I}\mathfrak{B}(X_j)),\ P)$ such that

$$P\Big(E_J \times \prod_{j\in I\setminus J} X_j\Big) = (\odot_{j\in J}P_j)(E_J), \qquad \forall\text{ finite } J \subset I,\ \ \forall E_J \in \odot_{j\in J}\mathfrak{B}(X_j). \tag{5.13}$$

We call P the *joint-* or *product- PVM of* the commuting family $(P_j)_{j\in I}$ and denote it by $\odot_{j\in I}P_j$.

(ii) Denote the functional calculus of P and P_j respectively, by Φ and Φ_j.[4] If $f((x_j)_{j\in I}) := g_k(x_k)$ for some $g_k : X_k \to \mathbb{C}$ measurable, then $\Phi(f) = \Phi_k(g_k)$ (with equal domains). In the integral notation, this reads:

$$\int_{(x_j)_{j\in I}\in X^\infty} g_k(x_k)\ d(\odot_{j\in I}P_j) = \int_{x_k\in X_k} g_k(x_k)\ dP_k. \tag{5.14}$$

◆

Proof: **Proof of Item (i):**

(Step 1.) For all finite $J \subseteq I$, the space $\prod_{j\in J} X_j$ with product topology is still locally compact (by Theorem 18.6 in (Willard, 2012)), second countable (by Theorem 16.2 in (Willard, 2012)) and Hausdorff (by Theorem 13.8 (b)).

(Step 2.) Fix an arbitrary *unit* $\psi \in \mathcal{H}$. For each finite $J \subset I$, the map $d^J\mu_\psi : \odot_{j\in J}\mathfrak{B}(X_j) \longrightarrow [0,+\infty)$, $d^J\mu_\psi(E_J) := \Big\langle \psi,\ (\odot_{j\in J}P_j)(E_J)\psi \Big\rangle$ is a Borel *probability* measure by Cor. 16.

- By Theorem 7.8 in (Folland, 1999), any finite Borel measure on a locally compact, second countable, Hausdorff space is regular and in particular, is *inner regular*. Therefore, $d^J\mu_\psi$ is inner regular for every unit $\psi \in \mathcal{H}$ and every finite $J \subseteq I$. Moreover, they satisfy the compatibility condition that given $J_1 \subseteq J_2 \subset I$ are finite nested index sets, for each $E_{J_1} \in \odot_{j\in J_1}X_j$,

$$d^{J_1}\mu_\psi(E_{J_1}) = \Big\langle \psi,\ (\odot_{j\in J_1}P)(E_{J_1})\psi \Big\rangle \overset{(5.10)}{=} \Big\langle \psi,\ (\odot_{j\in J_2}P)\Big(E_{J_1} \times \prod_{j\in J_2\setminus J_1} X_j\Big)\psi \Big\rangle = d^{J_2}\mu_\psi\Big(E_{J_1} \times \prod_{j\in J_2\setminus J_1} X_j\Big),$$

i.e., $d^{J_1}\mu_\psi = d^{J_2}\mu_\psi \circ (\pi_{J_1\leftarrow J_2})^{-1}$. Therefore, for each unit $\psi \in \mathcal{H}$, by Theorem 6, there exists a unique probability measure $d\mu_\psi$ on $\odot_{j\in\mathbb{N}}\mathfrak{B}(X_j)$ such that $d\mu_\psi \circ (\pi_{J\leftarrow I})^{-1} = d^J\mu_\psi$.

(Step 3.) Recall that the family $\mathfrak{A}_0$ of Prop. 3 is a Boolean algebra (for the present case put $\Sigma_j = \mathfrak{B}(X_j)$). Then, define the map $P_0 : \mathfrak{A}_0 \longrightarrow \mathcal{L}(\mathcal{H})$ such that

$$P_0\Big(E_J \times \prod_{j\in I\setminus J} X_j\Big) := (\odot_{j\in J}P_j)(E_J) \quad \forall \text{ finite } J \subset I,\ E_J \in \odot_{j\in J}\mathfrak{B}(X_j). \tag{5.15}$$

This is well-defined thanks to the consistency property (5.10) of the finite product PVM.

- $P_0(E)$ is an orthogonal projector for all $E \in \mathfrak{A}_0$ because all $\odot_{j\in J}P_j$ are PVMs. In particular, $P_0(X^\infty) = Id$. Hence, P_0 satisfies points (i) and (ii) in Def. 17 for a PV*pre*M. As such, by Lemma 22, if we prove that the set function $E \in \mathfrak{A}_0 \longmapsto \langle \psi, P_0(E)\psi \rangle$ is σ-additive $\forall \psi \in \mathcal{H}$, then $(\mathcal{H}, (X^\infty, \mathfrak{A}_0), P_0)$ will be a PV*pre*-M.

(Step 4.) Fix an arbitrary $\psi \in \mathcal{H} \setminus \{\vec{0}\}$ (not necessarily of unit norm). For all $E \in \mathfrak{A}_0$, one can write $E = E_J \times \prod_{j\in I\setminus I} X_j$ for some finite $J \subset I$ and $E_J \in \odot_{j\in J}\mathfrak{B}(X_j)$. Thus,

$$\langle \psi,\ P_0(E)\psi \rangle \overset{\text{(by def.)}}{=} \langle \psi,\ (\odot_{j\in J}P_j)(E_J)\ \psi \rangle = \|\psi\|^2 \Big\langle \frac{\psi}{\|\psi\|},\ (\odot_{j\in J}P_j)(E_J)\ \frac{\psi}{\|\psi\|} \Big\rangle \overset{\left(\substack{\psi/\|\psi\| \text{ is} \\ \text{unit, so by def.}}\right)}{=}$$

$$= \|\psi\|^2 d^J\mu_{\frac{\psi}{\|\psi\|}}(E_J) \overset{\text{(by def.)}}{=} \|\psi\|^2 d\mu_{\frac{\psi}{\|\psi\|}}(E).$$

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[4] Note that Φ takes measurable functions $f : X^\infty \to \mathbb{C}$ and outputs unbounded operators in $\mathcal{H}$.

But $d\mu_{\frac{\psi}{\|\psi\|}}$ is σ-additive in the whole $\odot_{j\in I}\mathfrak{B}(X_j) = \sigma(\mathfrak{A}_0)$, after all, it is a measure there. Hence, it is also σ-additive in $\mathfrak{A}_0$ (which is a sub-algebra). Therefore, the measure $E \in \mathfrak{A}_0 \longmapsto \langle\psi, P_0(E)\psi\rangle = \|\psi\|^2 d\mu_{\frac{\psi}{\|\psi\|}}(E)$ is indeed σ-additive, proving that[a] P_0 is a PV*pre*M.

(Step 5.) By Proposition 31 (and the fact that $\sigma(\mathfrak{A}_0) = \odot_{j\in I}\mathfrak{B}(X_j)$ by Prop. 3), there exists a unique PVM $(\mathcal{H}, (X^\infty, \odot_{j\in I}\mathfrak{B}(X_j)), P)$ extending the PVpreM $(\mathcal{H}, (X^\infty, \mathfrak{A}_0), P_0)$, i.e., such that $P(E) = P_0(E)$ for all $E \in \mathfrak{A}_0$.

- **Proof of Item (ii):**

The proof is exactly the same as the one we gave for Theorem 14.(ii). One just needs to change $\prod_{j=1}^n \mapsto \prod_{j\in I}$, $\prod_{j\in\{1,\ldots,n\}\setminus J} \mapsto \prod_{j\in I\setminus J}$, $\prod_{\ell=1,\neq k}^n \mapsto \prod_{\ell\in I\setminus\{k\}}$ and $(x_1, \ldots, x_n) \mapsto (x_j)_{j\in I}$.
o.ε.δ.

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[a] The $\psi = 0$ case makes the trivial σ-finite spectral measure $d\mu_\psi = 0$.

5.2.3 The Spectral Subspaces of an Arbitrary PVM

First, let us recall the properties of a multiplication operator in the L^2-space of an arbitrary measure space.

Lemma 23. Given a measure space $(X, \Sigma, d\mu)$ and a measurable function $f : X \to \mathbb{C} \cup \{\infty\}$ that is $d\mu$-almost-everywhere $\neq \infty$, the operator that acts multiplying by f:

$$\begin{array}{rccc} \mathcal{M}_f: & L^2(X, d\mu) & \longrightarrow & L^2(X, d\mu) \\ & \psi & \longmapsto & \Big[x \in X \longmapsto f(x)\psi(x)\Big] \end{array} \tag{5.16}$$

with $D(\mathcal{M}_f) := \{\psi \in L^2(X, d\mu) \mid \|\mathcal{M}_f\psi\|_{L^2(X,d\mu)} < +\infty\}$ (i.e., the maximal domain where it makes sense), is a densely defined operator and we call it the *multiplication operator of f*. It satisfies $(\mathcal{M}_f)^* = \mathcal{M}_{\overline{f}}$ with $D((\mathcal{M}_f)^*) = D(\mathcal{M}_f)$ and $\|\mathcal{M}_f\psi\| = \|(\mathcal{M}_f)^*\psi\|$ for all $\psi \in D(\mathcal{M}_f)$. Hence, $(\mathcal{M}_f, D(\mathcal{M}_f))$ is a normal operator. If f is real-valued, then it is a self-adjoint operator. If $d\mu$ is σ-finite, then the spectrum is $\sigma(\mathcal{M}_f) = \text{ess range}(f)$. ♦

Proof: See Example 3.8 in (Schmüdgen, 2012). ☐

About the following theorem, note that all items except (iii) are proven roughly the same way as when a PVM is associated to a separable Hilbert space and takes values on $\mathbb{R}$ (see for instance (Porta, 2019)). However, since we did not find the proof for our more general case anywhere, we provide it here. Item (iii) on the other hand is proven inspired by the usual proof for the existence of ONBs in non-separable Hilbert spaces.

Theorem 16. Let $\mathcal{H}$ be a Hilbert space, (X, Σ) a measurable space and $(\mathcal{H}, (X, \Sigma), P)$ a PVM.

(i) For each $\psi \in \mathcal{H}$,
$$\Big\{\Phi(f)\psi \;\Big|\; f \in L^2(X, d\mu_\psi)\Big\} = \overline{\text{span}\Big\{P(E)\psi \;\Big|\; E \in \Sigma\Big\}}. \tag{5.17}$$
We denote it by $\mathcal{H}^\psi$ and call it the *spectral subspace generated by ψ*. In particular, $\forall\varphi \in \mathcal{H}^\psi$, $d\mu_\varphi \ll d\mu_\psi$.

(ii) Given $\varphi, \psi \in \mathcal{H}$, if $\varphi \perp \mathcal{H}^\psi$, then $\mathcal{H}^\varphi \perp \mathcal{H}^\psi$.

(iii) There exists an orthonormal family $(\psi_{\mathfrak{N}})_{\mathfrak{N}\in\mathfrak{X}} \subseteq \mathcal{H}$ with $|\mathfrak{X}| \leq \dim(\mathcal{H})$ such that $(\mathcal{H}^{\psi_{\mathfrak{N}}})_{\mathfrak{N}\in\mathfrak{X}}$ are mutually orthogonal and their joint span's closure is $\mathcal{H}$. In particular,

$$\mathcal{H} = \bigoplus_{\mathfrak{N}\in\mathfrak{X}} \mathcal{H}^{\psi_{\mathfrak{N}}}. \tag{5.18}$$

We call such a family of vectors $(\psi_{\mathfrak{N}})_{\mathfrak{N}\in\mathfrak{X}} \subseteq \mathcal{H}$ a *spectral basis of the PVM P.*

(iv) For each fixed $\psi \in \mathcal{H}$, the map

$$\begin{array}{rccc} U: & \mathcal{H}^{\psi} & \longrightarrow & L^2(X, d\mu_{\psi}) \\ & \varphi = \Phi(f)\psi & \longmapsto & f \end{array} \tag{5.19}$$

is well-defined and it is a unitary isomorphism. Moreover, $|f|^2$ is the Radon-Nikodym derivative $d\mu_{\varphi}/d\mu_{\psi}$, i.e., $d\mu_{\varphi} = |f|^2 d\mu_{\psi}$.

(v) For each $\psi \in \mathcal{H}$, and each measurable $f: X \to \mathbb{C}$, the spectral subspace $\mathcal{H}^{\psi}$ *reduces* $\Phi(f)$ (see Def.44). Hence, the reduced parts $\Phi(f)|_{\mathcal{H}^{\psi}}$, $\Phi(f)|_{(\mathcal{H}^{\psi})^{\perp}}$ satisfy $\Phi(f) = \Phi(f)|_{\mathcal{H}^{\psi}} \oplus \Phi(f)|_{(\mathcal{H}^{\psi})^{\perp}}$ (block reduction on $\mathcal{H} = \mathcal{H}^{\psi} \oplus (\mathcal{H}^{\psi})^{\perp}$).

(vi) For each $\psi \in \mathcal{H}$ and each measurable $f: X \to \mathbb{C}$, defining U as in (iv),

$$\Phi(f)|_{\mathcal{H}^{\psi}} = U^{-1} \circ \mathcal{M}_f \circ (U \restriction_{\mathcal{H}^{\psi} \cap D_f}).$$

◆

Proof: **Item (i):** First, note that the l.h.s of (5.17) is well-defined because by definition,

$$f \in L^2(X, d\mu_{\psi}) \iff \int_{x\in X} |f(x)|^2 d\mu_{\psi} < +\infty \iff \psi \in D_f, \tag{5.20}$$

such that it always makes sense to compute $\Phi(f)\psi$ when $f \in L^2(X, d\mu_{\psi})$.

- **Claim:** The l.h.s of (5.17), which we denote by $\mathcal{H}^{\psi}$, is a *vector subspace of $\mathcal{H}$.*

Check: If $\varphi_1, \varphi_2 \in \mathcal{H}^{\psi}$, by definition, $\exists f_1, f_2 \in L^2(X, d\mu_{\psi})$ s.th. $\varphi_k = \Phi(f_k)\psi$. Then, $\forall \alpha, \beta \in \mathbb{C}$

$$\int_{x\in X} |\alpha f_1(x) + \beta f_2(x)|^2 d\mu_{\psi} = \int_{x\in X} \Big(|\alpha f_1(x)|^2 + |\beta f_2(x)|^2 + 2\,\mathrm{Re}\{\overline{\alpha f_1(x)}\beta f_2(x)\} \Big) d\mu_{\psi} =$$

$$= |\alpha|^2 \|f_1\|^2 + |\beta|^2 \|f_2\|^2 + 2\,\mathrm{Re}\langle \alpha f_1, \beta f_2 \rangle_{L^2(d\mu_{\psi})} \leq |\alpha|^2 \|f_1\|^2 + |\beta|^2 \|f_2\|^2 + 2|\alpha||\beta| \|f_1\| \|f_2\|_{L^2(d\mu_{\psi})} < \infty.$$

Thus, $\alpha f_1 + \beta f_2 \in L^2(X, d\mu_{\psi})$, i.e., $\psi \in D_{\alpha f_1 + \beta f_2}$. But then,

$$\alpha\varphi_1 + \beta\varphi_2 = \alpha\Phi(f_1)\psi + \beta\Phi(f_2)\psi \overset{(\text{Prop. } 30)}{=} \Phi(\alpha f_1 + \beta f_2)\psi \tag{5.21}$$

and hence, $\alpha\varphi_1 + \beta\varphi_2 \in \mathcal{H}^{\psi}$.

- **Claim:** $\mathcal{H}^{\psi}$ is a *closed* subset of $\mathcal{H}$.

Check: Let $(\varphi_n)_{n\in\mathbb{N}} \subset \mathcal{H}^{\psi}$ be an arbitrary sequence converging to $\mathcal{H}$, i.e., such that $\lim^{\mathcal{H}}_{n\to\infty} \varphi_n = \varphi$ for some $\varphi \in \mathcal{H}$. Then, by definition of $\mathcal{H}^{\psi}$, there exists $(f_n)_{n\in\mathbb{N}} \subseteq L^2(X, d\mu_{\psi})$ such that $\varphi_n = \Phi(f_n)\psi$ and thus, $\lim^{\mathcal{H}}_{n\to\infty} \Phi(f_n)\psi = \varphi$. In particular, $(\Phi(f_n)\psi)_{n\in\mathbb{N}}$ is a Cauchy sequence.

But then, uniformly in n, m,

$$0 = \lim_{n,m\to\infty} \|\Phi(f_n)\psi - \Phi(f_m)\psi\|^2 \overset{(5.21)}{=} \lim_{n,m\to\infty} \|\Phi(f_n - f_m)\psi\|^2 =$$

$$= \lim_{n,m\to\infty} \int_{x\in X} |f_n - f_m|^2 d\mu_\psi = \lim_{n,m\to\infty} \|f_n - f_m\|_{L^2(X,d\mu_\psi)},$$

so, $(f_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in the Hilbert space $L^2(X, d\mu_\psi)$. As such, it must converge to some $f \in L^2(X, d\mu_\psi)$, i.e., $\lim^{L^2(d\mu_\psi)}_{n\to\infty} f_n = f$. Therefore,

$$\lim_{n\to\infty} \Phi(f_n)\psi \overset{(\star)}{=} \Phi(f)\psi \overset{\text{(uniqueness of limit)}}{=} \varphi,$$

and thus, by definition, $\varphi \in \mathcal{H}^\psi$, proving that $\mathcal{H}^\psi$ is closed. We check $(\star)$:

$$\lim_{n\to\infty} \|\Phi(f_n)\psi - \Phi(f)\psi\| \overset{(5.21)}{=} \lim_{n\to\infty} \|\Phi(f_n - f)\psi\| = \lim_{n\to\infty} \|f_n - f\|_{L^2(d\mu_\psi)} = 0.$$

- Now, trivially, $\operatorname{span}\Big\{P(E)\psi \;\Big|\; E\in\Sigma\Big\} \subseteq \mathcal{H}^\psi$ because (i) $P(E) = \Phi(\mathbb{1}_E)$ for all $E\in\Sigma$ and (ii) $\mathbb{1}_E \in L^2(X, d\mu_\psi)$ ($d\mu_\psi$ is a finite measure). But $\mathcal{H}^\psi$ is closed, so, $\overline{\operatorname{span}\Big\{P(E)\psi \;\Big|\; E\in\Sigma\Big\}} \subseteq \mathcal{H}^\psi$.

- For the reverse inclusion, consider an arbitrary $\Phi(f)\psi \in \mathcal{H}^\psi$. By definition of functional calculus, $\Phi(f)\psi = \lim_{n\to\infty} \Phi(f\mathbb{1}_{\Omega_n})$ for $\Omega_n := |f|^{-1}([0,n))$. In particular, $(f\cdot\mathbb{1}_{\Omega_n}) \in \mathcal{M}_b(X,\Sigma)$, so by Thm. 2.10 in (Folland, 1999), it is a uniform limit of simple functions $(s^k_n)_{k\in\mathbb{N}}$. Then, by continuity of $\Phi(\cdot) : (\mathcal{M}_b, \|\cdot\|_\infty) \to \mathcal{L}(\mathcal{H})$, $\Phi(f)\psi = \lim_{n\to\infty}\lim_{k\to\infty} \Phi(s^k_n)\psi$. But $\Phi(s^k_n)\psi \in \operatorname{span}\Big\{P(E)\psi \;\Big|\; E\in\Sigma\Big\}$, so $\Phi(f)\psi \in \overline{\operatorname{span}\Big\{P(E)\psi \;\Big|\; E\in\Sigma\Big\}}$.

- For the last claim, let $\varphi = \Phi(g)\psi \in \mathcal{H}^\psi$ arbitrary (such that $g \in L^2(d\mu_\psi)$). Then, for each $E\in\Sigma$

$$d\mu_\varphi(E) = \|P(E)\varphi\|^2 = \|\Phi(\mathbb{1}_E)\Phi(g)\psi\|^2 \overset{(\mathbb{1}_E\in\mathcal{M}_b)}{=} \|\Phi(\mathbb{1}_E g)\psi\|^2 = \int_E |g|^2(x)d\mu_\psi = |g|^2 d\mu_\psi(E).$$

Hence, $d\mu_\varphi \ll d\mu_\psi$ and $|g|^2$ is precisely the Radon-Nikodym derivative $d\mu_\varphi/d\mu_\psi$ —this also proves one of the claims in (iv).

Item (ii): Let $f \in L^2(X, d\mu_\varphi)$ be such that $f \in \mathcal{M}_b(X,\Sigma)$. We prove that $\Phi(f)\varphi \perp \Phi(g)\psi$ for all $\Phi(g)\psi \in \mathcal{H}^\psi$:

$$\langle \Phi(f)\varphi, \Phi(g)\psi\rangle \overset{\text{(Prop. 30)}}{=} \langle \varphi,\ \Phi(\overline{f})\Phi(g)\psi\rangle \overset{(f\in\mathcal{M}_b)}{=} \langle \varphi, \Phi(\overline{f}g)\psi\rangle \overset{(\varphi\perp\mathcal{H}^\psi)}{=} 0.$$

Now let $f \in L^2(X, d\mu_\varphi)$ arbitrary. Then, given $f_n := \mathbb{1}_{\Omega_n} f$, $\Omega_n : |f|^{-1}([0,n))$, for each $\eta \in \mathcal{H}^\psi$,

$$\langle \Phi(f)\varphi, \eta\rangle \overset{\left(\substack{\text{def of }\Phi\ \&\\ \langle,\rangle\text{ conts.}}\right)}{=} \lim_{n\to\infty} \langle \Phi(f_n)\varphi, \eta\rangle \overset{(f_n\in\mathcal{M}_b)}{=} 0 \implies \Phi(f)\varphi \perp \mathcal{H}^\psi \implies \mathcal{H}^\varphi \perp \mathcal{H}^\psi.$$

Item (iii): Each $\psi \in \mathcal{H}$ has an associated $\mathcal{H}^\psi$. Now, define

$$\mathfrak{S} := \Big\{\ B \subseteq \mathcal{H} \ \Big|\ \forall\psi\in B,\ \|\psi\| = 1 \quad \text{and if} \quad \psi \neq \varphi \in B \quad \text{then} \quad \mathcal{H}^\psi \perp \mathcal{H}^\varphi\ \Big\}. \tag{5.22}$$

This is a non-empty family because $\emptyset \in \mathfrak{S}$. Note that every $B \in \mathfrak{S}$ is trivially an orthonormal family.[a] Now, the relation $\subseteq$ is a *partial order* in $\mathfrak{S}$, i.e.,

(a) Reflexive: $B \subseteq B\ \ \forall B\in\mathfrak{S}$ *(b) Antisymmetric:* $\forall B_1, B_2 \in \mathfrak{S} : \Big(B_1 \subseteq B_2\ \&\ B_2 \subseteq B_1\Big) \Longrightarrow B_1 = B_2$,

(c) Transitive: $\forall B_1, B_2, B_3 \in \mathfrak{S}: \; B_1 \subseteq B_2 \subseteq B_3 \Longrightarrow B_1 \subseteq B_3$.

- In particularr, every subfamily $\mathfrak{Y} \subseteq \mathfrak{S}$ that is a *chain* (i.e., such that $\forall B_1, B_2 \in \mathfrak{Y}$ either $B_1 \subseteq B_2$ or $B_2 \subseteq B_1$) also has an upper bound (i.e., $\exists A \in \mathfrak{S}: \; \forall B \in \mathfrak{Y}, \; B \subseteq A$): the upper bound of $\mathfrak{Y}$ is precisely $A := \bigcup_{B \in \mathfrak{Y}} B$.

Check that the claimed $A \in \mathfrak{S}$*:* If $A = \emptyset$ or it consists of a single element (since it must be unit) we are done. Else, let $\psi, \varphi \in A$ different. By definition, $\exists B_1, B_2 \in \mathfrak{Y}: \; \psi \in B_1, \varphi \in B_2$. Because $\mathfrak{Y}$ is a chain, either $B_1 \subseteq B_2$ or $B_2 \subseteq B_1$. Let us say it is the first one. Then, $\psi, \varphi \in B_2$ so, by definition, $\|\psi\| = 1 = \|\varphi\|$ and $\mathcal{H}^\psi \perp \mathcal{H}^\varphi$. Hence, $A \in \mathfrak{S}$.

- *Zorn's lemma* (see 0.2 in (Folland, 1999)): if $(X, \leq)$ is a partially ordered set where every chain has an upper bound, then X has a maximal element (i.e., an element $M \in X$ such that $\nexists x \in X \setminus M : \; M \leq x$).

- Then, by Zorn's lemma, $\mathfrak{S}$ has at least one maximal element, i.e., there exists an orthonormal family of vectors $B_{max} \subseteq \mathcal{H}$ such that for all different $\psi, \varphi \in B_{max}$, $\mathcal{H}^\psi \perp \mathcal{H}^\varphi$ with the property that there is no $\eta \in \mathcal{H} \setminus \{\vec{0}\}$ such that $\mathcal{H}^\eta \perp \mathcal{H}^\psi \; \forall \psi \in B_{max}$.

- **Claim:** $\overline{\operatorname{span}(\cup_{\psi \in B_{max}} \mathcal{H}^\psi)} = \mathcal{H}$.

Check: Assume this was not the case. Then, there would exist a non-zero vector $\eta \in \overline{\operatorname{span}\{\mathcal{H}^\psi \mid \psi \in B_{max}\}}^\perp$. But then, $\eta \perp \mathcal{H}^\psi \; \forall \psi \in B_{max}$, so by item (ii) $\mathcal{H}^\eta \perp \mathcal{H}^\psi \; \forall \psi \in B_{max}$, contradicting the proven maximality.

- Note that the cardinality of B_{max} cannot be greater than that of an ONB (which defines $\dim(\mathcal{H})$), because an ONB is a maximal orthonormal family of vectors. Finally, Prop. 15 gives the remaining part of statement (iii).

Item (iv): It is well-defined because,

- **Claim:** $\forall \varphi \in \mathcal{H}^\psi$ there exists a *unique* $f \in L^2(X, d\mu_\psi)$ such that $\varphi = \Phi(f)\psi$.

Check: Let $f_1, f_2 \in L^2(X, d\mu_\psi)$. Then, $\psi \in D_{f_1} \cap D_{f_2}$ and thus,

$$\Phi(f_1)\psi = \Phi(f_2)\psi \iff \|(\Phi(f_1) - \Phi(f_2))\psi\|^2 = 0 \overset{\left(\substack{\psi \in D_{f_1} \cap D_{f_2} \\ + \text{ Prop } 30}\right)}{\iff} \|\Phi(f_1 - f_2)\Psi\|^2 = 0 \overset{(\text{Prop. } 30)}{\Leftrightarrow}$$

$$\Leftrightarrow \int_{x \in X} |f_1(x) - f_2(x)|^2 d\mu_\psi = 0 \iff \|f_1 - f_2\|^2_{L^2(X, d\mu_\psi)} = 0 \iff f_1(x) = f_2(x) \; a.e. \; x \in X.$$

- *U is liner:* Let $\varphi_1, \varphi_2 \in \mathcal{H}^\psi$, $\alpha, \beta \in \mathbb{C}$. Then, $\exists f_1, f_2 \in L^2(X, d\mu_\psi)$ such that $\varphi_k = \Phi(f_k)\psi$. We proved when checking that $\mathcal{H}^\psi$ is a vector space that $\alpha\varphi_1 + \beta\varphi_2 = \alpha\Phi(f_1)\psi + \beta\Phi(f_2)\psi = \Phi(\alpha f_1 + \beta f_2)\psi$. Hence, $U(\alpha\varphi_1 + \beta\varphi_2) = \alpha f_1 + \beta f_2 = \alpha U(\varphi_1) + \beta U(\varphi_2)$.

- *U is isometric —and hence injective*: Let $\varphi \in \mathcal{H}^\psi$ with $\varphi = \Phi(f)\psi$. Then,

$$\|U\varphi\|^2_{L^2(d\mu_\psi)} = \|f\|^2_{L^2(d\mu_\psi)} = \int_{x \in X} |f(x)|^2 d\mu_\psi \overset{(\text{Prop. } 30)}{=} \|\Phi(f)\psi\|^2_{\mathcal{H}} = \|\varphi\|_{\mathcal{H}}.$$

- *U is surjective:* Let $g \in L^2(d\mu_\psi)$, then $\psi \in D_g$ and $\varphi := \Phi(g)\psi$ is such that $U\varphi = g$.

Item (v): Let $Q \in \mathcal{L}$ be the orthogonal projector with range $\mathcal{H}^\psi$ and $Q^\perp := I - Q$. Assume first that $f \in \mathcal{M}_b(X, \Sigma)$. Then, for an arbitrary $\varphi \in \mathcal{H}$,

- One the one hand, $\Phi(f)Q^\perp\varphi \in (\mathcal{H}^\psi)^\perp$ because for any $h \in L^2(d\mu_\psi)$

$$\langle \Phi(f)Q^\perp\varphi, \Phi(h)\psi\rangle = \langle Q^\perp\varphi,\ \Phi(\overline{f})\Phi(h)\psi\rangle \overset{(f\in\mathcal{M}_b)}{=} \langle Q^\perp\varphi, \underbrace{\Phi(\overline{f}h)\psi}_{\in\mathcal{H}^\psi}\rangle = 0.$$

This shows that $\Phi(f)Q^\perp = Q^\perp\Phi(f)Q^\perp$ for $f \in \mathcal{M}_b(X, \Sigma)$.

- On the other hand, $\Phi(f)Q\varphi \in \mathcal{H}^\psi$ because given $Q\varphi = \Phi(g)\psi$ for some $g \in L^2(d\mu_\psi)$,

$$Q\Phi(f)Q\varphi = Q\Phi(f)\Phi(g)\varphi \overset{(D_g\cap D_{fg}=D_g)}{=} Q\Phi(fg)\psi = \Phi(fg)\psi = \Phi(f)\Phi(g)\psi = \Phi(f)Q\varphi$$

This shows that $\Phi(f)Q = Q\Phi(f)Q$ for $f \in \mathcal{M}_b(X, \Sigma)$. With all,

$$Q\Phi(f) = Q\Phi(f)(Q + Q^\perp) = Q\Phi(f)Q = \Phi(f)Q,$$

and thus, $\mathcal{H}^\psi$ reduces $\Phi(f)$.

- Now let $f : X \to \mathbb{C}$ be an arbitrary measurable function, such that $\Omega_n := |f|^{-1}([0, n))$ and $f_n := f\mathbb{1}_{\Omega_n} \in \mathcal{M}_b(X, \Sigma)$. Let $\varphi \in D_f$ be arbitrary. One cannot freely plug $Q\varphi$ or $Q^\perp\varphi$ to $\Phi(f)$ anymore, after all, they might not be in D_f. Instead, we proceed with the following trick. Define $\eta_n := \Phi(\mathbb{1}_{\Omega_n})Q\varphi$ and note that now $\eta_n \in D_f$.[b] Then,

$$\Phi(f)\eta_n = \Phi(f)\Phi(\mathbb{1}_{\Omega_n})Q\varphi \overset{(\mathbb{1}_{\Omega_n}\in\mathcal{M}_b)}{=} \Phi(f_n)Q\varphi \overset{(f_n\in\mathcal{M}_b)}{=} Q\Phi(f_n)\varphi. \tag{5.23}$$

$$\text{such that } \lim_{n\to\infty}\Phi(f)\eta_n \overset{(5.23)}{=} \lim_{n\to\infty} Q\Phi(f_n)\varphi \overset{(Q\text{ conts.})}{=} Q\lim_{n\to\infty}\Phi(f_n)\varphi \overset{(\varphi\in D_f)}{=} Q\Phi(f)\varphi.$$

Moreover, given the $g \in L^2(d\mu_\psi)$ such that $Q\varphi = \Phi(g)\psi$,

$$\lim_{n\to\infty}\eta_n = \lim_{n\to\infty}\Phi(\mathbb{1}_{\Omega_n})\Phi(g)\psi = \lim_{n\to\infty}\Phi(\mathbb{1}_{\Omega_n}g)\psi \overset{[c]}{=} \Phi(g)\psi = Q\varphi.$$

But, $\Phi(f)$ is a closed operator by Prop. 30, so, the above two results together imply that $Q\varphi \in D_f$ and $\Phi(f)Q\varphi = Q\Phi(f)\varphi$. Thus, $Q\Phi(f) \subseteq \Phi(f)Q$ and $\mathcal{H}^\psi$ reduces $\Phi(f)$.

Item (vi): First, note that for arbitrary measurable $f : X \to \mathbb{C}$ and $g \in L^2(X, d\mu_\psi)$, given that $s_n := \sum_{k=1}^{N_n}\alpha_k^n\mathbb{1}_{E_k^n}$ (with $n \in \mathbb{N},\ N_n \in \mathbb{N}, \alpha_k^n \geq 0,\ E_k^n \in \Sigma$) is a monotonously increasing sequence of simple functions converging to $|f|^2$ (as in Thm. 2.10 of (Folland, 1999)),

$$\int_X |f|^2(x)d\mu_{\Phi(g)\psi} \overset{(\text{mont. conv})}{=} \lim_{n\to\infty}\sum_{k=1}^{N_n}\alpha_k^n\Big\langle\Phi(g)\psi,\ AP(E_k^n)\Phi(g)\psi\Big\rangle \overset{\left(\substack{P^2=P,\ P(E_k^n)=\Phi(\mathbb{1}_{E_k^n})\\ \Phi(\mathbb{1}_{E_k^n})\Phi(g)=\Phi(\mathbb{1}_{E_k^n}g)}\right)}{=}$$

$$= \lim_{n\to\infty}\sum_{k=1}^{N_n}\alpha_k^n\Big\|\Phi(\mathbb{1}_{E_k^n}g)\psi\Big\|^2 \overset{(\text{Prop } 30)}{=} \lim_{n\to\infty}\sum_{k=1}^{N_n}\alpha_k^n\int_X |g|^2(x)\mathbb{1}_{E_k^n}(x)d\mu_\psi \overset{(\text{mont. conv})}{=} \int_X |g|^2(x)|f|^2(x)d\mu_\psi. \tag{5.24}$$

- Using this we can check that the domains of $\Phi(f)|_{\mathcal{H}^\psi}$ (i.e., $D_f \cap \mathcal{H}^\psi$) and $U^{-1}D(\mathcal{M}_f)$ agree:

$$D_f \cap \mathcal{H}^\psi = \Big\{\varphi \in \mathcal{H}\ \Big|\ \Big(\exists g \in L^2(d\mu_\psi) : \varphi = \Phi(g)\psi\Big)\ \&\ f \in L^2(d\mu_\varphi)\Big\} =$$

$$= \Big\{\Phi(g)\psi \mid g \in L^2(d\mu_\psi)\ \&\ f \in L^2(d\mu_{\Phi(g)\psi})\Big\} \overset{(5.24)}{=} \Big\{\Phi(g)\psi \mid g,\ f\cdot g \in L^2(d\mu_\psi)\Big\} \Longrightarrow$$

$$\Rightarrow U(D_f \cap \mathcal{H}^{\psi}) = \Big\{ g \mid g,\ f \cdot g \in L^2(d\mu_{\psi}) \Big\} = \Big\{ g \in L^2(d\mu_{\psi}) |\ \|\mathcal{M}_f g\|_{L^2(d\mu_{\psi})} < \infty \Big\} = D(\mathcal{M}_f).$$

- Now we check the action of the operators. Let $\varphi = \Phi(g)\psi \in \mathcal{H}^{\psi} \cap D_f$. Then,

$$U\Phi(f)\varphi = U\Phi(f)\Phi(g)\psi \overset{\left(\substack{\varphi \in D_f \cap D_{fg} + \\ \text{Prop. } 30.\text{v}}\right)}{=} U\Phi(f \cdot g)\psi = f \cdot g = \mathcal{M}_f g = \mathcal{M}_f U\varphi.$$

Hence, $U\Phi(f)|_{\mathcal{H}^{\psi}} = \mathcal{M}_f(U \restriction_{D_f \cap \mathcal{H}^{\psi}})$. Applying U^{-1} from the left we have proven (vi).

o.ε.δ.

---

[a] In particular, $\forall \psi \in \mathcal{H}$, $\psi \in \mathcal{H}^{\psi}$ because $\Phi(\mathbb{1}_X)\psi = P(X)\psi = \psi$.

[b] Using that $\Phi(\mathbb{1}_{\Omega_n}) = P(\Omega_n)$,

$$\int_X |f|^2 d\mu_{\eta_n} = \int_X |f|^2 d\mu_{\underbrace{P(\Omega_n)}_{\Phi(\mathbb{1}_{\Omega_n})} Q\varphi} \overset{(5.24)}{=} \int_X |f|^2 \mathbb{1}_{\Omega_n} d\mu_{Q\varphi} \leq n^2 \int_X d\mu_{Q\varphi} < \infty.$$

[c] Note that $\|\Phi(\mathbb{1}_{\Omega_n} g)\psi - \Phi(g)\|^2 = \int_X |\mathbb{1}_{\Omega_n(x)} - 1|^2 |g(x)|^2 d\mu_{\psi}$. Then, we prove that this goes to 0 as $n \to \infty$ using the dominated convergence theorem: (a) point-wise $|\mathbb{1}_{\Omega_n}(x) - 1||g(x)| \xrightarrow[n\to\infty]{} 0$ because Ω_n is a nested exhaustion of $\mathbb{R}$; (b) $|g|^2$ is a dominating function in $L^1(d\mu_{\psi})$: $|\mathbb{1}_{\Omega_n}(x) - 1||g(x)|^2 \leq |g(x)|^2$.

5.2.4 The Joint Diagonalization of Arbitrarily Many Operators

Theorem 17. Given an arbitrary PVM $(\mathcal{H}, (X, \Sigma), P)$, there exist:

- an orthonormal family $(\psi_{\mathfrak{N}})_{\mathfrak{N} \in \mathfrak{X}} \subseteq \mathcal{H}$ with $|\mathfrak{X}| \leq \dim\mathcal{H}$ satisfying $\mathcal{H} = \bigoplus_{\mathfrak{N} \in \mathfrak{X}} \mathcal{H}^{\psi_{\mathfrak{N}}}$ and
- a unitary isomorphism

$$\begin{aligned} \mathscr{U} :\ \mathcal{H} = & \bigoplus_{\mathfrak{N} \in \mathfrak{X}} \mathcal{H}^{\psi_{\mathfrak{N}}} & \longrightarrow & \bigoplus_{\mathfrak{N} \in \mathfrak{X}} L^2(X, d\mu_{\psi_{\mathfrak{N}}}) \\ \varphi = & \Big(\Phi(f_{\mathfrak{N}})\psi_{\mathfrak{N}}\Big)_{\mathfrak{N} \in \mathfrak{X}} & \longmapsto & (f_{\mathfrak{N}})_{\mathfrak{N} \in \mathfrak{X}} \end{aligned} \tag{5.25}$$

such that for each measurable $f : X \to \mathbb{C}$,

$$\Phi(f) = \mathscr{U}^{-1} \circ \bigoplus_{\mathfrak{N} \in \mathfrak{X}} \mathcal{M}_f \circ (\mathscr{U} \restriction_{D_f}). \tag{5.26}$$

That is, the operator $\Phi(f)$ is a multiplication operator within the isomorphic space $\oplus_{\mathfrak{N} \in \mathfrak{X}} L^2(X, d\mu_{\psi_{\mathfrak{N}}})$ —namely, it is a diagonal operator in that space. One calls this construction a *diagonalization for the functional calculus* Φ.

♦

Proof: The first item is Theorem 16.(iii). Next, using Theorem 16.(iv) within each sector $\mathcal{H}^{\psi_{\mathfrak{N}}}$ we get a unitary $U_{\mathfrak{N}} : \mathcal{H}^{\psi} \to L^2(X, d\mu_{\psi})$. Putting them together as $\mathscr{U} := \oplus_{\mathfrak{N} \in \mathfrak{X}} U_{\mathfrak{N}}$, they make a unitary map —see the proof of Theorem 13.(i). Finally,

$$\Phi(f) \overset{(\text{Th.}16.(\text{v}))}{=} \bigoplus_{\mathfrak{N} \in \mathfrak{X}} \Phi(f)|_{\mathcal{H}^{\psi_{\mathfrak{N}}}} \overset{(\text{Th.}16.(\text{vi}))}{=} \bigoplus_{\mathfrak{N} \in \mathfrak{X}} U^{-1} \circ \mathcal{M}_f \circ (U \restriction_{\mathcal{H}^{\psi_{\mathfrak{N}}} \cap D_f}) \overset{(\text{by def.})}{=} \mathscr{U}^{-1} \circ \bigoplus_{\mathfrak{N} \in \mathfrak{X}} \mathcal{M}_f \circ (\mathscr{U} \restriction_{D_f}).$$

o.ε.δ.

Definition 19. In the setting of Lemma 23, if $X = \prod_{j \in I} X_j$ for arbitrary I and $(X_j)_{j \in I}$, we will denote the multiplication operator of the k-th coordinate map $\pi_k : (x_j)_{j \in I} \mapsto x_k$, $\mathcal{M}_{\pi_k}$, by $\hat{x}_k$ (since $\hat{x}_k f((x_j)_{j \in I}) = x_k f((x_j)_{j \in I})$).

♦

Theorem 18 (*Spectral Theorem*)**.** Let $(\mathcal{A}, D(\mathcal{A}))$ be a self-adjoint operator in a Hilbert space $\mathcal{H}$. Then, its spectrum satisfies $\sigma(\mathcal{A}) \subseteq \mathbb{R}$ and there *exists a unique* PVM $(\mathcal{H},\ (\sigma(\mathcal{A}), \mathfrak{B}(\sigma(\mathcal{A}))),\ P)$ such that

$$\mathcal{A} = \Phi\Big(a \in \sigma(\mathcal{A}) \mapsto a\Big) \equiv \int_{a\in\sigma(\mathcal{A})} a\ dP. \tag{5.27}$$

We call it *the (spectral) PVM of* $(\mathcal{A}, D(\mathcal{A}))$. We denote its associated functional calculus by $f(\mathcal{A}) := \Phi(f)$. Moreover, there exist:

- an orthonormal family $(\psi_{\mathfrak{N}})_{\mathfrak{N}\in\mathfrak{X}} \subseteq \mathcal{H}$ with $|\mathfrak{X}| \leq \dim\mathcal{H}$ satisfying $\mathcal{H} = \bigoplus_{\mathfrak{N}\in\mathfrak{X}} \mathcal{H}^{\psi_{\mathfrak{N}}}$ and
- a unitary isomorphism

$$\begin{array}{rrcc} \mathscr{U}:\ \ \mathcal{H} = & \bigoplus\limits_{\mathfrak{N}\in\mathfrak{X}} \mathcal{H}^{\psi_{\mathfrak{N}}} & \longrightarrow & \bigoplus\limits_{\mathfrak{N}\in\mathfrak{X}} L^2(\sigma(\mathcal{A})\ , d\mu_{\psi_{\mathfrak{N}}}) \\ \varphi = & \Big(f_{\mathfrak{N}}(\mathcal{A})\psi_{\mathfrak{N}}\Big)_{\mathfrak{N}\in\mathfrak{X}} & \longmapsto & (f_{\mathfrak{N}})_{\mathfrak{N}\in\mathfrak{X}} \end{array} \tag{5.28}$$

such that for each measurable $f : \sigma(\mathcal{A}) \to \mathbb{C}$,

$$\Phi(f) = \mathscr{U}^{-1} \circ \bigoplus_{\mathfrak{N}\in\mathfrak{X}} \mathcal{M}_f \circ (\mathscr{U} \restriction_{D_f}). \tag{5.29}$$

In particular, $\mathscr{U} \circ \mathcal{A} \circ \mathscr{U}^{-1} = \bigoplus\limits_{\mathfrak{N}\in\mathfrak{X}} \hat{x}$, i.e., $\mathcal{A}$ is the canonical diagonal (multiplication) operator in the isomorphic space. We call this space the *diagonalization space of the operator* $\mathcal{A}$. The wavefunction $\Psi \in \bigoplus\limits_{\mathfrak{N}\in\mathfrak{X}} L^2(\sigma(\mathcal{A}), d\mu_{\psi_{\mathfrak{N}}})$ is said to "be" the state-vector $\mathscr{U}^{-1}\Psi \in \mathcal{H}$ *in* $\mathcal{A}$*-representation* (e.g., the "energy representation"). ♦

Proof: See Theorem 5.7 in (Schmüdgen, 2012) for P's existence over $\mathbb{R}$ (instead of $\sigma(\mathcal{A})$). Applying Thm. 17 we get the rest over $\mathbb{R}$ (instead of $\sigma(\mathcal{A})$). Now, we define the PVM's support $\operatorname{supp}P$ to be the smallest closed subset $B \subseteq \mathbb{R}$ such that $P(B) = I$.[a] By definition, for all $\psi \in \mathcal{H}$, $d\mu_\psi(B) = \langle\psi, P(B)\psi\rangle = 0$ if $B \cap \operatorname{supp}P = \emptyset$. Hence, any Borel subset of $\mathbb{R} \setminus \operatorname{supp}P$ has $d\mu_\psi$-measure 0 for all spectral measures. As such, by restricting the σ-algebra, we can substitute $\mathbb{R}$ by $\operatorname{supp}P$. Finally, by Prop. 5.10.(i) in (Schmüdgen, 2012), $\operatorname{supp}P = \sigma(\mathcal{A})$. **o.ε.δ.**

---

[a] After Def. 4.3. in (Schmüdgen, 2012), this definition of $\operatorname{supp}P$ is proven to be equivalent to more conventional definitions whenever the space (X, Σ) over which the PVM is defined is (the Borel σ-algebra of) a Hausdorff second countable topological space.

Theorem 19. For an arbitrary index set I, let $(\mathcal{A}_j, D(\mathcal{A}_j))_{j\in I}$ be a family of self-adjoint operators acting on a Hilbert space $\mathcal{H}$, with the property that they *commute strongly*, i.e., that their respective spectral PVMs $(\mathcal{H},\ (\mathbb{R}, \mathfrak{B}(\mathbb{R})),\ P_j)_{j\in I}$ (given by Thm. 18) commute with each other:

$$P_j(B_j) \circ P_k(B_k) = P_k(B_k) \circ P_j(B_j) \qquad \forall j, k \in I, \forall B_j, B_k \in \mathfrak{B}(\mathbb{R}). \tag{5.30}$$

Then, the PVM $(\mathcal{H},\ (\prod_{j\in I}\mathbb{R},\ \odot_{j\in I}\mathfrak{B}(\mathbb{R})),\ \odot_{j\in I}P_j)$ is called the *joint spectral PVM of the family* $(\mathcal{H}_j, D(\mathcal{H}_j))_{j\in I}$. Given Φ, Φ_k are the functional calculus of $\odot_{j\in I}P_j$ and P_k respectively, for all $g_k : \mathbb{R} \to \mathbb{C}$ measurable, defining $f((x_j)_{j\in I}) := g_k(x_k)$, we get that $\Phi(f) = \Phi_k(g_k)$. In particular,

$$\mathcal{A}_k = \int_{(a_j)_{j\in I}\in\prod_{j\in I}\mathbb{R}} a_k\ d(\odot_{j\in I}P_j). \tag{5.31}$$

Moreover, there exist

- an orthonormal family $(\psi_{\mathfrak{N}})_{\mathfrak{N}\in\mathfrak{X}} \subseteq \mathcal{H}$ with $|\mathfrak{X}| \leq \dim\mathcal{H}$ satisfying $\mathcal{H} = \bigoplus_{\mathfrak{N}\in\mathfrak{X}} \mathcal{H}^{\psi_{\mathfrak{N}}}$, and
- a unitary isomorphism

$$\begin{array}{rrcll} \mathscr{U}: & \mathcal{H} = & \bigoplus\limits_{\mathfrak{N}\in\mathfrak{X}} \mathcal{H}^{\psi_{\mathfrak{N}}} & \longrightarrow & \bigoplus\limits_{\mathfrak{N}\in\mathfrak{X}} L^2(\prod_{j\in I}\mathbb{R},\ d\mu_{\psi_{\mathfrak{N}}}) \\ & \varphi = & \Big(\Phi(f_{\mathfrak{N}})\psi_{\mathfrak{N}}\Big)_{\mathfrak{N}\in\mathfrak{X}} & \longmapsto & (f_{\mathfrak{N}})_{\mathfrak{N}\in\mathfrak{X}} \end{array} \tag{5.32}$$

such that for each measurable $f : \prod_{j\in I}\mathbb{R} \to \mathbb{C}$,

$$\Phi(f) = \mathscr{U}^{-1} \circ \bigoplus_{\mathfrak{N}\in\mathfrak{X}} \mathcal{M}_f \circ (\mathscr{U} \restriction_{\mathcal{D}_f}). \tag{5.33}$$

In particular, $\mathcal{A}_j = \mathscr{U}^{-1} \circ \bigoplus\limits_{\mathfrak{N}\in\mathfrak{X}} \hat{x}_j \circ \mathscr{U}$. We call this construction the *joint diagonalization of the family* $(\mathcal{A}_j)_{j\in I}$. The wavefunction $\Psi \in \bigoplus\limits_{\mathfrak{N}\in\mathfrak{X}} L^2(\prod_{j\in I}\mathbb{R}, d\mu_{\psi_{\mathfrak{N}}})$ is said to be $\mathscr{U}^{-1}\Psi \in \mathcal{H}$ *in* $(\mathcal{A}_j)_{j\in I}$*-representation* (e.g., the configuration representation). ♦

Proof: Because $\mathbb{R}$ is a locally compact, second countable and Hausdorff space, the hypotheses imply, by Theorem 15, that there exists a unique product PVM $(\mathcal{H},\ (\prod_{j\in I}\mathbb{R},\ \odot_{j\in I}\mathfrak{B}(\mathbb{R})),\ \odot_{j\in I}P_j)$. Moreover, given Φ, Φ_k are the functional calculus of P, P_k respectively, the theorem also gives that $\Phi((a_j)_{j\in I} \in \prod_{j\in I}\mathbb{R} \mapsto a_k) = \Phi_k(a_k \in \mathbb{R} \mapsto a_k)$, which is equal to $\mathcal{A}_k$ by Theorem 18. The rest of the statement follows from Theorem 17. ***o.ε.δ.***

If we defined the *joint spectrum* $\sigma((\mathcal{A}_j)_{j\in I})$ to be the (suitably defined) support of the joint PVM $\odot_{j\in I}P_j$ (as it is done for finite products), it would be interesting to relate it to the product of the spectra $\prod_{j\in I}\sigma(\mathcal{A}_j)$. We leave this as future work.

5.3 The Decomposition into $L^2(\mathbb{R}^\infty, d^\infty\mu_{\mathfrak{C}})$ Layers was the Joint Diagonalization of All Position Operators

Let us manifest a very suggestive analogy. If we applied (the joint-diagonalization) Theorem 19 to $\bigotimes_{k\in\mathbb{N}} L^2(\mathbb{R}, dx)$ for countably many PVMs $(\otimes_{k\in\mathbb{N}}L^2(\mathbb{R}, dx),\ (\mathbb{R}, \mathfrak{B}(\mathbb{R})),\ P_j)_{j\in\mathbb{N}}$ (of associated self-adjoint operators $(\Phi_j(x_j \mapsto x_j),\ D_{x_j})$), we would get the existence of some vectors $(\eta_{\mathfrak{N}})_{\mathfrak{N}\in\mathfrak{X}}$, each generating a subspace $\mathcal{H}_{\mathfrak{N}} := \mathcal{H}^{\eta_{\mathfrak{N}}}$, such that

$$\begin{array}{ccc} \bigotimes_{k\in\mathbb{N}} L^2(\mathbb{R}, dx) = \bigoplus\limits_{\mathfrak{N}\in\mathfrak{X}} \mathcal{H}_{\mathfrak{N}} & \xrightarrow{\text{Block diagonal unitary}} & \bigoplus\limits_{\mathfrak{N}\in\mathfrak{X}} L^2(\mathbb{R}^\infty, d\mu_{\eta_{\mathfrak{N}}}) \\ \Psi = (\Psi_{\mathfrak{N}})_{\mathfrak{N}\in\mathfrak{X}} & \longmapsto & (g_{\mathfrak{N}})_{\mathfrak{N}\mathfrak{X}} \end{array}$$

for some probability measures $d\mu_{\eta_{\mathfrak{N}}}$ satisfying $d\mu_{\Psi_{\mathfrak{N}}} = |g_j|^2 d\mu_{\eta_{\mathfrak{N}}}$ (the spectral measure of $\Psi_{\mathfrak{N}}$). Meanwhile, we found in Chapter 4 (under a very different concern) that there exists a family of vectors $(\rho_{\mathfrak{C}})_{\mathfrak{C}\in\Gamma}$ each generating a subspace $\otimes^{\mathfrak{C}}_{k\in\mathbb{N}}L^2(\mathbb{R}, dx)$ such that

$$\begin{array}{ccc} \bigotimes_{k\in\mathbb{N}} L^2(\mathbb{R}, dx) = \bigoplus\limits_{\mathfrak{C}\in\Gamma} \Big(\otimes^{\mathfrak{C}}_{k\in\mathbb{N}} L^2(\mathbb{R}, dx)\Big) & \xrightarrow{\text{Block diagonal unitary}} & \bigoplus\limits_{\mathfrak{C}\in\Gamma} L^2(\mathbb{R}^\infty, d^\infty\mu_{\rho_{\mathfrak{C}}}) \\ \Psi = (\Psi_{\mathfrak{C}})_{\mathfrak{C}\in\,\Gamma} & \longmapsto & (g_{\mathfrak{C}})_{\mathfrak{C}\in\Gamma} \end{array}$$

for some probability measures $d^\infty\mu_{\rho_\mathfrak{C}}$ satisfying that for elementary tensor product Ψ's, $d^\infty\mu_{\Psi_\mathfrak{C}} = |g_\mathfrak{C}|^2 d^\infty\mu_{\rho_\mathfrak{C}}$.

The reader will agree that this cannot be a coincidence. But then the question is, which are the PVMs whose joint diagonalization space is this? Which is the joint functional calculus and which are the self-adjoint operators working the lever behind the curtain? Our claim (in view of the discussion at the beginning of the chapter) is that the operators behind the scenes are the "lifts" $Id \otimes \cdots \otimes \hat{q} \otimes Id \otimes \cdots$ of the position operator $\hat{q}$ acting on each factor $L^2(\mathbb{R}, dx)$. In order to make this precise, we start by checking what such a lift could rigorously mean in the ITP setting.

5.3.1 The Lift of Unbounded Operators to the ITP

Lemma 24. Let I be an arbitrary index set and let $(\mathcal{H}_j)_{j\in I}$ be a family of Hilbert spaces. For some $k \in I$ let $(A_k, D(A_k))$ be a densely defined operator on $\mathcal{H}_k$. Then, the set

$$D_0(\widehat{\mathcal{A}_k}) := \operatorname{span}\left\{ \bigotimes_{j\in I} f_j \in \bigotimes_{j\in I} \mathcal{H}_j \;\middle|\; f_k \in D(A_k) \right\} \tag{5.34}$$

is a dense vector subspace of $\bigotimes_{j\in I} \mathcal{H}_j$. Defining the map $\widehat{\mathcal{A}_k} : D_0(\widehat{\mathcal{A}_k}) \to \bigotimes_{j\in I} \mathcal{H}_j$ as the linear extension of

$$\widehat{\mathcal{A}_k}\Big(\bigotimes_{j\in I} f_j\Big) = \bigotimes_{j\in I\setminus\{k\}} f_j \otimes (A_k f_k)$$

(i.e.,*"putting identities in all entries except the k-th, where we put* A_k"), makes $(\widehat{\mathcal{A}_k}, D_0(\widehat{\mathcal{A}_k}))$ a densely defined linear operator on $\bigotimes_{j\in I} \mathcal{H}_j$. We call $\widehat{\mathcal{A}_k}$ the *lift of* A_k *to the ITP.* ♦

Proof: $D_0(\widehat{\mathcal{A}_k})$ is a vector space by definition. To prove density, consider an arbitrary elementary tensor product $\bigotimes_{j\in I} g_j \in \bigotimes_{j\in I} \mathcal{H}_j$. Then, by density of $D(A_k)$ in $\mathcal{H}_k$, $\exists(\phi_n)_{n\in\mathbb{N}} \subseteq \mathcal{H}_k$ such that $\lim_{n\to\infty}^{\|\cdot\|_{\mathcal{H}_k}} \phi_n = g_k$. But then, by Prop. 53, $(\bigotimes_{j\in I\setminus\{k\}} g_j \otimes \phi_n) \xrightarrow[n\to\infty]{\|\cdot\|_\otimes} \bigotimes_{j\in I} g_j$. Since by definition, $(\bigotimes_{j\in I\setminus\{k\}} g_j \otimes \phi_n)_{n\in\mathbb{N}} \subseteq D_0(\widehat{\mathcal{A}_k})$, the set of elementary tensor products $S := \{\otimes_{j\in I} g_j \in \otimes_{j\in I}\mathcal{H}_j\}$ is such that $S \subseteq \overline{D_0(\widehat{\mathcal{A}_k})}$. Because $D_0(\widehat{\mathcal{A}_k})$ is a vector space, $\operatorname{span} S \subseteq \overline{D_0(\widehat{\mathcal{A}_k})}$, but then, we defined the ITP such that $\overline{\operatorname{span} S} = \bigotimes_{j\in I} \mathcal{H}_j$, so $\overline{D_0(\widehat{\mathcal{A}_k})} = \bigotimes_{j\in I} \mathcal{H}_j$. **o.ε.δ.**

Proposition 32. In the setting of Lemma 24,

(i) If $(A_k, D(A_k))$ is a *symmetric* operator, then so is $(\widehat{\mathcal{A}_k}, D_0(\widehat{\mathcal{A}_k}))$.

(ii) If $(A_k, D(A_k))$ is essentially self-adjoint (e.g., if it self-adjoint), then $(\widehat{\mathcal{A}_k}, D_0(\widehat{\mathcal{A}_k}))$ is *essentially self-adjoint* (i.e., the operator closure $\overline{\widehat{\mathcal{A}_k}}$ is the unique self-adjoint extension of $\widehat{\mathcal{A}_k}$).

(iii) Each and every layer $\otimes^\mathfrak{C}_{j\in I}\mathcal{H}_j$, $\mathfrak{C} \in \Gamma$, *reduces* $(\widehat{\mathcal{A}_k}, D_0(\widehat{\mathcal{A}_k}))$ (see Def. 44). As such, defining $D_0^\mathfrak{C}(\widehat{\mathcal{A}_k}) := D_0(\widehat{\mathcal{A}_k}) \cap \otimes^\mathfrak{C}_{j\in I}\mathcal{H}_j$, and $\widehat{\mathcal{A}_k}|_\mathfrak{C} := \widehat{\mathcal{A}_k}|_{\otimes^\mathfrak{C}_{j\in I}\mathcal{H}_j}$,

$$\Big(\widehat{\mathcal{A}_k},\ D_0(\widehat{\mathcal{A}_k})\Big) = \Big(\bigoplus_{\mathfrak{C}\in\Gamma} \widehat{\mathcal{A}_k}|_\mathfrak{C},\ \oplus^{\mathscr{F}}_{\mathfrak{C}\in\Gamma} D_0^\mathfrak{C}(\widehat{\mathcal{A}_k})\Big), \tag{5.35}$$

where $\oplus^{\mathscr{F}}_{\mathfrak{C}\in\Gamma} D_0^\mathfrak{C}(\widehat{\mathcal{A}_k}) := \oplus_{\mathfrak{C}\in\Gamma} D_0^\mathfrak{C}(\widehat{\mathcal{A}_k}) \;\bigcap\; \oplus^{\mathscr{F}}_{\mathfrak{C}\in\Gamma}(\otimes^\mathfrak{C}_{j\in I}\mathcal{H}_j)$ (which are the elements in $\oplus_{\mathfrak{C}\in\Gamma} D^\mathfrak{C}(\widehat{\mathcal{A}_k})$ —as in Prop. 18— for which only finitely many sectors are non-zero). ♦

Proof: **Item (i):** Let $\Phi, \Psi \in D_0(\widehat{\mathcal{A}_k})$ arbitrary, then $\Phi = \sum_{\ell=1}^{N} c_\ell \bigotimes_{j\in I} f_j^\ell$, $\Psi = \sum_{\ell=1}^{M} b_\ell \bigotimes_{j\in I} g_j^\ell$ with $N, M \in \mathbb{N}$, $b_\ell, c_\ell \in \mathbb{C}$, and $g_j^\ell, f_j^\ell \in \mathcal{H}_j$ if $j \neq k$, else, $g_k^\ell, f_k^\ell \in D(A_k)$. As such,

$$\langle \Phi, \widehat{\mathcal{A}_k}\Psi\rangle = \sum_{\ell=1}^{N}\sum_{r=1}^{M} \overline{c_\ell} b_r \langle f_k^\ell,\ A_k g_k^r\rangle \prod_{j\in I\setminus\{k\}} \langle f_j^\ell,\ g_j^r\rangle \overset{(A_k \text{ sym})}{=} \sum_{\ell=1}^{N}\sum_{r=1}^{M} \overline{c_\ell} b_r \langle A_k f_k^\ell,\ g_k^r\rangle \prod_{j\in I\setminus\{k\}} \langle f_j^\ell,\ g_j^r\rangle = \langle \widehat{\mathcal{A}_k}\Phi, \Psi\rangle.$$

Item (ii): By Prop. 53 (linearity of $\otimes$), the lift of the operator $(A_k + \alpha Id,\ D(A_k))$ for any $\alpha \in \mathbb{C}$ is simply $(\widehat{\mathcal{A}_k} + \alpha Id,\ D_0(\widehat{\mathcal{A}_k}))$. By Theorem 3.92 in (Teufel, 2021), a symmetric densely defined operator like $(A_k, D(A_k))$ is essentially self-adjoint *if and only if* $range(A_k + iId)$ and $range(A_k - iId)$ are dense in $\mathcal{H}_k$. Now, let S denote the set of elementary tensor products of $\otimes_{j\in I}\mathcal{H}_j$ and let there be an arbitrary $\bigotimes_{j\in I} g_j \in S$. By the density of range$(A_k \pm iId)$, there exists $(\phi_n)_{n\in\mathbb{N}} \subseteq D(A_k)$ such that $(A_k \pm iId)\phi_n \xrightarrow[n\to\infty]{\|\cdot\|_{\mathcal{H}_k}} g_k$. Hence, the sequence $(\bigotimes_{j\in I\setminus\{k\}} g_j \otimes \phi_n)_n \subseteq D_0(\widehat{\mathcal{A}_k})$ is such that

$$(\widehat{\mathcal{A}_k} \pm iId)\left(\bigotimes_{j\in I\setminus\{k\}} g_j \otimes \phi_n\right) = \bigotimes_{j\in I\setminus\{k\}} g_j \otimes [(A_k \pm iId)\phi_n] \xrightarrow[n\to\infty]{\text{Prop. } 53} \bigotimes_{j\in I} g_j.$$

Thereby, $S \subseteq \overline{\text{range}(\widehat{\mathcal{A}_k} \pm iId)}$. Since the range of a linear operator is a vector space, span$S \subseteq \overline{\text{range}(\widehat{\mathcal{A}_k} \pm iId)}$. But, span$S$ is dense in $\bigotimes_{j\in I}\mathcal{H}_j$ so, range$(\widehat{\mathcal{A}_k} \pm iId)$ is dense as well. Therefore, by the aforementioned theorem, $(\widehat{\mathcal{A}_k}, D_0(\widehat{\mathcal{A}_k}))$ is essentially self-adjoint.

Item (iii): By definition, any $\Psi \in D_0(\widehat{\mathcal{A}_k})$ is a finite linear combination of elementary tensor products with a constraint only on the k-th factor,

$$\Psi = \sum_{\ell=1}^{N} c_\ell \bigotimes_{j\in I} \psi_j^\ell \quad \text{for some } c_\ell \in \mathbb{C},\ \bigotimes_{j\in I} \psi_j^\ell \in \bigotimes_{j\in I} \mathcal{H}_j,\ \psi_k^\ell \in D(A_k). \tag{5.36}$$

But every elementary tensor product belongs to *one and only one* subspace $\otimes_{j\in I}^{\mathfrak{C}}\mathcal{H}_j$, so we can always group the summands belonging to each class $\mathfrak{C} \in \Gamma$, say, with an extra index indicating the class. That is, for any $\Psi \in D_0(\widehat{\mathcal{A}_k})$, there is an $n \in \mathbb{N}$ and some $\{\mathfrak{C}_1, ..., \mathfrak{C}_n\} \subseteq \Gamma$ such that given $P^{\mathfrak{C}}$ is the orthogonal projector to $\otimes_{j\in I}^{\mathfrak{C}}\mathcal{H}_j$,

$$\Psi = \sum_{k=1}^{n} \Psi^{\mathfrak{C}_k} \qquad \text{for} \qquad \Psi^{\mathfrak{C}} := P^{\mathfrak{C}}\Psi = \begin{cases} \sum_{\ell=1}^{N_{\mathfrak{C}}} c_\ell^{\mathfrak{C}} \bigotimes_{j\in I} \psi_j^{\ell,\mathfrak{C}} & \text{if } \mathfrak{C} \in \{\mathfrak{C}_1, ..., \mathfrak{C}_n\} \\ 0 & \text{else} \end{cases}, \tag{5.37}$$

In particular, this shows that for $\Psi \in D_0(\widehat{\mathcal{A}_k})$, $P^{\mathfrak{C}}\Psi \in D_0(\widehat{\mathcal{A}_k})$ for all $\mathfrak{C} \in \Gamma$, such that $P^{\mathfrak{C}} D_0(\widehat{\mathcal{A}_k}) \subseteq D_0(\widehat{\mathcal{A}_k})$. Then, using in $(\star)$ that trivially, $\otimes_{j\in I\setminus\{k\}} f_j \otimes f_k \approx \otimes_{j\in I\setminus\{k\}} f_j \otimes g_k$ for all $k \in I$, $g_k \in \mathcal{H}_k$ and $(f_j)_{j\in I} \in \mathscr{C}$,

$$\widehat{\mathcal{A}_k} P^{\mathfrak{C}}\Psi = \widehat{\mathcal{A}_k}\left(\sum_{\ell=1}^{N_{\mathfrak{C}}} c_\ell^{\mathfrak{C}} \bigotimes_{j\in I} \psi_j^{\ell,\mathfrak{C}}\right) = \sum_{\ell=1}^{N_{\mathfrak{C}}} c_\ell^{\mathfrak{C}} \bigotimes_{j\in I\setminus\{k\}} \psi_j^{\ell,\mathfrak{C}} \otimes (A_k \psi_k^{\ell,\mathfrak{C}}) \overset{(\star)}{=}$$

$$= P^{\mathfrak{C}}\left(\sum_{\ell=1}^{N_{\mathfrak{C}}} c_\ell^{\mathfrak{C}} \bigotimes_{j\in I\setminus\{k\}} \psi_j^{\ell,\mathfrak{C}} \otimes (A_k \psi_k^{\ell,\mathfrak{C}})\right) = P^{\mathfrak{C}}\widehat{\mathcal{A}_k}\Psi.$$

Hence, $P^{\mathfrak{C}}\widehat{\mathcal{A}_k} \subseteq \widehat{\mathcal{A}_k} P^{\mathfrak{C}}$ and $\otimes_{j\in I}^{\mathfrak{C}}\mathcal{H}_j$ reduces $\widehat{\mathcal{A}_k}$.

- Finally, let $\Psi \in D_0(\widehat{\mathcal{A}_k})$ be arbitrary. In view of (5.37), is is also an element of $\oplus_{\mathfrak{C}\in\Gamma} D_0^{\mathfrak{C}}(\widehat{\mathcal{A}_k})$, i.e., each $\Psi^{\mathfrak{C}} \in D_0^{\mathfrak{C}}(\widehat{\mathcal{A}_k})$ and $\sum_{\mathfrak{C}\in\Gamma} \left\| \widehat{\mathcal{A}_k} \Psi^{\mathfrak{C}} \right\|^2 = \sum_{k=1}^{n} \left\| \widehat{\mathcal{A}_k} \Psi^{\mathfrak{C}_k} \right\|^2 < +\infty$. Therefore, $D_0(\widehat{\mathcal{A}_k}) \subseteq \oplus_{j\in I} D_0^{\mathfrak{C}}(\widehat{\mathcal{A}_k})$. At the same time, by (5.37), $\Psi \in \oplus^{\mathscr{F}}_{\mathfrak{C}\in\Gamma}(\otimes^{\mathfrak{C}}_{j\in I}\mathcal{H}_j)$ because only finitely many sectors are occupied. Combining both, we get that $D_0(\widehat{\mathcal{A}_k}) \subseteq \oplus_{\mathfrak{C}\in\Gamma} D_0^{\mathfrak{C}}(A_k) \bigcap \oplus^{\mathscr{F}}_{\mathfrak{C}\in\Gamma}(\otimes^{\mathfrak{C}}_{j\in I}\mathcal{H}_j)$. The inclusion in reverse is trivial by (5.37), so, $D_0^{\mathfrak{C}}(\widehat{\mathcal{A}_k}) = \oplus_{\mathfrak{C}\in\Gamma} D_0^{\mathfrak{C}}(A_k) \bigcap \oplus^{\mathscr{F}}_{\mathfrak{C}\in\Gamma}(\otimes^{\mathfrak{C}}_{j\in I}\mathcal{H}_j)$. ***ο.ε.δ.***

For later use, we provide also the following method to construct densely defined operators in $\bigotimes_{j\in I}\mathcal{H}_j$, this time assembling operators that act on each layer of the ITP.

Proposition 33. Let I be an arbitrary index set and let $(\mathcal{H}_j)_{j\in I}$ be a family of Hilbert spaces. Given the the decomposition $\bigotimes_{j\in I}\mathcal{H}_j = \oplus_{\mathfrak{C}\in\Gamma}(\otimes^{\mathfrak{C}}_{j\in I}\mathcal{H}_j)$ of Theorem 25 and given a densely defined operator $(A_{\mathfrak{C}}, D(A_{\mathfrak{C}}))$ acting on each layer $\otimes^{\mathfrak{C}}_{j\in I}\mathcal{H}_j$,

$$(\mathcal{A}, D(\mathcal{A})) := \Big(\oplus_{\mathfrak{C}\in\Gamma} A_{\mathfrak{C}},\ \oplus_{\mathfrak{C}\in\Gamma} D(A_{\mathfrak{C}}) \Big) \quad \text{(see Prop. 18)} \tag{5.38}$$

is a densely-defined operator on $\bigotimes_{j\in I}\mathcal{H}_j$.

(i) If all $(A_{\mathfrak{C}}, D(A_{\mathfrak{C}}))$ are *symmetric*, then $(\mathcal{A}, D(\mathcal{A}))$ is symmetric.

(ii) If all $(A_{\mathfrak{C}}, D(A_{\mathfrak{C}}))$ are *closed* operators, then $(\mathcal{A}, D(\mathcal{A}))$ is a closed operator.

(iii) If all $(A_{\mathfrak{C}}, D(A_{\mathfrak{C}}))$ are *self-adjoint*, then $(\mathcal{A}, D(\mathcal{A}))$ is self-adjoint.

Proof: Just apply Proposition 18. □

5.3.2 The "Coordinate" Strategy: use the Identifications with $L^2(\mathbb{R}^\infty, d^\infty\mu_{\mathfrak{C}})$ to Find the PVM of the Position Operators in $\bigotimes_{k\in\mathbb{N}} L^2(\mathbb{R}, dx)$

Corollary 17. Let $(\hat{q}, D(\hat{q}))$ be the usual position operator acting on $L^2(\mathbb{R}, dx)$ (such that $\hat{q}\psi(q) = q\psi(q)$). For each $k \in \mathbb{N}$, let us denote by $\widehat{q}_k = \underbrace{Id \otimes \cdots \otimes \hat{q}}_{k} \otimes Id \otimes \cdots$ its lift (as in Lemma 24) acting on the k-th factor of $\bigotimes_{k\in\mathbb{N}} L^2(\mathbb{R}, dx)$. Then, $(\overline{\widehat{q}_k})_{k\in\mathbb{N}}$ is a family of self-adjoint operators on $\bigotimes_{k\in\mathbb{N}} L^2(\mathbb{R}, dx)$, acting as

$$\overline{\widehat{q}_k}(\psi_1 \otimes \psi_2 \otimes \cdots) = \psi_1 \otimes \cdots \otimes (\hat{q}\psi_k) \otimes \psi_{k+1} \otimes \cdots . \tag{5.39}$$

We will call them *the (lifted) position operators of the ITP*. For each of them there exists a unique PVM $(\bigotimes_{k\in\mathbb{N}} L^2(\mathbb{R}, dx),\ (\mathbb{R}, \mathfrak{B}(\mathbb{R})),\ Q_k)$ such that its functional calculus Φ_k yields $\Phi_k(q \in \mathbb{R} \mapsto q) = \overline{\widehat{q}_k}$. ♦

Proof: Apply Prop. 32 and Thm. 18 to each operator.□

Now, we would like to apply (the joint diagonalization) Theorem 19 to these lifted position operators $\{\overline{\widehat{q}_k}\}_{k\in\mathbb{N}}$ in order to check if the structure we found in Theorem 13 is indeed their joint diagonalization space.

The reason why one would expect to be able to apply Theorem 19 is that the position operators $\widehat{q}_k = Id \otimes \cdots \otimes \hat{q} \otimes Id \otimes \cdots$ commute pairwise with each other in their common domains (after all, each one acts in a different factor of the elementary tensor products). However, commutation is a weaker condition than *strong* commutation (i.e., that the spectral PVMs commute). And we need strong commutation for Theorem 19. There exist ways to identify strong commutativity without explicitly knowing the PVMs —as we find in Appendix B.

Alternatively, one could find a way to lift PVMs from factor spaces of the ITP to the product space and then check that they commute strongly —this is also developed in Appendix B.

Instead, in the main text, we are going to apply a "clever" roundabout by "bootstrapping" the tools we have developed so-far. In particular, we are going to employ the "dissection" of the abstract space $\bigotimes_{k\in\mathbb{N}} L^2(\mathbb{R}, dx)$ into the more "concrete" $L^2(\mathbb{R}^\infty, d^\infty\mu_{\mathfrak{C}})$ spaces from Theorem 13. The idea is to use them as a "coordinate space" where technical things like finding PVMs is easier and to pullback the results to the abstract ITP only at the very end.

Our "coordinate" strategy goes as follows. At each $L^2(\mathbb{R}^\infty, d^\infty\mu_{\mathfrak{C}})$, there are obvious position operators $(\hat{x}_k, D(\hat{x}_k))$ —of Def. 19— and there are obvious "ansatzes" for their PVMs —namely, $P_k(B_k) = \mathcal{M}_{\mathbb{1}_{B_k}}$, $B_k \in \mathfrak{B}(\mathbb{R})$. There, it is easy to check whether they are PVMs, whether they commute with each other and even their functional calculi can be easily guessed. We will also find their joint PVM explicitly (still in $L^2(\mathbb{R}^\infty, d^\infty\mu_{\mathfrak{C}})$) and only at the very end we will map everything to the layers of the ITP. Lastly, we will check that the direct sum of those PVMs gives the joint PVM of the position operators $\widehat{q}_k$ from above.

We start by checking that the "ansatz" PVMs of the first step are right.

Proposition 34. Let I be an arbitrary index set and denote $\mathbb{R}^I := \prod_{j\in I}\mathbb{R}$. Equip it with the product σ-algebra $\odot_{j\in I}\mathfrak{B}(\mathbb{R})$ and let $d\mu$ be an arbitrary measure. Then, for each $k \in I$,

- **(i)** the map $P_k : \mathfrak{B}(\mathbb{R}) \longrightarrow \mathcal{L}(L^2(\mathbb{R}^I, d\mu))$ such that for each $B_k \in \mathfrak{B}(\mathbb{R})$, $P_k(B_k) := \mathcal{M}_{\mathbb{1}_B}$, with $B := B_k \times \prod_{j\in I\setminus\{k\}}\mathbb{R}$, *is* a PVM.
- **(ii)** The map Φ_k taking arbitrary measurable functions $g : \mathbb{R} \to \mathbb{C}$ and yielding densely defined operators $\Phi_k(g) := \mathcal{M}_{g\circ\pi_k}$ of domain $D(\mathcal{M}_{g\circ\pi_k})$ (as defined in Lemma 23) is the *functional calculus of* P_k.
- **(iii)** P_k and Φ_k are the spectral PVM and functional calculus of the multiplication operator $(\hat{x}_k, D(\hat{x}_k))$ from Def. 19, i.e., $\Phi_k(x_k \in \mathbb{R} \mapsto x_k) = \hat{x}_k$. ♦

Proof: **Item (i):** First, for all $B_k \in \mathfrak{B}(\mathbb{R})$, $P_k(B_k)$ is a bounded operator: let $\psi \in L^2(\prod_{j\in I}\mathbb{R},\, d\mu)$, then

$$\|P_k(B_k)\psi\|^2_{L^2} = \|\mathcal{M}_{\mathbb{1}_B}\psi\|^2_{L^2} = \int_{x\in\mathbb{R}^I} |\mathbb{1}_B(x)\psi(x)|^2 d\mu \leq \int_{x\in\mathbb{R}^I} |\psi(x)|^2 d\mu = \|\psi\|.$$

- Every, $P_k(B_k)$ is idempotent:

$$P_k(B_k)^2\psi = \mathbb{1}_B \cdot \mathbb{1}_B \cdot \psi = \mathbb{1}_B \cdot \psi = P_k(B_k)\psi.$$

By Lemma 23, $P_k(B_k)$ are self-adjoint because they are real-valued multiplication operators. Hence, $P_k(B_k)$ are all orthogonal projectors.

- Now, let $(B_k^\ell)_{\ell\in\mathbb{N}} \subset \mathfrak{B}(\mathbb{R})$ be pairwise disjoint. Then, $\forall\psi \in L^2(\mathbb{R}^I, d\mu)$,

$$\left\| P_k\Big(\bigsqcup_{\ell=1}^{\infty} B_k^\ell\Big)\psi - \sum_{\ell=1}^{N} P_k(B_k^\ell)\psi \right\|^2 = \int_{x\in\mathbb{R}^I} \left| \mathbb{1}_{\bigsqcup_{\ell=1}^\infty B_k^\ell \times \prod_{I\setminus k}\mathbb{R}}(x) - \sum_{\ell=1}^{N} \mathbb{1}_{B_k^\ell \times \prod_{I\setminus k}\mathbb{R}}(x) \right|^2 |\psi|^2(x) d\mu =$$

$$\overset{(\text{Disjoint } B_k^\ell)}{=} \int_{x\in\mathbb{R}^I} \left|\mathbb{1}_{\bigsqcup_{\ell=1}^\infty B_k^\ell\times\prod_{I\setminus k}\mathbb{R}}(x) - \mathbb{1}_{\bigsqcup_{\ell=1}^N B_k^\ell\times\prod_{I\setminus k}\mathbb{R}}(x)\right|^2 |\psi|^2(x)d\mu.$$

Point-wise, the integrand converges to 0 almost everywhere, since there is an N for every $x_k \in \bigsqcup_{\ell\in\mathbb{N}} B_k^\ell$, such that $x_k \in \bigsqcup_{\ell=1}^n B_k^\ell \ \forall n > N$. Moreover, by the triangle inequality, the integrand is dominated by $2|\psi|^2$, which is in L^2. Hence, by the dominated convergence theorem, $\left\|P_k\Big(\bigsqcup_{\ell=1}^\infty B_k^\ell\Big)\psi - \sum_{\ell=1}^N P_k(B_k^\ell)\psi\right\|^2 \xrightarrow{N\to\infty} 0$ and P_k is strongly σ-additive. With all, P_k is a PVM.

Item (ii) : By Lemma 23, $(\Phi_k(g), D(\mathcal{M}_{g\circ\pi_k}))$ is a well-defined densely defined operator for any measurable $g:\mathbb{R}\to\mathbb{C}$. In particular, if $g\in\mathcal{M}_b(\mathbb{R},\mathfrak{B}(\mathbb{R}))$,

$$\|\Phi_k(g)\|_{op} = \sup_{\substack{\psi\in L^2(\mathbb{R}^I,d\mu),\\ \|\psi\|=1}} \|\mathcal{M}_{g\circ\pi_k}\psi\| = \sup_{\substack{\psi\in L^2(\mathbb{R}^I,d\mu),\\ \|\psi\|=1}} \Big(\int_{x\in\mathbb{R}^I} |g(x_k)\psi(x)|^2 d\mu\Big)^{1/2} \leq \|g\|_\infty. \quad (5.40)$$

- For simple functions $s\in\mathcal{M}_b(\mathbb{R},\mathfrak{B}(\mathbb{R}))$, say, $s=\sum_{\ell=1}^N \alpha_\ell \mathbb{1}_{B_k^\ell}$, with $N\in\mathbb{N}$, $B_k^\ell\in\mathfrak{B}(\mathbb{R})$, $\alpha_\ell\in\mathbb{C}$, using in $(\star)$ that for any $A\in\mathfrak{B}(\mathbb{R})$, $\mathbb{1}_A\circ\pi_k = \mathbb{1}_{A\times\prod_{I\setminus k}\mathbb{R}}$,

$$\Phi_k(s) = \mathcal{M}_{\sum_{\ell=1}^N \alpha_\ell \mathbb{1}_{B_k^\ell}\circ\pi_k} \overset{(\star)}{=} \mathcal{M}_{\sum_{\ell=1}^N \alpha_\ell \mathbb{1}_{B_k^\ell\times\prod_{I\setminus k}\mathbb{R}}} = \sum_{\ell=1}^N \alpha_\ell \mathcal{M}_{B_k^\ell\times\prod_{I\setminus k}\mathbb{R}} = \sum_{\ell=1}^N \alpha_\ell P_k(B_k^\ell).$$

Hence, Φ_k agrees with the action of the functional calculus of P_j on the simple functions. Now, as a map from $\mathcal{M}_b(\mathbb{R},\mathfrak{B}(\mathbb{R}))$ to $\mathcal{L}(L^2(\mathbb{R}^I,d\mu))$, Φ_k is clearly linear and it is bounded because (5.40) implies that $\|\Phi_k\|_{op}\leq 1$. Hence, $\Phi_k\restriction_{\mathcal{M}_b}$ is continuous. But the simple functions are dense in $\mathcal{M}_b(\mathbb{R},\mathfrak{B}(\mathbb{R}))$, so, Φ must act as the functional calculus of P_k everywhere in $\mathcal{M}_b(\mathbb{R},\mathfrak{B}(\mathbb{R}))$ —after all, two continuous maps between Hausdorff spaces that agree on a dense subset agree everywhere (see Lemma 32).

- Denote by D_g^k the domain of the functional calculus of P_k on an arbitrary measurable $g:\mathbb{R}\to\mathbb{C}$.[a] Then, since ϕ_k agrees with the functional calculus of P_k on $\mathcal{M}_b$, the only thing we miss to check is that $D_g^k = D(\mathcal{M}_{g\circ\pi_k})$. Let $s_n=\sum_{\ell=1}^{N_n}\alpha_\ell^n \mathbb{1}_{B_\ell^n}$ be a strictly increasing sequence of non-negative simple functions approximating $|g|^2$ (Thm. 2.10 (Folland, 1999)). Then,

$$\int_{x\in\mathbb{R}^I} |\psi(x)|^2|g(x_k)|^2 d\mu \overset{(\text{mont. conv.})}{=} \lim_{n\to\infty}\sum_{\ell=1}^{N_n}\alpha_\ell \int_{x\in\mathbb{R}^I} |\psi(x)|^2 \mathbb{1}_{B_\ell^n}(x_k)d\mu = \lim_{n\to\infty}\sum_{\ell=1}^{N_n}\alpha_\ell \langle\psi, P_k(B_\ell^n)\psi\rangle =$$

$$= \lim_{n\to\infty}\sum_{\ell=1}^{N_n}\alpha_\ell\, d\mu_\psi^k(B_\ell^n) = \lim_{n\to\infty}\sum_{\ell=1}^{N_n}\alpha_\ell \int_{x_k\in\mathbb{R}} \mathbb{1}_{B_\ell^n}(x_k)d\mu_\psi^k \overset{(\text{mont. conv.})}{=} \int_{x_k\in\mathbb{R}} |g(x_k)|^2 d\mu_\psi^k.$$

Therefore, $\psi\in D(\mathcal{M}_{g\circ\pi_k}) \iff \psi\in D_g^k$.

Item (iii): By definition, $\Phi_k(x_k\in\mathbb{R}\mapsto x_k)\psi(x) = x_k\psi(x) = \hat{x}_k\psi(x)$ and we just checked that the domains agree. By Theorem 18 there is a unique PVM with this property for $(\hat{x}_k, D(\hat{x}_k))$ because by Lemma 23 it is a self-adjoint operator. Hence, it must be P_k. ***o.ε.δ.***

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[a] That is $D_g^k := \{\ \psi\in L^2(\mathbb{R}^I,d\mu)\mid g\in L^2(\mathbb{R},d\mu_\psi^k)\ \}$ for $d\mu_\psi^k(B) := \langle\psi,\ P_k(B)\psi\rangle,\ B\in\mathfrak{B}(\mathbb{R})$.

Now we make explicit which is the joint diagonalization PVM within the L^2-space (before transporting the results to the ITP).

Proposition 35. In the setting of Proposition 34,

- **(i)** The map $P : \odot_{j\in I}\mathfrak{B}(\mathbb{R}) \to \mathcal{L}(L^2(\mathbb{R}^I, d\mu))$ such that $P(B) := \mathcal{M}_{\mathbb{1}_B}$ for $B \in \odot_{j\in I}\mathfrak{B}(\mathbb{R})$ *is a PVM.*
- **(ii)** The map Φ taking arbitrary measurable functions $f : \mathbb{R}^I \to \mathbb{C}$ and outputting densely defined operators $\Phi(f) := \mathcal{M}_f$ of domain $D(\mathcal{M}_f)$ (as defined in Lemma 23) is the *functional calculus* of P.
- **(iii)** All operators $(\hat{x}_k, D(\hat{x}_k))_{k\in I}$ commute strongly (i.e., their PVMs commute) and the joint spectral PVM and functional calculus are exactly P and Φ. Hence, $P = \odot_{j\in I}P_j$. ♦

Proof: The proof of item (i) and (ii) are almost exactly the same as those of Proposition 34. We include them merely for completeness.

Item (i): First, for all $B \in \odot_{j\in I}\mathfrak{B}(\mathbb{R})$, $P(B)$ is a bounded operator: let $\psi \in L^2(\mathbb{R}^I,\ d\mu)$, then

$$\|P(B)\psi\|_{L^2}^2 = \|\mathcal{M}_{\mathbb{1}_B}\psi\|_{L^2}^2 = \int_{x\in\mathbb{R}^I} |\mathbb{1}_B(x)\psi(x)|^2 d\mu \le \int_{x\in\mathbb{R}^I} |\psi(x)|^2 d\mu = \|\psi\|.$$

- Then, $P(B)$ is idempotent: $P(B)^2\psi = \mathbb{1}_B \cdot \mathbb{1}_B \cdot \psi = \mathbb{1}_B \cdot \psi = P(B)\psi$, and it is self-adjoint by Lemma 23 because it is a real-valued multiplication operator. Hence, $P(B)$ are all orthogonal projectors.

- Now, let $(B^\ell)_{\ell\in\mathbb{N}} \subset \odot_{j\in I}\mathfrak{B}(\mathbb{R})$ be pairwise disjoint. Then, $\forall\psi \in L^2(\mathbb{R}^I, d\mu)$,

$$\left\| P\Big(\bigsqcup_{\ell=1}^{\infty} B^\ell\Big)\psi - \sum_{\ell=1}^{N} P(B^\ell)\psi \right\|^2 = \int_{x\in\mathbb{R}^I} \left| \mathbb{1}_{\bigsqcup_{\ell=1}^{\infty} B^\ell}(x) - \sum_{\ell=1}^{N} \mathbb{1}_{B^\ell}(x) \right|^2 |\psi|^2(x) d\mu =$$

$$\overset{(\text{Disjoint } B^\ell)}{=} \int_{x\in\mathbb{R}^I} \left| \mathbb{1}_{\bigsqcup_{\ell=1}^{\infty} B^\ell}(x) - \mathbb{1}_{\bigsqcup_{\ell=1}^{N} B^\ell}(x) \right|^2 |\psi|^2(x) d\mu.$$

Point-wise, the integrand converges to 0 almost everywhere, since there is an N for every $x \in \bigsqcup_{\ell\in\mathbb{N}} B^\ell$ such that $x \in \bigsqcup_{\ell=1}^{n} B^\ell\ \forall n \ge N$. Moreover, by the triangle inequality, the integrand is dominated by $2|\psi|^2$, which is in L^2. Hence, by the dominated convergence theorem, $\left\| P\Big(\bigsqcup_{\ell=1}^{\infty} B^\ell\Big)\psi - \sum_{\ell=1}^{N} P(B^\ell)\psi \right\|^2 \xrightarrow{N\to\infty} 0$ and P is strongly σ-additive. With all, P is a PVM.

Item (ii) : By Lemma 23, $(\Phi(f), D(\mathcal{M}_f))$ is a well-defined densely defined operator for each measurable $f : \mathbb{R}^I \to \mathbb{C}$. In particular, if $f \in \mathcal{M}_b(\mathbb{R}^I, \odot_{j\in I}\mathfrak{B}(\mathbb{R}))$,

$$\|\Phi(f)\|_{op} = \sup_{\substack{\psi\in L^2(\mathbb{R}^I,d\mu),\\ \|\psi\|=1}} \|\mathcal{M}_f\psi\| = \sup_{\substack{\psi\in L^2(\mathbb{R}^I,d\mu),\\ \|\psi\|=1}} \left(\int_{\mathbb{R}^I} |f(x)\psi(x)|^2 d\mu \right)^{1/2} \le \|f\|_\infty. \tag{5.41}$$

- Now, for simple functions $s \in \mathcal{M}_b(\mathbb{R}, \odot_{j\in I}\mathfrak{B}(\mathbb{R}))$, say, $s = \sum_{\ell=1}^{N} \alpha_\ell \mathbb{1}_{B^\ell}$, $N \in \mathbb{N}$, $B^\ell \in \odot_{j\in I}\mathfrak{B}(\mathbb{R})$, $\alpha_\ell \in \mathbb{C}$,

$$\Phi(s) = \mathcal{M}_{\sum_{\ell=1}^{N} \alpha_\ell \mathbb{1}_{B^\ell}} = \sum_{\ell=1}^{N} \alpha_\ell \mathcal{M}_{B^\ell} = \sum_{\ell=1}^{N} \alpha_\ell P(B^\ell).$$

So Φ agrees with the action of the functional calculus of P on simple functions. As a map from $\mathcal{M}_b(\mathbb{R}^I, \odot_{j\in I}\mathfrak{B}(\mathbb{R}))$ to $\mathcal{L}(L^2(\mathbb{R}^I, d\mu))$, Φ is bounded, because by (5.41), $\|\Phi\|_{op} \leq 1$. Hence, $\Phi \restriction_{\mathcal{M}_b}$ is continuous. Since the simple functions are dense in $(\mathcal{M}_b(\mathbb{R}^I, \odot_{j\in I}\mathfrak{B}(\mathbb{R})), \|\cdot\|)$ and two continuous maps between Hausdorff spaces that agree on a dense subset agree everywhere, then Φ must act as the functional calculus of P on $\mathcal{M}_b$.

- The only missing thing to prove that Φ is P's functional calculus is that for an arbitrary measurable $f : \mathbb{R}^\infty \to \mathbb{C}$, $D_f = \mathcal{M}_f$. Let $s_n = \sum_{\ell=1}^{N_n} \alpha_\ell^n \mathbb{1}_{B_\ell^n}$ be a strictly increasing sequence of non-negative simple functions approximating $|f|^2$ (Thm. (Folland, 1999)). Then,

$$\int_{x\in\mathbb{R}^I} |\psi(x)|^2|f(x)|^2 d\mu \overset{\text{(mont. conv.)}}{=} \lim_{n\to\infty}\sum_{\ell=1}^{N_n}\alpha_\ell \int_{x\in\mathbb{R}^I} |\psi(x)|^2 \mathbb{1}_{B_\ell^n}(x) d\mu = \lim_{n\to\infty}\sum_{\ell=1}^{N_n}\alpha_\ell \langle\psi, P(B_\ell^n)\psi\rangle =$$

$$= \lim_{n\to\infty}\sum_{\ell=1}^{N_n}\alpha_\ell d\mu_\psi(B_\ell^n) = \lim_{n\to\infty}\sum_{\ell=1}^{N_n}\alpha_\ell \int_{x\in\mathbb{R}^I} \mathbb{1}_{B_\ell^n}(x) d\mu_\psi \overset{\text{(mont. conv.)}}{=} \int_{x\in\mathbb{R}^I} |f(x)|^2 d\mu_\psi.$$

Therefore, $\psi \in D(\mathcal{M}_f) \iff \psi \in D_f$.

Item (iii): Trivially, all the PVMs $(P_j)_{j\in I}$ from above commute with each other:

$$(P_j(B_j)P_k(B_k)\psi)(x) = \mathbb{1}_{B_j}(x_j)\mathbb{1}_{B_k}(x_k)\psi(x) = (P_k(B_k)P_j(B_j)\psi)(x). \tag{5.42}$$

Hence, by Theorem 19, there exists a joint PVM $\odot_{j\in I}P_j$. Now, P satisfies that if I is finite, say, if $I = \{1, ..., n\}$,

$$P(B_1 \times \cdots \times B_n) = \mathcal{M}_{\mathbb{1}_{B_1\times\cdots\times B_1}} = \mathcal{M}_{\mathbb{1}_{B_1}\circ\pi_1} \cdots \mathcal{M}_{\mathbb{1}_{B_n}\circ\pi_n} = P_1(B_1)\cdots P_n(B_n)$$

for all $B_j \in \mathfrak{B}(\mathbb{R})$. But by Theorem 14, the product PVM, $\odot_{j=1}^n P_j$, is the unique PVM with this property. Hence $P = \odot_{j=1}^n P_j$ and thus, by construction it is the joint diagonalization PVM of $(\hat{x}_j, D(\hat{x}_j))_{j\in\{1,...,n\}}$.

- If I is not finite, then, for any finite $J \subseteq I$ and $E_J \in \odot_{j\in J}\mathfrak{B}(\mathbb{R})$

$$P\left(E_J \times \prod_{j\in I\setminus J}\mathbb{R}\right) = \mathcal{M}_{\mathbb{1}_{(E_J\times\prod_{j\in I\setminus J}\mathbb{R})}} = \mathcal{M}_{\mathbb{1}_{E_J}\circ\pi_{J\leftarrow I}} \overset{\left(\substack{\text{this is the } P \text{ for} \\ I=J \text{ which we just} \\ \text{proved equals } \odot_{j\in J}P_j}\right)}{=} (\odot_{j\in J}P_j)(E_J).$$

But, by Theorem 15, there is a unique PVM with this property and it is $\odot_{j\in I}P_j$. Hence, $P = \odot_{j\in I}P_j$ and thus, by definition, P is the joint diagonalization PVM of $(\hat{x}_j, D(\hat{x}_j))_{j\in I}$.

o.ε.δ.

5.3.3 The Technical Checks for the "Coordinate" Strategy

Finally, we want to transport the PVMs and product PVM obtained in the generic $L^2(\mathbb{R}^\infty, d^\infty\mu)$-space to each layer of the ITP, assemble them together and check that we get indeed the PVMs of $\widehat{q}_k$. For that, however, several technical checks need to be made first (e.g., that a unitary maps a PVM to a PVM, that an uncountable direct sum of PVMs is still a PVM, that the uncountable direct sum commutes with the infinite product of PVMs etc.). We gather these technical checks in this section.

Lemma 25. Let $\mathcal{H}, \mathcal{K}$ be Hilbert spaces with unitary isomorphism $U : \mathcal{H} \to \mathcal{K}$. Then, given a PVM $(\mathcal{H}, (X, \Sigma), P^{\mathcal{H}})$ the isomorphism induces a PVM $(\mathcal{K}, (X, \Sigma), P^{\mathcal{K}})$ via $P^{\mathcal{K}}(\cdot) := UP^{\mathcal{H}}U^{-1}$. We call it the *push-forwarded PVM.* In particular, the functional calculus of $P^{\mathcal{K}}$ is such that for all measurable $f : X \to \mathbb{C}$,

$$(\ \Phi^{\mathcal{K}}(f), \ D_f^{\mathcal{K}} \ ) = (\ U\Phi^{\mathcal{H}}(f)U^{-1}, \ UD_f^{\mathcal{H}} \ ). \tag{5.43}$$

Given a commuting family of PVMs $(\mathcal{H}, (X_j, \mathfrak{B}(X_j)), P_j^{\mathcal{H}})_{j\in I}$ (with X_j locally compact, second countable and Hausdorff topological spaces), their push-forwarded PVMs $(P_j^{\mathcal{K}})_{j\in I}$ are also a commuting family. In such a case, (the product PVM is well-defined by Theorems 14 and 15 and) the product $\odot$ of PVMs commutes with the push-forward, i.e.,

$$\odot_{j\in I} P_j^{\mathcal{K}} = U \ (\odot_{j\in I} P_j^{\mathcal{H}}) \ U^{-1}.$$

♦

Proof: We check that $P^{\mathcal{K}}$ is a PVM. Let $B \in \Sigma$ arbitrary. Each $P^{\mathcal{K}}(B)$ is bounded and idempotent: $(P^{\mathcal{K}}(B))^2 = UP^{\mathcal{H}}(B)U^{-1}UP^{\mathcal{H}}(B)U = P^{\mathcal{K}}(B)$. Each $P^{\mathcal{K}}(B)$ is self-adjoint $P^{\mathcal{K}}(B)^* = (U^{-1})^*P^{\mathcal{H}}(B)^*U^* = UP^{\mathcal{H}}(B)U^{-1} = P^{\mathcal{K}}(B)$. Given $(B_k)_{k\in\mathbb{N}} \subseteq \Sigma$ pairwise disjoint,

$$P^{\mathcal{K}}\Big(\bigsqcup_{k\in\mathbb{N}} B_k\Big)\psi = UP^{\mathcal{H}}\Big(\bigsqcup_{k\in\mathbb{N}} B_k\Big)U^{-1}\psi = U\Big(\lim_{N\to\infty}\Big(\sum_{k=1}^{N} P^{\mathcal{H}}(B_k)U^{-1}\psi\Big)\Big) \overset{(U \text{ conts})}{=}$$

$$= \lim_{N\to\infty}\sum_{k=1}^{N}\Big(UP^{\mathcal{H}}(B_k)U^{-1}\psi\Big) = \sum_{k=1}^{\infty} P^{\mathcal{K}}(B_k)\psi.$$

Therefore, $P^{\mathcal{K}}$ is indeed a PVM.

- Now we check the relation between the spectral measures: let $\psi \in \mathcal{H}$,

$$d\mu_{U\psi}^{\mathcal{K}}(\cdot) = \langle U\psi, P^{\mathcal{K}}(\cdot)U\psi\rangle = \langle U\psi, UP^{\mathcal{H}}(\cdot)U^{-1}U\psi\rangle \overset{(U \text{ isometry})}{=} \langle \psi, P^{\mathcal{H}}(\cdot)\psi\rangle = d\mu_{\psi}^{\mathcal{H}}(\cdot).$$

Hence, for $f : X \to \mathbb{C}$,

$$\psi \in D_f^{\mathcal{H}} \iff \int_{x\in X} |f|^2 d\mu_{\psi}^{\mathcal{H}} < +\infty \iff \int_{x\in X} |f|^2 d\mu_{U\psi}^{\mathcal{K}} < +\infty \iff U\psi \in D_f^{\mathcal{K}}.$$

And therefore, $D_f^{\mathcal{K}} = UD_f^{\mathcal{H}}$. Finally, let $f_n := f\mathbb{1}_{\Omega_n}$ for $\Omega_n = |f|^{-1}([0, n))$ and let $s_n^k = \sum_{j=1}^{N_k} c_j^{n,k}\mathbb{1}_{B_j^{n,k}}$ be a sequence of simple functions uniformly approximating $f_n \in \mathcal{M}_b(X, \Sigma)$ as $k \to \infty$ (Thm. 2.10(Folland, 1999)). Then, for $\varphi \in D_f^{\mathcal{K}}$,

$$\Phi^{\mathcal{K}}(f)\varphi = \lim_{n\to\infty}\Phi^{\mathcal{K}}(f_n)\varphi = \lim_{n\to\infty}\lim_{k\to\infty}\sum_{j=1}^{N_k} c_j^{n,k}P^{\mathcal{K}}(B_j^{n,k})\varphi = \lim_{n\to\infty}\lim_{k\to\infty}\sum_{j=1}^{N_n} c_j^{n,k}UP^{\mathcal{H}}(B_j^{n,k})U^{-1}\varphi =$$

$$\overset{(U \text{ conts.})}{=} U\lim_{n\to\infty}\lim_{k\to\infty}\sum_{j=1}^{N_k} c_j^{n,k}P^{\mathcal{H}}(B_j^{n,k})U^{-1}\varphi = U\lim_{n\to\infty}\Phi^{\mathcal{H}}(f_n)(U^{-1}\varphi) \overset{(\text{by def})}{=} U\Phi^{\mathcal{H}}(f)U^{-1}\varphi.$$

- Next, we check that the push-forwarded PVMs $(P_j^{\mathcal{K}})_{j\in I}$ commute whenever $(P_j^{\mathcal{H}})_{j\in I}$ do.

Let $j, k \in I, \ B_k \in \Sigma_k, B_j \in \Sigma_j$. Then,

$$P_j^{\mathcal{K}}(B_j)P_k^{\mathcal{K}}(B_k) = UP_j^{\mathcal{H}}(B_j)\cancel{U^{-1}U}P_k^{\mathcal{H}}(B_k)U^{-1} = UP_k^{\mathcal{H}}(B_k)P_j^{\mathcal{H}}(B_j)U^{-1} = P_k^{\mathcal{K}}(B_k)P_j^{\mathcal{K}}(B_j).$$

- Assume I is finite, say $I = \{1, ..., n\}$. Then, if $B_j \in \Sigma_j \ \forall j \in \{1, ..., n\}$,

$$(\odot_{j=1}^{n} P_j^{\mathcal{K}})(B_1 \times \cdots \times B_n) = P_1^{\mathcal{K}}(B_1) \cdots P_n^{\mathcal{K}}(B_n) = UP_1^{\mathcal{H}}(B_1)U^{-1} \cdots UP_n^{\mathcal{H}}(B_n)U^{-1} =$$

$$= UP_1^{\mathcal{H}}(B_1) \cdots P_n^{\mathcal{H}}(B_n)U^{-1} = U(\odot_{j=1}^{n} P_j^{\mathcal{H}})(B_1 \times \cdots \times B_n)U^{-1}.$$

By the uniqueness characterization of the finite product PVM given in Thm. 14, $\odot_{j=1}^{n} P_j^{\mathcal{K}} = U(\odot_{j=1}^{n} P_j^{\mathcal{H}})U^{-1}$. Assume now that I is infinite, then for finite $J \subset I$, let $B_J \in \odot_{j \in J} \Sigma_j$.

$$(\odot_{j \in I} P_j^{\mathcal{K}})\Big(B_J \times \prod_{j \in I \setminus J} X_j\Big) = (\odot_{j \in J} P_j^{\mathcal{K}})(B_J) \overset{\left(\substack{\text{for finite products push-forward} \\ \text{and product commute}}\right)}{=}$$

$$= U(\odot_{j \in J} P_j^{\mathcal{H}})(B_J)U^{-1} = U(\odot_{j \in I} P_j^{\mathcal{H}})\Big(B_J \times \prod_{j \in I \setminus J} X_j\Big)U^{-1}.$$

By the uniqueness characterization of the arbitrary product PVM given in Thm. 15, $\odot_{j \in I} P_j^{\mathcal{K}} = U(\odot_{j \in I} P_j^{\mathcal{H}})U^{-1}$. ***o.ε.δ.***

Proposition 36. Given a class $\mathfrak{C} \in \Gamma$ and a choice of generator $\bigotimes_{j \in \mathbb{N}} \rho_j^{\mathfrak{C}}$, with associated unitary identification $W_{\mathfrak{C}}^{-1} : L^2(\mathbb{R}^\infty, \odot_{j \in \mathbb{N}} |\rho_j^{\mathfrak{C}}|^2 dx) \longrightarrow \otimes_{k \in \mathbb{N}}^{\mathfrak{C}} L^2(\mathbb{R}, dx)$ as in Theorem 10,

(i) the push-forwards by $W_{\mathfrak{C}}^{-1}$ of the PVMs found in Propositions 34 make up the (unique) PVMs $(\otimes_{k \in \mathbb{N}}^{\mathfrak{C}} L^2(\mathbb{R}, dx),\ (\mathbb{R}, \mathfrak{B}(\mathbb{R})),\ P_j^{\mathfrak{C}})_{j \in \mathbb{N}}$ that act on elementary tensor products $\psi_1 \otimes \psi_2 \otimes \cdots \in \otimes_{k \in \mathbb{N}}^{\mathfrak{C}} L^2(\mathbb{R}, dx)$ as

$$P_j^{\mathfrak{C}}(B_j)(\psi_1 \otimes \psi_2 \otimes \cdots) = \psi_1 \otimes \psi_2 \otimes \cdots \otimes (\mathbb{1}_{B_j} \psi_j) \otimes \psi_{j+1} \otimes \cdots, \qquad \forall B_j \in \mathfrak{B}(\mathbb{R}). \tag{5.44}$$

Moreover, the functional calculus $\Phi_k^{\mathfrak{C}}$ of $P_k^{\mathfrak{C}}$ is such that for arbitrary measurable $g : \mathbb{R} \to \mathbb{C}$, $(\Phi_k^{\mathfrak{C}}(g),\ D_{k,g}^{\mathfrak{C}}) = (W_{\mathfrak{C}}^{-1} \mathcal{M}_{g \circ \pi_k} W_{\mathfrak{C}},\ W_{\mathfrak{C}}^{-1} D(\mathcal{M}_{g \circ \pi_k}))$.

(ii) $\{P_j^{\mathfrak{C}}\}_{j \in \mathbb{N}}$ is a commuting family of PVMs and their product PVM $(\otimes_{k \in \mathbb{N}}^{\mathfrak{C}} L^2(\mathbb{R}, dx),\ (\mathbb{R}^\infty, \mathfrak{B}(\mathbb{R}^\infty)),\ \odot_{j \in \mathbb{N}} P_j^{\mathfrak{C}})$ acts on elementary tensor products $\psi_1 \otimes \psi_2 \otimes \cdots \in \otimes_{k \in \mathbb{N}}^{\mathfrak{C}} L^2(\mathbb{R}, dx)$ as

$$(\odot_{j \in \mathbb{N}} P_j^{\mathfrak{C}})(B_1 \times \cdots B_n \times \mathbb{R} \times \cdots)(\psi_1 \otimes \psi_2 \otimes \cdots) = (\mathbb{1}_{B_1} \psi_1) \otimes \cdots \otimes (\mathbb{1}_{B_n} \psi_n) \otimes \psi_{n+1} \otimes \cdots$$

for all $n \in \mathbb{N}$, $B_j \in \mathfrak{B}(\mathbb{R})$.

(iii) Define for an arbitrary measurable $f : \mathbb{R}^\infty \to \mathbb{C}$, the domain

$$D_f^{\mathfrak{C}} := W_{\mathfrak{C}}^{-1} D(\mathcal{M}_f) = \left\{ \Psi \in \otimes_{k \in \mathbb{N}}^{\mathfrak{C}} L^2(\mathbb{R}, dx) \ \middle| \ \int_{x \in \mathbb{R}^\infty} |f(x_1, x_2, ...)|^2 |(W_{\mathfrak{C}} \Psi)(x_1, x_2, ...)|^2 d^\infty \mu_{\rho^{\mathfrak{C}}} < \infty \right\}.$$

Then, the functional calculus $\Phi^{\mathfrak{C}}$ associated to the joint PVM $(\odot_{j \in \mathbb{N}} P_j^{\mathfrak{C}})$ satisfies $\Phi^{\mathfrak{C}}(f) = W_{\mathfrak{C}}^{-1} \mathcal{M}_f W_{\mathfrak{C}} \restriction_{D_f^{\mathfrak{C}}}$. Given a measurable $g : \mathbb{R}^\infty \longrightarrow \mathbb{C}$, $g(x_1, x_2, ...) = g_1(x_1) \cdots g_n(x_n)$, an elementary tensor product $\psi_1 \otimes \cdots \otimes \psi_n \otimes \rho_{n+1}^{\mathfrak{C}} \otimes \cdots \in \otimes_{k \in \mathbb{N}}^{\mathfrak{C}} L^2(\mathbb{R}, dx)$ is in $D_g^{\mathfrak{C}}$ *if and only if*

$$\int_{x \in \mathbb{R}^\infty} \left| f(x_1, x_2, ...) \frac{\psi_1(x_1) \cdots \psi_n(x_n)}{\rho_1^{\mathfrak{C}}(x_1) \cdots \rho_n^{\mathfrak{C}}(x_n)} \right|^2 d^\infty \mu_{\rho^{\mathfrak{C}}} < +\infty, \tag{5.45}$$

and if so,

$$\Phi^{\mathfrak{C}}(g)(\psi_1 \otimes \cdots \otimes \psi_n \otimes \rho_{n+1}^{\mathfrak{C}} \otimes \cdots) = (g_1 \psi_1) \otimes \cdots \otimes (g_n \psi_n) \otimes \rho_{n+1}^{\mathfrak{C}} \otimes \cdots. \tag{5.46}$$

♦

Proof: **Item (i) :** By Lemma 25, for $B_k \in \mathfrak{B}(\mathbb{R})$, the push-forwarded PVMs are such that $P_j^{\mathfrak{C}}(B_j) = W_{\mathfrak{C}}^{-1} P_j(B_j) W_{\mathfrak{C}}$. On elementary tensor products $\psi_1 \otimes \psi_2 \otimes \cdots \in \otimes_{k\in\mathbb{N}}^{\mathfrak{C}} L^2(\mathbb{R}, dx)$ they act as:

$$P_j^{\mathfrak{C}}(B_j)(\psi_1 \otimes \psi_2 \otimes \cdots) = \Big(W_{\mathfrak{C}}^{-1} P_j(B_j) W_{\mathfrak{C}}\Big)(\psi_1 \otimes \psi_2 \otimes \cdots) =$$

$$(\text{ use that by Prop. 24}, \quad (\psi_1 \otimes \cdots \otimes \psi_N \otimes \rho_{N+1}^{\mathfrak{C}} \otimes \cdots) \xrightarrow[N\to\infty]{\text{norm}} \psi_1 \otimes \psi_2 \otimes \cdots \ . \tag{5.47}$$

$$\overset{(W_{\mathfrak{C}}, P_j \text{ conts.})}{=} \lim_{N\to\infty} \left(W_{\mathfrak{C}}^{-1} P_j(B_j) \Big[(x_1, x_2, \ldots) \longmapsto \frac{\psi_1(x_1)\cdots\psi_N(x_N)}{\rho_1^{\mathfrak{C}}(x_1)\cdots\rho_N^{\mathfrak{C}}(x_N)} \Big] \right) =$$

$$= \lim_{N\to\infty} W_{\mathfrak{C}}^{-1} \Big[(x_1, x_2, \ldots) \longmapsto \frac{\psi_1(x_1)\cdots \mathbb{1}_{B_j}(x_j)\psi_j(x_j)\cdots\psi_N(x_N)}{\rho_1^{\mathfrak{C}}(x_1)\cdots\rho_N^{\mathfrak{C}}(x_N)} \Big] = \psi_1 \otimes \cdots \otimes (\mathbb{1}_{B_j}\psi_j) \otimes \psi_{j+1} \otimes \cdots$$

- Uniqueness comes from the fact that this fully specifies how each $P_j^{\mathfrak{C}}(B_j)$ acts on linear combinations of elementary tensor products, which are dense in $\bigotimes_{j\in\mathbb{N}}^{\mathfrak{C}} L^2(\mathbb{R}, dx)$. Since $P_j^{\mathfrak{C}}(B_j)$ is continuous, any other continuous operator that acts like $P_j^{\mathfrak{C}}(B_j)$ in (5.44), must coincide everywhere else too (Lemma 32).

- Finally, the claim on the functional calculus follows from Lemma 24 and Proposition 34.

Item (ii) : By Lemma 25, $\odot_{j\in I} P_j^{\mathfrak{C}} = W_{\mathfrak{C}}^{-1}(\odot_{j\in I} P_j) W_{\mathfrak{C}}$. Hence, for $n \in \mathbb{N}$ and $B_j \in \mathfrak{B}(\mathbb{R})$,

$$(\odot_{j\in I} P_j^{\mathfrak{C}})(B_1 \times \cdots \times B_n \times \mathbb{R} \times \cdots)(\psi_1 \otimes \psi_2 \otimes \cdots) =$$

$$= \left(W_{\mathfrak{C}}^{-1} (\odot_{j\in I} P_j)(B_1 \times \cdots \times B_n \times \mathbb{R} \times \cdots) W_{\mathfrak{C}} \right) (\psi_1 \otimes \psi_2 \otimes \cdots) \overset{((5.47)\& \ W_{\mathfrak{C}}, \ (\odot_j P_j)(B) \text{ conts.})}{=}$$

$$= \lim_{N\to\infty} \left(W_{\mathfrak{C}}^{-1} (\odot_{j\in I} P_j)(B_1 \times \cdots \times B_n \times \mathbb{R} \times \cdots)) \Big[(x_1, x_2, \ldots) \longmapsto \frac{\psi_1(x_1)\cdots\psi_N(x_N)}{\rho_1^{\mathfrak{C}}(x_1)\cdots\rho_N^{\mathfrak{C}}(x_N)} \Big] \right) =$$

$$\overset{(\odot_{j\in I} P_j(B) = \mathcal{M}_{\mathbb{1}_B})}{=} \lim_{N\to\infty} W_{\mathfrak{C}}^{-1} \Big[(x_1, x_2, \ldots) \longmapsto \frac{\mathbb{1}_{B_1}(x_1)\psi_1(x_1)\cdots \mathbb{1}_{B_n}\psi_n(x_n)\psi_{n+1}(x_{n+1})\cdots\psi_N(x_N)}{\rho_1^{\mathfrak{C}}(x_1)\cdots\rho_N^{\mathfrak{C}}(x_N)} \Big] =$$

$$= (\mathbb{1}_{B_1}\psi_1) \otimes \cdots \otimes (\mathbb{1}_{B_n}\psi_n) \otimes \psi_{n+1} \otimes \cdots \ .$$

Item (iii): The claim on the domain is proven in Lemma 25 and Proposition 35. In particular, by Lemma 25, $\Phi^{\mathfrak{C}}(f) = W_{\mathfrak{C}}^{-1} \Phi(f) W_{\mathfrak{C}} = W_{\mathfrak{C}}^{-1} \mathcal{M}_f W_{\mathfrak{C}}$. Hence, for factorizable g as in the statement:

$$\Phi^{\mathfrak{C}}(g)(\psi_1 \otimes \cdots \otimes \psi_n \otimes \rho_{n+1}^{\mathfrak{C}} \otimes \cdots) \cdots = W_{\mathfrak{C}}^{-1} \mathcal{M}_g W_{\mathfrak{C}} (\psi_1 \otimes \cdots \otimes \psi_n \otimes \rho_{n+1}^{\mathfrak{C}} \otimes \cdots) =$$

$$= W_{\mathfrak{C}}^{-1} \mathcal{M}_g \Big[(x_1, x_2, \ldots) \longmapsto \frac{\psi_1(x_1)\cdots\psi_n(x_n)}{\rho_1^{\mathfrak{C}}(x_1)\cdots\rho_n^{\mathfrak{C}}(x_n)} \Big] =$$

$$= W_{\mathfrak{C}}^{-1} [(x_1, x_2, \ldots) \longmapsto \frac{g_1(x_1)\psi_1(x_1)\cdots g_n(x_n)\psi_n(x_n)}{\rho_1^{\mathfrak{C}}(x_1)\cdots\rho_n^{\mathfrak{C}}(x_n)} \Big] = g_1\psi_1 \otimes \cdots \otimes g_n\psi_n \otimes \rho_{n+1}^{\mathfrak{C}} \cdots \ .$$

ο.ε.δ.

Proposition 37. Let $\mathcal{H}$ be a Hilbert space with orthogonal decomposition $\mathcal{H} = \bigoplus_{\mathfrak{N}\in\mathfrak{X}} \mathcal{H}_{\mathfrak{N}}$ for an arbitrary $\mathfrak{X}$, and let $(\mathcal{H}_{\mathfrak{N}}, \ (X, \Sigma), \ P_{\mathfrak{N}})$ be a PVM for each sector $\mathfrak{N} \in \mathfrak{X}$.

(i) The map $P : \Sigma \to \mathcal{L}(\mathcal{H})$, $P(B) := \oplus_{\mathfrak{N}\in\mathfrak{X}} P_{\mathfrak{N}}(B)$ for $B \in \Sigma$ is a PVM on $\mathcal{H}$. We will denote it by $\oplus_{\mathfrak{N}\in\mathfrak{X}} P_{\mathfrak{N}}$.

(ii) Given Φ and $\Phi_{\mathfrak{N}}$ are the functional calculus of P and $P_{\mathfrak{N}}$ respectively, for each measurable $f : X \to \mathbb{C}$, following the notation in Prop. 18,

$$(\Phi(f),\ D_f) = \Big(\oplus_{\mathfrak{N}\in\mathfrak{X}}\ \Phi_{\mathfrak{N}}(f),\ \oplus_{\mathfrak{N}\in\mathfrak{X}} D_f^{\mathfrak{N}}\Big), \tag{5.48}$$

where $(\Phi_{\mathfrak{N}}(f), D_f^{\mathfrak{N}})$ is the operators given in $\mathcal{H}_{\mathfrak{N}}$ by the PVM $P_{\mathfrak{N}}$.

(iii) The spectral measure associated with $\Psi = (\psi_{\mathfrak{N}})_{\mathfrak{N}\in\mathfrak{X}} \in \oplus_{\mathfrak{N}\in\mathfrak{X}}\mathcal{H}_{\mathfrak{N}}$ is such that for any $B \in \Sigma$,

$$d\mu_{\Psi}(B) = \sum_{\mathfrak{N}\in\mathfrak{X}} d^{\mathfrak{N}}\mu_{\psi_{\mathfrak{N}}}(B) \qquad \text{(in the sense of Def. 30)}, \tag{5.49}$$

where $d^{\mathfrak{N}}\mu_{\psi_{\mathfrak{N}}}$ is the spectral measure that $P_{\mathfrak{N}}$ attributes to $\psi_{\mathfrak{N}} \in \mathcal{H}_{\mathfrak{N}}$. ◆

Proof: **Item (i)** First, by Prop. 18.(i), $\oplus_{\mathfrak{N}\in\mathfrak{X}}P_{\mathfrak{N}}$ is a bounded operator because $D(P_{\mathfrak{N}}(B)) = \mathcal{H}_{\mathfrak{N}}$ for all $\mathfrak{N} \in \mathfrak{X}$. It is also idempotent since, given $F = (f_{\mathfrak{N}})_{\mathfrak{N}\in\mathfrak{X}} \in \oplus_{\mathfrak{N}\in\mathfrak{X}}\mathcal{H}_{\mathfrak{N}}$,

$$\left(\oplus_{\mathfrak{N}\in\mathfrak{X}}P_{\mathfrak{N}}(B)\right)^2 F = (\oplus_{\mathfrak{N}\in\mathfrak{X}}P_{\mathfrak{N}}(B))(P_{\mathfrak{N}}(B)f_{\mathfrak{N}})_{\mathfrak{N}\in\mathfrak{X}} = (P_{\mathfrak{N}}(B)^2 f_{\mathfrak{N}})_{\mathfrak{N}\in\mathfrak{X}} = (P_{\mathfrak{N}}(B)f_{\mathfrak{N}})_{\mathfrak{N}\in\mathfrak{X}}, \tag{5.50}$$

which equals $(\oplus_{\mathfrak{N}\in\mathfrak{X}}P_{\mathfrak{N}}(B))F$ by definition. It is self-adjoint by Prop. 18.(iii) because each $P_{\mathfrak{N}}(B)$ is so. Finally, given $(B_n)_{n\in\mathbb{N}} \subseteq \Sigma$ pairwise disjoint,

$$\overset{\oplus_{\mathfrak{N}}\mathcal{H}_{\mathfrak{N}}}{\lim_{N\to\infty}} \sum_{n=1}^{N} \Big(\oplus_{\mathfrak{N}\in\mathfrak{X}}\ P_{\mathfrak{N}}(B_n)\Big)F \overset{\text{(by def)}}{=} \overset{\oplus_{\mathfrak{N}}\mathcal{H}_{\mathfrak{N}}}{\lim_{N\to\infty}} \sum_{n=1}^{N} \Big(P_{\mathfrak{N}}(B_n)f_{\mathfrak{N}}\Big)_{\mathfrak{N}\in\mathfrak{X}} \overset{(\star)}{=}$$

$$= \Big(P_{\mathfrak{N}}\Big(\bigsqcup_{n\in\mathbb{N}} B_n\Big)f_{\mathfrak{N}}\Big)_{\mathfrak{N}\in\mathfrak{X}} \overset{\text{(by def)}}{=} \Big(\oplus_{\mathfrak{N}\in\mathfrak{X}}\ P_{\mathfrak{N}}\Big(\bigsqcup_{n\in\mathbb{N}} B_n\Big)\Big)F. \tag{5.51}$$

if $(\star)$ holds true, this implies that $P(\cdot) := \oplus_{\mathfrak{N}\in\mathfrak{X}}P_{\mathfrak{N}}(\cdot)$ is strongly σ-additive and hence a PVM.

- To prove the $(\star)$, we need to show that the following quantity tends to 0 as $N \to \infty$:

$$\left\|\sum_{n=1}^{N}\Big(P_{\mathfrak{N}}(B_n)f_{\mathfrak{N}}\Big)_{\mathfrak{N}\in\mathfrak{X}} - \Big(P_{\mathfrak{N}}\Big(\bigsqcup_{n\in\mathbb{N}} B_n\Big)f_{\mathfrak{N}}\Big)_{\mathfrak{N}\in\mathfrak{X}}\right\|^2_{\oplus_{\mathfrak{N}}\mathcal{H}_{\mathfrak{N}}} = \left\|\sum_{n=1}^{N}\Big(P_{\mathfrak{N}}(B_n)f_{\mathfrak{N}}\Big)_{\mathfrak{N}\in\mathfrak{X}}\right\|^2 +$$

$$+\left\|\Big(P_{\mathfrak{N}}\Big(\bigsqcup_{n\in\mathbb{N}} B_n\Big)f_{\mathfrak{N}}\Big)_{\mathfrak{N}\in\mathfrak{X}}\right\|^2 - 2\,\mathrm{Re}\left\{\Big\langle\sum_{n=1}^{N}\Big(P_{\mathfrak{N}}(B_n)f_{\mathfrak{N}}\Big)_{\mathfrak{N}\in\mathfrak{X}},\ \Big(P_{\mathfrak{N}}\Big(\bigsqcup_{n\in\mathbb{N}} B_n\Big)f_{\mathfrak{N}}\Big)_{\mathfrak{N}\in\mathfrak{X}}\Big\rangle\right\}. \tag{5.52}$$

Observe that for $n, m \in \mathbb{N}$, $n \neq m$, $\Big(P_{\mathfrak{N}}(B_n)f_{\mathfrak{N}}\Big)_{\mathfrak{N}\in\mathfrak{X}} \perp \Big(P_{\mathfrak{N}}(B_m)f_{\mathfrak{N}}\Big)_{\mathfrak{N}\in\mathfrak{X}}$ because

$$\Big\langle\Big(P_{\mathfrak{N}}(B_n)f_{\mathfrak{N}}\Big)_{\mathfrak{N}\in\mathfrak{X}},\ \Big(P_{\mathfrak{N}}(B_m)f_{\mathfrak{N}}\Big)_{\mathfrak{N}\in\mathfrak{X}}\Big\rangle = \sum_{\mathfrak{N}\in\mathfrak{X}}\Big\langle P_{\mathfrak{N}}(B_n)f_{\mathfrak{N}},\ P_{\mathfrak{N}}(B_m)f_{\mathfrak{N}}\Big\rangle =$$

$$= \sum_{\mathfrak{N}\in\mathfrak{X}}\Big\langle f_{\mathfrak{N}},\ P_{\mathfrak{N}}(B_n)P_{\mathfrak{N}}(B_m)f_{\mathfrak{N}}\Big\rangle = 0,$$

where the last equality holds because $B_n \cap B_m = \emptyset$ and $P_{\mathfrak{N}}$ are PVMs —such that $P_{\mathfrak{N}}(B_n)P_{\mathfrak{N}}(B_m) = P_{\mathfrak{N}}(B_n \cap B_m) = 0$. Hence, by the Pythagorean theorem, the sum $\sum_{n=1}^{N}$ in the first term of (5.52)'s r.h.s can be taken out of the norm. Similarly, we can simplify the third term of that r.h.s as follows (using in (†) that $\sum_{n=1}^{N}\Big(P_{\mathfrak{N}}(B_n)f_{\mathfrak{N}}\Big)_{\mathfrak{N}\in\mathfrak{X}} = \Big(\sum_{n=1}^{N}P_{\mathfrak{N}}(B_n)f_{\mathfrak{N}}\Big)_{\mathfrak{N}\in\mathfrak{X}} = \Big(P_{\mathfrak{N}}(\bigcup_{j=1}^{N}B_n)f_{\mathfrak{N}}\Big)_{\mathfrak{N}\in\mathfrak{X}}$):

$$\Big\langle \sum_{n=1}^{N}\Big(P_{\mathfrak{N}}(B_n)f_{\mathfrak{N}}\Big)_{\mathfrak{N}\in\mathfrak{X}},\ \Big(P_{\mathfrak{N}}\Big(\bigsqcup_{n\in\mathbb{N}}B_n\Big)f_{\mathfrak{N}}\Big)_{\mathfrak{N}\in\mathfrak{X}}\Big\rangle \overset{(\dagger)}{=} \sum_{\mathfrak{N}\in\mathfrak{X}}\Big\langle P_{\mathfrak{N}}\Big(\bigsqcup_{n=1}^{N}B_n\Big)f_{\mathfrak{N}},\ P_{\mathfrak{N}}\Big(\bigsqcup_{n=1}^{\infty}B_n\Big)f_{\mathfrak{N}}\Big\rangle =$$

$$= \sum_{\mathfrak{N}\in\mathfrak{X}}\Big\langle f_{\mathfrak{N}},\ P_{\mathfrak{N}}\bigg(\Big(\bigsqcup_{n=1}^{N}B_n\Big)\bigcap\Big(\bigsqcup_{n=1}^{\infty}B_n\Big)\bigg)f_{\mathfrak{N}}\Big\rangle = \sum_{\mathfrak{N}\in\mathfrak{X}}\Big\langle f_{\mathfrak{N}},\ P_{\mathfrak{N}}\Big(\bigsqcup_{n=1}^{N}B_n\Big)f_{\mathfrak{N}}\Big\rangle = \sum_{\mathfrak{N}\in\mathfrak{X}}\Big\|P\Big(\bigsqcup_{n=1}^{N}B_n\Big)f_{\mathfrak{N}}\Big\|^2 =$$

$$= \sum_{\mathfrak{N}\in\mathfrak{X}}\Big\|\sum_{n=1}^{N}P(B_n)f_{\mathfrak{N}}\Big\|^2 \overset{(P(B_n)f_{\mathfrak{N}}\perp P(B_m)f_{\mathfrak{N}})}{=} \sum_{\mathfrak{N}\in\mathfrak{X}}\sum_{n=1}^{N}\|P(B_n)f_{\mathfrak{N}}\|^2.$$

With all, (5.52) equals

$$\sum_{n=1}^{N}\|(P_{\mathfrak{N}}(B_n)f_{\mathfrak{N}})_{\mathfrak{N}\in\mathfrak{X}}\|^2 + \Big\|\Big(P_{\mathfrak{N}}\Big(\bigsqcup_{n\in\mathbb{N}}B_n\Big)f_{\mathfrak{N}}\Big)_{\mathfrak{N}\in\mathfrak{X}}\Big\|^2 - 2\sum_{\mathfrak{N}\in\mathfrak{X}}\sum_{n=1}^{N}\|P(B_n)f_{\mathfrak{N}}\|^2 =$$

$$= \sum_{n=1}^{N}\sum_{\mathfrak{N}\in\mathfrak{X}}\|P_{\mathfrak{N}}(B_n)f_{\mathfrak{N}}\|^2 + \sum_{\mathfrak{N}\in\mathfrak{X}}\Big\|P_{\mathfrak{N}}\Big(\bigsqcup_{n\in\mathbb{N}}B_n\Big)f_{\mathfrak{N}}\Big\|^2 - 2\sum_{n=1}^{N}\sum_{\mathfrak{N}\in\mathfrak{X}}\|P(B_n)f_{\mathfrak{N}}\|^2 =$$

$$= \sum_{\mathfrak{N}\in\mathfrak{X}}\Big\|P_{\mathfrak{N}}\Big(\bigsqcup_{n\in\mathbb{N}}B_n\Big)f_{\mathfrak{N}}\Big\|^2 - \sum_{n=1}^{N}\sum_{\mathfrak{N}\in\mathfrak{X}}\|P(B_n)f_{\mathfrak{N}}\|^2. \tag{5.53}$$

We claim that

$$\lim_{N\to\infty}\sum_{n=1}^{N}\sum_{\mathfrak{N}\in\mathfrak{X}}\|P_{\mathfrak{N}}(B_n)f_{\mathfrak{N}}\|^2 = \sum_{\mathfrak{N}\in J}\Big\|P_{\mathfrak{N}}\Big(\bigsqcup_{n=1}^{\infty}B_n\Big)f_{\mathfrak{N}}\Big\|^2. \tag{5.54}$$

If this is true, because (5.53) equals (5.52), the $(\star)$ will be proven.

- Therefore, we just miss to check (5.54). By Lemma 14, $\exists J := \{\mathfrak{N}_k\}_{k\in\mathbb{N}} \subseteq \mathfrak{X}$ such that $f_{\mathfrak{N}} = 0$ if $\mathfrak{N} \in \mathfrak{X}\setminus J$. Hence, wherever we had $\sum_{\mathfrak{N}\in\mathfrak{X}}$ we could have put $\sum_{k=1}^{\infty}$ if we changed $\mathfrak{N} \mapsto \mathfrak{N}_k$. Now, consider this latter series as a Lebesgue integral over $\mathbb{N}$ relative to the counting measure $d\nu$. Then, the claimed equation (5.54) can be rewritten as

$$\lim_{N\to\infty}\sum_{n=1}^{N}\int_{k\in\mathbb{N}}\|P_{\mathfrak{N}_k}(B_n)f_{\mathfrak{N}_k}\|^2 d\nu = \int_{k\in\mathbb{N}}\Big\|P_{\mathfrak{N}_k}\Big(\bigcup_{n=1}^{\infty}B_n\Big)f_{\mathfrak{N}_k}\Big\|^2 d\nu. \tag{5.55}$$

But this equality holds by the monotone convergence theorem because:

- $\sum_{n=1}^{N}\|P_{\mathfrak{N}_k}(B_n)f_{\mathfrak{N}_k}\|^2$ is strictly increasing as N increases for every $k\in\mathbb{N}$
- for each $k\in\mathbb{N}$ (point-wise) $\sum_{n=1}^{N}\|P_{\mathfrak{N}_k}(B_n)f_{\mathfrak{N}_k}\|^2 \xrightarrow[N\to\infty]{} \|P_{\mathfrak{N}_k}(\bigcup_{n\in\mathbb{N}}B_n)f_{\mathfrak{N}_k}\|^2$. To see why: recall that $P_{\mathfrak{N}_k}(B_n)f_{\mathfrak{N}_k} \perp P_{\mathfrak{N}_k}(B_m)f_{\mathfrak{N}_k}$, so, $\sum_{n=1}^{N}\|P_{\mathfrak{N}_k}(B_n)f_{\mathfrak{N}_k}\|^2 = \Big\|\sum_{n=1}^{N}P_{\mathfrak{N}}(B_n)f_{\mathfrak{N}}\Big\|^2$ and because $P_{\mathfrak{N}_k}$ is a PVM, $\sum_{n=1}^{N}P_{\mathfrak{N}_k}(B_n)f_{\mathfrak{N}_k} \xrightarrow[N\to\infty]{} P_{\mathfrak{N}_k}(\sqcup_{n\in\mathbb{N}}B_n)f_{\mathfrak{N}_k}$ in $\mathcal{H}_{\mathfrak{N}_k}$-norm.

Item (ii): First of all, by Prop. 18, $\Big(\oplus_{\mathfrak{N}\in\mathfrak{X}}\Phi_{\mathfrak{N}}(f),\ \oplus_{\mathfrak{N}\in\mathfrak{X}}D_f^{\mathfrak{N}}\Big)$ is a well-defined densely defined operator. Now, let $f : X \to \mathbb{C}$ be an arbitrary measurable function. By Theorem 2.10 in (Folland, 1999), $|f|^2 : X \to [0,+\infty)$ is the point-wise limit of a monotonously increasing sequence of simple functions $(\sum_{k=1}^{N_n}\alpha_k^n\mathbb{1}_{E_k^n})_{n\in\mathbb{N}}$ for some $N_n\in\mathbb{N}$, $E_k^n\in\Sigma$, $\alpha_k^n \geq 0$. Fix an arbitrary $\Psi := (\psi_{\mathfrak{N}})_{\mathfrak{N}\in\mathfrak{X}} \in \oplus_{\mathfrak{N}\in\mathfrak{X}}\mathcal{H}_{\mathfrak{N}}$. Then, denoting by $d\mu_\Psi$ and $d^{\mathfrak{N}}\mu_{\psi_{\mathfrak{N}}}$ the spectral measures of Ψ and $\psi_{\mathfrak{N}}$ respectively, given by the PVMs $\oplus_{\mathfrak{N}\in\mathfrak{X}}P_{\mathfrak{N}}$ and $P_{\mathfrak{N}}$,

$$\int_{x\in X}|f|^2(x)d\mu_\Psi \overset{\text{(mont. conv)}}{=} \lim_{n\to\infty}\int_{x\in X}\sum_{k=1}^{N_n}\alpha_k^n\mathbb{1}_{E_k^n}d\mu_\Psi = \lim_{n\to\infty}\sum_{k=1}^{N_n}\alpha_k^n d\mu_\Psi(E_k^n) =$$

$$= \lim_{n\to\infty}\sum_{k=1}^{N_n}\alpha_k^n\Big\langle\Psi,(\oplus_{\mathfrak{N}\in\mathfrak{X}}P_{\mathfrak{N}})(E_k^n)\Psi\Big\rangle = \lim_{n\to\infty}\sum_{k=1}^{N_n}\alpha_k^n\sum_{\mathfrak{N}\in\mathfrak{X}}\Big\langle\psi_{\mathfrak{N}},P_{\mathfrak{N}}(E_k^n)\psi_{\mathfrak{N}}\Big\rangle = \lim_{n\to\infty}\sum_{\mathfrak{N}\in\mathfrak{X}}\sum_{k=1}^{N_n}\alpha_k^n\; d^{\mathfrak{N}}\mu_{\psi_{\mathfrak{N}}}(E_k^n) =$$

$$\overset{(\star\star)}{=}\sum_{\mathfrak{N}\in\mathfrak{X}}\lim_{n\to\infty}\sum_{k=1}^{N_n}\alpha_k^n\; d^{\mathfrak{N}}\mu_{\psi_{\mathfrak{N}}}(E_k^n) \overset{(\star\star\star)}{=} \sum_{\mathfrak{N}\in\mathfrak{X}}\int_{x\in X}|f|^2(x)d^{\mathfrak{N}}\mu_{\psi_{\mathfrak{N}}} \overset{\text{(Prop. 30)}}{=} \sum_{\mathfrak{N}\in\mathfrak{X}}\|\Phi_{\mathfrak{N}}(f)\psi_{\mathfrak{N}}\|^2. \tag{5.56}$$

If $(\star\star)$ and $(\star\star\star)$ hold true, then

$$\Psi\in D_f \overset{\text{(by def)}}{\iff} \int_{x\in X}|f|^2 d\mu_\psi < +\infty \overset{(5.56)}{\iff} \left(\begin{array}{c}\psi_{\mathfrak{N}}\in D_f^{\mathfrak{N}}\text{ —i.e., }\|\Phi_{\mathfrak{N}}(f)\psi_{\mathfrak{N}}\|^2<+\infty\text{—}\\ \forall\mathfrak{N}\in\mathfrak{X}\text{ and }\sum_{\mathfrak{N}\in\mathfrak{X}}\|\Phi_{\mathfrak{N}}(f)\psi_{\mathfrak{N}}\|^2<+\infty\end{array}\right) \overset{\text{(by def)}}{\iff} \Psi\in\oplus_{\mathfrak{N}\in\mathfrak{X}}D_f^{\mathfrak{N}}.$$

Hence, we would prove that indeed, $D_f = \oplus_{\mathfrak{N}\in\mathfrak{X}}D_f^{\mathfrak{N}}$. Now, $(\star\star\star)$ is just to apply for each summand $\mathfrak{N}\in\mathfrak{X}$ that

$$\lim_{n\to\infty}\underbrace{\sum_{k=1}^{N_n}\alpha_k^n\; d^{\mathfrak{N}}\mu_{\psi_{\mathfrak{N}}}(E_k^n)}_{g_n(\mathfrak{N})} \overset{\text{(by def)}}{=} \lim_{n\to\infty}\int_{x\in X}\Big(\sum_{k=1}^{N_n}\alpha_k^n\mathbb{1}_{E_k^n}(x)\Big)\,d^{\mathfrak{N}}\mu_{\psi_{\mathfrak{N}}} \overset{\text{(mont. conv)}}{=} \underbrace{\int_{x\in X}|f|^2(x)d^{\mathfrak{N}}\mu_{\psi_{\mathfrak{N}}}}_{g(\mathfrak{N})}.$$

At the same time, as we highlighted with the "under-braces", this last result is a statement of point-wise convergence (at each fixed $\mathfrak{N}\in\mathfrak{X}$) for the functions $g(\mathfrak{N}):\mathbb{N}\to[0,+\infty)$. Moreover, for each fixed $\mathfrak{N}\in\mathfrak{X}$, $g_n(\mathfrak{N})$ is monotonously increasing in $n\in\mathbb{N}$. As such, given the set $J:=\{\mathfrak{N}_k\}_{k\in\mathbb{N}}\subseteq\mathfrak{X}$ such that $f_{\mathfrak{N}}=0$ if $\mathfrak{N}\in\mathfrak{X}\setminus J$ (given by Lemma 14), we are allowed to apply the monotone convergence theorem to prove that

$$\lim_{n\to\infty}\int_{k\in\mathbb{N}}g_n(\mathfrak{N}_k)d\nu = \int_{k\in\mathbb{N}}g(\mathfrak{N}_k)d\nu, \tag{5.57}$$

where $d\nu$ is the counting measure. But writing the integrals as series and switching $\mathfrak{N}_k\mapsto\mathfrak{N}$, such that $\sum_{k=1}^{\infty}$ is exchanged by $\sum_{\mathfrak{N}\in\mathfrak{X}}$ (allowed, because $g(\mathfrak{N})=0$ if $\mathfrak{N}\in\mathfrak{X}\setminus J$), then (5.57) is exactly $(\star\star)$.

- Finally, we check the operator actions. Let $f:X\to\mathbb{C}$ be arbitrary measurable, $f_n:=f\mathbb{1}_{\Omega_n}$, $\Omega_n:=|f|^{-1}([0,n))$ and let $(s_n^\ell)_{\ell\in\mathbb{N}}$ be a monotonously increasing simple function approximation for f_n, say, $s_n^\ell:=\sum_{k=1}^{N_n^\ell}\alpha_k^{n,\ell}\mathbb{1}_{E_k^{n,\ell}}$. Then, for any $\Psi\in D_f$, by definition,

$$\Phi(f)\Psi = \overset{\oplus_{\mathfrak{N}}\mathcal{H}_{\mathfrak{N}}}{\lim_{n\to\infty}}\Phi(f_n)\Psi = \overset{\oplus_{\mathfrak{N}}\mathcal{H}_{\mathfrak{N}}}{\lim_{n\to\infty}}\;\overset{\|\cdot\|_{op}}{\lim_{\ell\to\infty}}\sum_{k=1}^{N_n^\ell}\alpha_k^{n,\ell}(\oplus_{\mathfrak{N}\in\mathfrak{X}}P_{\mathfrak{N}})(E_k^{n,\ell})\Psi \overset{\left(\substack{\text{if conv.}\\ \text{in op. norm}\\ \text{also strong}}\right)}{=}$$

$$= \overset{\oplus_{\mathfrak{N}}\mathcal{H}_{\mathfrak{N}}}{\lim_{n\to\infty}}\;\overset{\oplus_{\mathfrak{N}}\mathcal{H}_{\mathfrak{N}}}{\lim_{\ell\to\infty}}\sum_{k=1}^{N_n^\ell}\alpha_k^{n,\ell}(\oplus_{\mathfrak{N}\in\mathfrak{X}}P_{\mathfrak{N}})(E_k^{n,\ell})\Psi = \overset{\oplus_{\mathfrak{N}}\mathcal{H}_{\mathfrak{N}}}{\lim_{n\to\infty}}\;\overset{\oplus_{\mathfrak{N}}\mathcal{H}_{\mathfrak{N}}}{\lim_{\ell\to\infty}}\sum_{k=1}^{N_n^\ell}\alpha_k^{n,\ell}\Big(P_{\mathfrak{N}}(E_k^{n,\ell})\psi_{\mathfrak{N}}\Big)_{\mathfrak{N}\in\mathfrak{X}} =$$

$$= \overset{\oplus_{\mathfrak{N}}\mathcal{H}_{\mathfrak{N}}}{\lim_{n\to\infty}}\;\overset{\oplus_{\mathfrak{N}}\mathcal{H}_{\mathfrak{N}}}{\lim_{\ell\to\infty}}\left(\sum_{k=1}^{N_n^\ell}\alpha_k^{n,\ell}\;P_{\mathfrak{N}}(E_k^{n,\ell})\psi_{\mathfrak{N}}\right)_{\mathfrak{N}\in\mathfrak{X}}. \tag{5.58}$$

Now, note the following generality. If we know for some $G^\ell:=(g_{\mathfrak{N}}^\ell)_{\mathfrak{N}\in\mathfrak{X}}\in\oplus_{\mathfrak{N}\in\mathfrak{X}}\mathcal{H}_{\mathfrak{N}}$ that the limit $\lim_{\ell\to\infty}^{\oplus_{\mathfrak{N}}\mathcal{H}_{\mathfrak{N}}}G^\ell$ exists, then, it must be that its limit $(g_{\mathfrak{N}})_{\mathfrak{N}\in\mathfrak{X}}$ is such that $g_{\mathfrak{N}}=\lim_{\ell\to\infty}^{\mathcal{H}_{\mathfrak{N}}}g_{\mathfrak{N}}^\ell$.

Indeed: if $\left\|(g_{\mathfrak{N}})_{\mathfrak{N}\in\mathfrak{X}} - G^{\ell}\right\|^2 = \sum_{\mathfrak{N}\in\mathfrak{X}} \left\|g_{\mathfrak{N}} - g^{\ell}_{\mathfrak{N}}\right\|^2 \xrightarrow[\ell\to\infty]{} 0$ then, in particular for each $\mathfrak{N}\in\mathfrak{X}$, $\left\|g_{\mathfrak{N}} - g^{\ell}_{\mathfrak{N}}\right\|^2 \xrightarrow[\ell\to\infty]{} 0$.

Let us apply this twice. First, by (5.58), we know that for each $n\in\mathbb{N}$, $\lim^{\oplus}_{\ell\to\infty}(\sum_{k=1}^{N^{\ell}_n} \alpha^{n,\ell}_k P_{\mathfrak{N}}(E^{n,\ell}_k)\psi_{\mathfrak{N}})_{\mathfrak{N}\in\mathfrak{X}}$ exists. By the generality, this implies that its limit is $(\lim^{\mathcal{H}_{\mathfrak{N}}}_{\ell\to\infty} \sum_{k=1}^{N^{\ell}_n} \alpha^{n,\ell}_k P_{\mathfrak{N}}(E^{n,\ell}_k)\psi_{\mathfrak{N}})_{\mathfrak{N}\in\mathfrak{X}}$, which, by definition of functional calculus equals $(\Phi_{\mathfrak{N}}(f_n)\psi_{\mathfrak{N}})_{\mathfrak{N}\in\mathfrak{X}}$. Second, by (5.58), $\lim^{\oplus}_{n\to\infty}$ $(\Phi_{\mathfrak{N}}(f_n)\psi_{\mathfrak{N}})_{\mathfrak{N}\in\mathfrak{X}}$ exists as well, so by the generality, its limit must be $(\lim^{\mathcal{H}_{\mathfrak{N}}}_{n\to\infty} \Phi_{\mathfrak{N}}(f_n)\psi_{\mathfrak{N}})$, which by definition of functional calculus equals $(\Phi_{\mathfrak{N}}(f)\psi_{\mathfrak{N}})_{\mathfrak{N}\in\mathfrak{X}}$. With all, (5.58) implies that $\Phi(f)\Psi$ equals $(\phi_{\mathfrak{N}}(f)\psi_{\mathfrak{N}})_{\mathfrak{N}\in\mathfrak{X}}$, which is precisely $(\oplus_{\mathfrak{N}\in\mathfrak{X}}\phi_{\mathfrak{N}}(f))\Psi$.

Item (iii): We proved in (5.56) that for any $g : X \to [0,+\infty)$ measurable (back then it was $|f|^2$),

$$\int_{x\in X} g(x)d\mu_{\Psi} = \sum_{\mathfrak{N}\in\mathfrak{X}} \int_{x\in X} g(x)d^{\mathfrak{N}}\mu_{\psi_{\mathfrak{N}}}. \tag{5.59}$$

Putting $g = \mathbb{1}_B$ for $B\in\Sigma$, this proves that $d\mu_{\Psi} = \sum_{\mathfrak{N}\in\mathfrak{X}} d^{\mathfrak{N}}\mu_{\psi_{\mathfrak{N}}}$.

ο.ε.δ.

Lemma 26. Let $\mathfrak{X}$ and I be arbitrary index sets. Let $(X_j)_{j\in I}$ be locally compact, second countable and Hausdorff topological spaces, and let there be a Hilbert space $\mathcal{H}$ with orthogonal decomposition $\mathcal{H} = \bigoplus_{\mathfrak{N}\in\mathfrak{X}} \mathcal{H}_{\mathfrak{N}}$. Then, given that for each $\mathfrak{N}\in\mathfrak{X}$, $(\mathcal{H}_{\mathfrak{N}},\ (X_j, \mathfrak{B}(X_j)),\ P^{\mathfrak{N}}_j)_{j\in I}$ denotes a commuting family of PVMs, *the product of PVMs and the direct product commute* in the following sense:

$$\odot_{j\in I}\Big(\oplus_{\mathfrak{N}\in\mathfrak{X}} P^{\mathfrak{N}}_j\Big) = \oplus_{\mathfrak{N}\in\mathfrak{X}}\Big(\odot_{j\in I} P^{\mathfrak{N}}_j\Big), \qquad \text{i.e.,} \quad \Big[\odot_{j\in I}\Big(\oplus_{\mathfrak{N}\in\mathfrak{X}} P^{\mathfrak{N}}_j\Big)\Big](E) = \oplus_{\mathfrak{N}\in\mathfrak{X}}\Big(\Big[\odot_{j\in I} P^{\mathfrak{N}}_j\Big](E)\Big) \tag{5.60}$$

for all $E\in\odot_{j\in I}\mathfrak{B}(X_j)$.

◆

Proof: First, observe that in the following sense, *the direct sum of operators distributes along the composition*. Let $(A_{\mathfrak{N}})_{\mathfrak{N}\in\mathfrak{X}}$, $(B_{\mathfrak{N}})_{\mathfrak{N}\in\mathfrak{X}}$ be arbitrary bounded operator families such that $A_{\mathfrak{N}}, B_{\mathfrak{N}}\in\mathcal{L}(\mathcal{H}_{\mathfrak{N}})$. Then, for all $\Psi := (\psi_{\mathfrak{N}})_{\mathfrak{N}\in\mathfrak{X}} \in \oplus_{\mathfrak{N}\in\mathfrak{X}}\mathcal{H}_{\mathfrak{N}}$,

$$\Bigg(\bigoplus_{\mathfrak{N}\in\mathfrak{X}}\Big[A_{\mathfrak{N}}\circ B_{\mathfrak{N}}\Big]\Bigg)\Psi \overset{\text{(by def)}}{=} \Big(A_{\mathfrak{N}}\circ B_{\mathfrak{N}}\,\psi_{\mathfrak{C}}\Big)_{\mathfrak{N}\in\mathfrak{X}} = \Bigg(\bigoplus_{\mathfrak{N}\in\mathfrak{X}} A_{\mathfrak{N}}\Bigg)\Big(B_{\mathfrak{N}}\,\psi_{\mathfrak{C}}\Big)_{\mathfrak{N}\in\mathfrak{X}} = \Bigg(\bigoplus_{\mathfrak{N}\in\mathfrak{X}} A_{\mathfrak{N}}\Bigg)\circ\Bigg(\bigoplus_{\mathfrak{N}\in\mathfrak{X}} B_{\mathfrak{N}}\Bigg)\Psi. \tag{5.61}$$

- By Prop. 37, $\oplus_{\mathfrak{N}\in\mathfrak{X}}(\odot_{j\in I}P^{\mathfrak{N}}_j)$ and all $\oplus_{\mathfrak{N}\in\mathfrak{X}}(P^{\mathfrak{N}}_j)$ (with $j\in I$) are PVMs. Moreover, by (5.61), $\oplus_{\mathfrak{N}\in\mathfrak{X}}(P^{\mathfrak{N}}_j)$ commute with each other, so, by Theorem 15, there exists a product PVM $\odot_{j\in I}(\oplus_{\mathfrak{N}\in\mathfrak{X}}P^{\mathfrak{N}}_j)$. By Theorem 15, for each finite $J := \{j_1, ..., j_n\} \in I$ and any given $E_J \in \odot_{j\in J}\mathfrak{B}(X_j)$,

$$\Big[\odot_{j\in I}\left(\oplus_{\mathfrak{N}\in\mathfrak{X}}P^{\mathfrak{N}}_j\right)\Big]\Bigg(E_J \times \prod_{j\in I\setminus J} X_j\Bigg) = \Big[\odot_{j\in J}\left(\oplus_{\mathfrak{N}\in\mathfrak{X}}P^{\mathfrak{N}}_j\right)\Big](E_J). \tag{5.62}$$

By the same theorem,

$$\oplus_{\mathfrak{N}\in\mathfrak{X}}\Bigg(\Big[\odot_{j\in I} P^{\mathfrak{N}}_j\Big]\Big(E_J \times \prod_{j\in I\setminus J} X_j\Big)\Bigg) = \oplus_{\mathfrak{N}\in\mathfrak{X}}\Bigg(\Big[\odot_{j\in J} P^{\mathfrak{N}}_j\Big](E_J)\Bigg). \tag{5.63}$$

- Given $E_J = E_{j_1} \times \cdots \times E_{j_n}$ for arbitrary $E_j \in \mathfrak{B}(X_j)$, Theorem 14 implies that:

$$\oplus_{\mathfrak{N}\in\mathfrak{X}}\Big(\Big[\odot_{j\in J}P_j^{\mathfrak{N}}\Big](E_J)\Big) = \oplus_{\mathfrak{N}\in\mathfrak{X}}\Big(P_{j1}^{\mathfrak{N}}(E_{j_1})\cdots P_{jn}^{\mathfrak{N}}(E_{j_n})\Big) \overset{(5.61)}{=} \oplus_{\mathfrak{N}\in\mathfrak{X}}(P_{j1}^{\mathfrak{C}}(E_{j_1}))\cdots\oplus_{\mathfrak{N}\in\mathfrak{X}}(P_{jn}^{\mathfrak{C}}(E_{j_n})).$$

But again by Theorem 14, this is also the result of (5.62)'s r.h.s. Now, Theorem 14 says that there is a unique such PVM on $\odot_{j\in J}\mathfrak{B}(X_j)$. Hence, $\odot_{j\in J}\Big(\oplus_{\mathfrak{N}\in\mathfrak{X}}P_j^{\mathfrak{N}}\Big)(\cdot) = \oplus_{\mathfrak{N}\in\mathfrak{X}}\Big(\odot_{j\in J}P_j^{\mathfrak{N}}(\cdot)\Big)$. But then, the left hand sides of (5.62) and (5.63) agree for all E_J and all finite $J \subseteq I$. If so, the uniqueness statement of Theorem 15 implies that the two PVMs $\odot_{j\in I}\Big(\oplus_{\mathfrak{N}\in\mathfrak{X}}P_j^{\mathfrak{N}}\Big)(\cdot)$ and $\oplus_{\mathfrak{N}\in\mathfrak{X}}\Big(\odot_{j\in I}P_j^{\mathfrak{N}}(\cdot)\Big)$ must be exactly the same everywhere else. ***o.ε.δ.***

5.3.4 The "Grand Finale"

Proposition 38. Let $\mathscr{W}_{\mathfrak{R}} = \oplus_{\mathfrak{C}\in\Gamma}W_{\mathfrak{C}}$ be the mapping of Theorem 13 and let $(\overline{\widehat{q}_k})_{k\in\mathbb{N}}$ be the closures of the lifted position operators $(\widehat{q}_k, D_0(\widehat{q}_k))_{k\in\mathbb{N}}$ defined in Corollary 17 —where we found they are essentially self-adjoint. Then,

(i) the spectral PVM $(\bigotimes_{k\in\mathbb{N}}L^2(\mathbb{R},dx),\ (\mathbb{R},\mathfrak{B}(\mathbb{R})),\ Q_k)$ of $\overline{\widehat{q}_k}$ is exactly such that

$$Q_k(B_k) = \bigoplus_{\mathfrak{C}\in\Gamma}P_k^{\mathfrak{C}}(B_k) = \bigoplus_{\mathfrak{C}\in\Gamma}W_{\mathfrak{C}}^{-1}\mathcal{M}_{\mathbb{1}_{B_k}\circ\pi_k}W_{\mathfrak{C}} \tag{5.64}$$

for each $B_k \in \mathfrak{B}(\mathbb{R})$ (see Propositions 34, 35 and 36 for explicit definitions).

(ii) $\{\overline{\widehat{q}_k}\}_{k\in\mathbb{N}}$ commute strongly so there exists a joint diagonalization PVM $(\bigotimes_{k\in\mathbb{N}}L^2(\mathbb{R},dx),$ $(\mathbb{R}^\infty,\mathfrak{B}(R^\infty)),\ \odot_{k\in\mathbb{N}}Q_k)$ and it equals

$$(\odot_{k\in\mathbb{N}}Q_k)(B) = \oplus_{\mathfrak{C}\in\Gamma}((\odot_{k\in\mathbb{N}}P_k^{\mathfrak{C}})(B)) = \oplus_{\mathfrak{C}\in\Gamma}(W_{\mathfrak{C}}^{-1}\mathcal{M}_{\mathbb{1}_B}.W_{\mathfrak{C}}). \tag{5.65}$$

(iii) The joint PVM's functional calculus Ξ is such that given a measurable $\varphi : \mathbb{R}^\infty \to \mathbb{C}$,

$$(\Xi(\varphi), D_\varphi) = (\oplus_{\mathfrak{C}\in\Gamma}\Phi^{\mathfrak{C}},\ \oplus_{\mathfrak{C}\in\Gamma}D_\varphi^{\mathfrak{C}}) = (\oplus_{\mathfrak{C}\in\Gamma}W_{\mathfrak{C}}^{-1}\mathcal{M}_\varphi W_{\mathfrak{C}}^{-1},\ \oplus_{\mathfrak{C}\in\Gamma}W_{\mathfrak{C}}^{-1}D(\mathcal{M}_\varphi)). \tag{5.66}$$

(iv) For elementary tensor products $\Psi = \psi_1\otimes\psi_2\otimes\cdots$ with $\|\psi_j\|_{L^2(\mathbb{R},dx)} = 1$, the joint spectral measures are

$$d\mu_\Psi = \odot_{j\in\mathbb{N}}|\psi_j|^2dx_j.$$

♦

In order to prove the proposition we need a (quite technical) lemma that we separate from the main proof for its subtlety.

Lemma 27. Let $\bigotimes_{j\in I}\psi_j \in D_0(\widehat{q}_k)$ i.e., such that $\psi_k \in D(\mathcal{M}_\lambda)$ for $\lambda : x \in \mathbb{R} \mapsto x$. Let $\bigotimes_{k\in\mathbb{N}}\rho_k$ be a generator for the layer $\mathfrak{C}$ to which $\bigotimes_{j\in\mathbb{N}}\psi_j$ belongs, and let $W_{\mathfrak{C}} : \otimes_{k\in\mathbb{N}}^{\mathfrak{C}}L^2(\mathbb{R},dx) \longrightarrow L^2(\mathbb{R}^\infty, d^\infty\mu_{\mathfrak{C}})$ be its associated unitary of Theorem 10. Then,

(i) for almost every $x = (x_1, x_2, ...) \in \mathbb{R}^\infty$,

$$\Big[(W_{\mathfrak{C}}\circ\widehat{q}_k)\Big(\bigotimes_{j\in\mathbb{N}}\psi_j\Big)\Big](x) \quad = \quad x_k\cdot\Big[W_{\mathfrak{C}}\Big(\bigotimes_{j\in\mathbb{N}}\psi_j\Big)\Big](x). \tag{5.67}$$

(ii) Moreover, $\bigotimes_{j\in I}\psi_j \in D_0(\widehat{q}_k) \iff W_{\mathfrak{C}}(\bigotimes_{j\in\mathbb{N}}\psi_j) \in D(\mathcal{M}_{\lambda\circ\pi_k})$ and

$$W_{\mathfrak{C}}\circ\widehat{q}_k(\bigotimes_{j\in\mathbb{N}}\psi_j) = \mathcal{M}_{\lambda\circ\pi_k}\circ W_{\mathfrak{C}}\ (\bigotimes_{j\in\mathbb{N}}\psi_j). \tag{5.68}$$

♦

Proof: **Item (i) :** By Prop. 24, $(\psi_1 \otimes \cdots \otimes \psi_N \otimes \rho_{N+1} \otimes \cdots) \xrightarrow[N\to\infty]{} \bigotimes_{j\in\mathbb{N}} \psi_j$ in norm. $W_{\mathfrak{c}}$ is sequentially continuous so $W_{\mathfrak{c}}(\psi_1 \otimes \cdots \otimes \psi_N \otimes \rho_{N+1} \otimes \cdots) = \frac{\psi_1 \cdots \psi_N}{\rho_1 \cdots \rho_N} \xrightarrow[N\to\infty]{} W_{\mathfrak{c}}(\bigotimes_{j\in\mathbb{N}} \psi_j)$ in $L^2(\mathbb{R}^\infty, d^\infty \mu_{\mathfrak{c}})$-norm. By Theorem 3.12 in (Rudin, 1987) this implies that there is a subsequence indexed by some $(N_k)_{k\in\mathbb{N}}$ that converges point-wise almost everywhere in $x \in \mathbb{R}^\infty$. That is, for $d^\infty \mu_{\mathfrak{c}}$-almost every $x \in \mathbb{R}^\infty$,

$$\Big[W_{\mathfrak{c}}\Big(\bigotimes_{j\in\mathbb{N}} \psi_j\Big)\Big](x) = \lim_{k\to\infty} \frac{\psi_1(x_1)\cdots\psi_{N_k}(x_{N_k})}{\rho_1(x_1)\cdots\rho_{N_k}(x_{N_k})}. \tag{5.69}$$

- Observe that $\widehat{q}_k(\bigotimes_{j\in\mathbb{N}} \psi_j) = \psi_1 \otimes \cdots \otimes (\hat{q}\psi_k) \otimes \psi_{k+1} \otimes \cdots$. Even if we replace all ψ_j in $\otimes_{j\in\mathbb{N}}\psi_j$ with $j > k$ by ρ_j, the resulting vector is still in $D_0(\widehat{q_k})$ because the k-th entry is still $\psi_k \in D(\hat{q})$. Then, by Prop. 24, as $N \to \infty$, the sequence

$$\widehat{q}_k(\psi_1 \otimes \cdots \otimes \psi_N \otimes \rho_{N+1} \otimes \cdots) = \psi_1 \otimes \cdots \otimes (\hat{q}\psi_k) \otimes \psi_{k+1} \otimes \cdots \otimes \psi_N \otimes \rho_{N+1} \otimes \cdots$$

converges in $\otimes^{\mathfrak{c}}_{j\in\mathbb{N}} L^2(\mathbb{R}, dx)$'s norm to $\widehat{q}_k(\bigotimes_{j\in\mathbb{N}} \psi_j)$. Hence, because $W_{\mathfrak{c}}$ is sequentially continuous, as $N \to \infty$, the L^2-functions

$$x \longmapsto \Big[W_{\mathfrak{c}}\Big(\hat{q}_k(\psi_1 \otimes \cdots \otimes \psi_N \otimes \rho_{N+1} \otimes \cdots)\Big)\Big](x) = x_k \frac{\psi_1(x_1)\cdots\psi_N(x_N)}{\rho_1(x_1)\cdots\rho_N(x_N)}$$

converge in L^2-norm to $W_{\mathfrak{c}}\Big(\hat{q}_k(\bigotimes_{j\in\mathbb{N}} \psi_j)\Big)$. In particular, they will also converge for the subsequence indexed by $(N_k)_{k\in\mathbb{N}}$. But then, by Rudin's theorem above, there exists a subsequence of this subsequence, say, one indexed by $(N_\ell)_{\ell\in\mathbb{N}}$ such that for $d^\infty\mu_{\mathfrak{c}}$-almost every $x \in \mathbb{R}^\infty$ there is point-wise convergence:

$$\Big[W_{\mathfrak{c}}\Big(\hat{q}_k(\bigotimes_{j\in\mathbb{N}} \psi_j)\Big)\Big](x) = \lim_{\ell\to\infty} x_k \cdot \frac{\psi_1(x_1)\cdots\psi_{N_\ell}(x_{N_\ell})}{\rho_1(x_1)\cdots\rho_{N_\ell}(x_{N_\ell})}. \tag{5.70}$$

- Now, for each $x \in \mathbb{R}^\infty$, all subsequences of (5.69) must converge to the same limit point, so (5.69) multiplied by x_k equals (5.70).

Item (ii) : First, note that

$$\Big\|\widehat{q}_k(\bigotimes_{j\in\mathbb{N}} \psi_j)\Big\|^2 = \|\hat{q}\psi_k\|^2 \cdot \prod_{j\in\mathbb{N}\setminus\{k\}} \|\psi_j\|^2.$$

As such, $\Big\|\widehat{q}_k(\bigotimes_{j\in\mathbb{N}} \psi_j)\Big\|^2 < +\infty \iff \bigotimes_{j\in\mathbb{N}} \psi_j \in D_0(\widehat{q}_k)$. Second, note that

$$\int_{x\in\mathbb{R}^\infty} \Big|\lambda \circ \pi_k(x) \cdot \Big[W_{\mathfrak{c}}\Big(\bigotimes_{j\in\mathbb{N}} \psi_j\Big)\Big](x)\Big|^2 d^\infty\mu_{\rho^{\mathfrak{c}}} = \int_{x\in\mathbb{R}^\infty} \Big|x_k \cdot \Big[W_{\mathfrak{c}}\Big(\bigotimes_{j\in\mathbb{N}} \psi_j\Big)\Big](x)\Big|^2 d^\infty\mu_{\rho^{\mathfrak{c}}} \overset{\text{(item (i))}}{=}$$

$$= \int_{x\in\mathbb{R}^\infty} \Big|\Big[W_{\mathfrak{c}} \circ \widehat{q}_k\Big(\bigotimes_{j\in\mathbb{N}} \psi_j\Big)\Big](x)\Big|^2 d^\infty\mu_{\rho^{\mathfrak{c}}} = \Big\|W_{\mathfrak{c}} \circ \widehat{q}_k(\bigotimes_{j\in\mathbb{N}} \psi_j)\Big\|^2_{L^2} \overset{(W_{\mathfrak{c}}\text{ isom})}{=} \Big\|\widehat{q}_k(\bigotimes_{j\in\mathbb{N}} \psi_j)\Big\|^2.$$

Hence, $\Big\|\widehat{q}_k(\bigotimes_{j\in\mathbb{N}} \psi^\ell_j)\Big\|^2 < +\infty \iff W_{\mathfrak{c}}(\bigotimes_{j\in\mathbb{N}} \psi_j) \in D(\mathcal{M}_{\lambda\circ\pi_k})$. This proves the claim on domains. But then, by definition, point-wise $d^\infty\mu_{\mathfrak{c}}$-almost everywhere, $\mathcal{M}_{\lambda\circ\pi_k} \circ W_{\mathfrak{c}}(\bigotimes_{j\in\mathbb{N}} \psi_j)$ equals the r.h.s of (5.67). Hence, by (i), we get (5.68).

o.ε.δ.

Proof **of Proposition 38 :** **Item (i):** On the one hand, by Proposition 37.(i), for each $k \in \mathbb{N}$, $\oplus_{\mathfrak{C}\in\Gamma} P_k^{\mathfrak{C}}$ is a PVM. Its functional calculus Ξ_k satisfies for each $g : \mathbb{R} \to \mathbb{C}$ measurable,

$$\Xi_k(g) \overset{(\text{Prop. } 37.(\text{ii}))}{=} \oplus_{\mathfrak{C}\in\Gamma} \Phi_k^{\mathfrak{C}}(g) \overset{(\text{Prop. } 36.(\text{ii}))}{=} \oplus_{\mathfrak{C}\in\Gamma} (W_{\mathfrak{C}}^{-1} \mathcal{M}_{g\circ\pi_k} W_{\mathfrak{C}}) \upharpoonright_{D_g^k}, \tag{5.71}$$

where $D_g^k = \oplus_{\mathfrak{C}\in\Gamma}(W_{\mathfrak{C}}^{-1} D(\mathcal{M}_{g\circ\pi_k}))$ (for more explicit shapes see Propositions 34, 35 and 36).

- On the other hand, by Prop. 32,

$$(\widehat{q}_k,\ D_0(\widehat{q}_k)) = \Big(\bigoplus_{\mathfrak{C}\in\Gamma} \widehat{q}_k|_{\mathfrak{C}},\ \oplus^{\mathscr{F}}_{\mathfrak{C}\in\Gamma} D_0^{\mathfrak{C}}(\widehat{q}_k) \Big) \tag{5.72}$$

where $D_0^{\mathfrak{C}}(\widehat{q}_k) := D_0(\widehat{q}_k) \cap \otimes^{\mathfrak{C}}_{k\in\mathbb{N}} L^2(\mathbb{R}, dx)$ and $\oplus^{\mathscr{F}}_{\mathfrak{C}\in\Gamma} D_0^{\mathfrak{C}}(\widehat{q}_k) := \oplus_{\mathfrak{C}\in\Gamma} D_0^{\mathfrak{C}}(\widehat{q}_k) \ \bigcap \ \oplus^{\mathscr{F}}_{\mathfrak{C}\in\Gamma}(\otimes^{\mathfrak{C}}_{k\in\mathbb{N}} L^2(\mathbb{R}, dx))$.

- Now, in Lemma 27 we proved that for elementary tensor products, $\bigotimes_{j\in\mathbb{N}} \psi \in D_0^{\mathfrak{C}}(\widehat{q}_k) \iff \bigotimes_{j\in\mathbb{N}} \psi \in W_{\mathfrak{C}}^{-1} D(\mathcal{M}_{\lambda\circ\pi_k})$. But by equation (5.37), an arbitrary $\Psi \in D_0^{\mathfrak{C}}(\widehat{q}_k)$ is a *finite* linear combination of elementary tensors in $D_0^{\mathfrak{C}}(\widehat{q}_k)$. Hence, (defining $\lambda : x_k \in \mathbb{R} \longmapsto x_k$), $D_0^{\mathfrak{C}}(\widehat{q}_k) \subseteq W_{\mathfrak{C}}^{-1} D(\mathcal{M}_{\lambda\circ\pi_k})$. Furthermore, by Lemma 27, $\mathcal{M}_{\lambda\circ\pi_k} \circ W_{\mathfrak{C}}$ and $W_{\mathfrak{C}} \circ \widehat{q}_k$ agree on elementary tensor products. So, by linearity, they agree for all $\Psi \in D_0^{\mathfrak{C}}(\widehat{q}_k)$. Therefore, $W_{\mathfrak{C}} \widehat{q}_k|_{\mathfrak{C}} = \mathcal{M}_{\lambda\circ\pi_k} W_{\mathfrak{C}} \upharpoonright_{D_0^{\mathfrak{C}}(\widehat{q}_k)}$. That is, for each layer $\mathfrak{C} \in \Gamma$,

$$\widehat{q}_k|_{\mathfrak{C}} = W_{\mathfrak{C}}^{-1} \mathcal{M}_{\lambda\circ\pi_k} (W_{\mathfrak{C}} \upharpoonright_{D_0^{\mathfrak{C}}(\widehat{q}_k)}). \tag{5.73}$$

But then,

$$\hat{q}_k \overset{(5.72)}{=} \bigoplus_{\mathfrak{C}\in\Gamma} (\hat{q}_k|_{\mathfrak{C}}) \overset{(5.73)}{=} \bigoplus_{\mathfrak{C}\in\Gamma} W_{\mathfrak{C}}^{-1} \mathcal{M}_{\lambda\circ\pi_k} (W_{\mathfrak{C}} \upharpoonright_{D_0^{\mathfrak{C}}(\widehat{q}_k)}) \overset{(\text{by def})}{=} \Big(W_{\mathfrak{C}}^{-1} \mathcal{M}_{\lambda\circ\pi_k} W_{\mathfrak{C}} \Big) \upharpoonright_{D_0(\hat{q}_k)} \overset{(5.71)}{=} \Xi(\lambda) \upharpoonright_{D_0(\hat{q}_k)} .$$

- This proves that $(\Xi_k(\lambda), D_\lambda)$ is an extension of $(\widehat{q}_k,\ D_0(\widehat{q}_k))$. Now, $(\Xi_k(\lambda), D_\lambda)$ is a direct sum of self-adjoint operators, so by Prop. 18, it is itself self-adjoint —also because it is the functional calculus of a real-valued function. On the other hand, we proved in Corollary 17 that $(\widehat{q}_k,\ D_0(\widehat{q}_k))$ is essentially self-adjoint. Then, by uniqueness of self-adjoint extension for essentially self-adjoint operators (Prop. 1.21 in (Arai, 2018)), $(\Xi_k(\lambda), D_\lambda)$ must be the closure of $\hat{q}_k$. Finally, by Theorem 18, there is a unique PVM whose functional calculus satisfies the stated, so, $\oplus_{\mathfrak{C}\in\Gamma} P_k^{\mathfrak{C}}$ must be the spectral PVM of $\overline{\widehat{q}_k}$, i.e., Q_k.

Items (ii) & (iii): They follow trivially by Lem. 26, Prop. 37 and the previous results.

Item (iv): Let $\Psi = \bigotimes_{j\in\mathbb{N}} \psi_j$ and let $\mathfrak{C} \in \Gamma$ be its $\approx$-equivalence class. Let $B = B_1 \times \cdots \times B_n \times \prod_{j\in\mathbb{N}\setminus\{1,\dots,n\}} \mathbb{R} \in \mathfrak{B}(\mathbb{R}^\infty)$ for some $n \in \mathbb{N}$ and $B_j \in \mathfrak{B}(\mathbb{R})$. Then, the spectral measure associated to Ψ by $\odot_{k\in\mathbb{N}} Q_k$ (equivalently, after Prop. 37, by $\oplus_{\mathfrak{C}\in\Gamma}(\odot_{k\in\mathbb{N}} P_k^{\mathfrak{C}})$) is

$$d\mu_\Psi(B) \overset{\left(\substack{\text{Prop. } 37.(\text{iii}) \ + \\ \text{only } \odot_{j\in\mathbb{N}} P_j^{\mathfrak{C}} \text{ yields } \neq 0}\right)}{=} d^{\mathfrak{C}}\mu_\Psi(B) = \Big\langle \Psi, (\odot_{j\in\mathbb{N}} P_j^{\mathfrak{C}})(B)\Psi \Big\rangle = \Big\langle \Psi,\ W_{\mathfrak{C}}^{-1} \mathcal{M}_{\mathbb{1}_B} W_{\mathfrak{C}} \Psi \Big\rangle \overset{\left(\substack{W_{\mathfrak{C}} \text{ isom. } \& \\ \mathcal{M}^2_{\mathbb{1}_B} = \mathcal{M}_{\mathbb{1}_B}}\right)}{=}$$

$$(\text{ use in } (\star) \text{ that by Prop. } 24, \quad (\psi_1 \otimes \cdots \otimes \psi_N \otimes \rho^{\mathfrak{C}}_{N+1} \otimes \cdots) \xrightarrow[N\to\infty]{\text{norm}} \psi_1 \otimes \psi_2 \otimes \cdots\ ) \tag{5.74}$$

$$= \|\mathcal{M}_{\mathbb{1}_B} W_{\mathfrak{C}} \Psi\|^2 \overset{(\star\ \&\ \mathcal{M}_{\mathbb{1}_B},\ W_{\mathfrak{C}} \text{ conts})}{=} \lim_{N\to\infty} \Big\| \mathcal{M}_{\mathbb{1}_B} W_{\mathfrak{C}} (\psi_1 \otimes \cdots \otimes \psi_N \otimes \rho^{\mathfrak{C}}_{N+1} \otimes \cdots) \Big\|^2 =$$

$$= \lim_{N\to\infty} \int_{x\in\mathbb{R}^\infty} \mathbb{1}_B(x) \frac{|\psi_1|^2(x_1)\cdots|\psi_N|^2(x_N)}{|\rho_1^{\mathfrak{C}}|^2(x_1)\cdots|\rho_N^{\mathfrak{C}}|^2(x_N)} d^\infty\mu_{\rho^{\mathfrak{C}}} \overset{\left(\substack{\mathbb{1}_B(x)=\mathbb{1}_{B_1}(x_1)\cdots\mathbb{1}_{B_n}(x_n) \\ +\text{ Prop. } 20}\right)}{=}$$

$$= \lim_{N\to\infty} \prod_{j=1}^{n} \int_{x_j\in\mathbb{R}} \mathbb{1}_{B_j}(x_j) \frac{|\psi_j|^2(x_j)}{|\rho_j^{\mathfrak{C}}|^2(x_j)} d\mu_{\rho_j^{\mathfrak{C}}} \prod_{j=n+1}^{N} \int_{x_j\in\mathbb{R}} \frac{|\psi_j|^2(x_j)}{|\rho_j^{\mathfrak{C}}|^2(x_j)} d\mu_{\rho_j^{\mathfrak{C}}} =$$

$$\overset{\left(\substack{d\mu_{\rho_j^{\mathfrak{C}}}:=|\rho_j^{\mathfrak{C}}|^2dx \\ \|\psi_j\|=1}\right)}{=} \prod_{j=1}^{n} \int_{x_j\in\mathbb{R}} \mathbb{1}_{B_j}(x_j)|\psi_j|^2(x_j)dx = \prod_{j=1}^{n}(|\psi_j|^2dx)(B_j).$$

- Putting a finite disjoint union of sets like B, namely, an element of the Boolean algebra $\mathfrak{A}_0$ of Prop. 3, say, $B = \bigsqcup_{\ell=1}^{N} B_1^\ell \times\cdots\times B_{n_\ell}^\ell \times\mathbb{R}\times\cdots$, we get by additivity of $d\mu_\Psi$ that

$$d\mu_\Psi(B) = \sum_{\ell=1}^{N}\prod_{j=1}^{n_\ell}(|\psi_j|^2dx)(B_j^\ell).$$

- $\|\psi_j\| = 1$ implies that $|\psi_j|^2dx$ are probability measures. Hence, by Cor. 6, there exists an infinite product measure $\odot_{j\in\mathbb{N}}|\psi_j|^2dx_j$ satisfying

$$(\odot_{j\in\mathbb{N}}|\psi_j|^2dx_j)(B) = \sum_{\ell=1}^{N}(\odot_{j=1}^{n}|\psi_j|^2dx_j)(B_1^\ell\times\cdots\times B_{n_\ell}^\ell) = \sum_{\ell=1}^{N}\prod_{j=1}^{n_\ell}(|\psi_j|^2dx_j)(B_j^\ell).$$

But then, $(\odot_{j\in\mathbb{N}}|\psi_j|^2dx_j)$ and $d\mu_\Psi$ coincide on the Boolean algebra $\mathfrak{A}_0$ and they are both probability measures —hence, σ-finite. As such, by Theorem 1 they must coincide everywhere, i.e., $d\mu_\Psi = \odot_{j\in\mathbb{N}}|\psi_j|^2dx_j$. ***o.ε.δ.***

Theorem 20. The unitary isomorphism we found in Theorem 13 —which we called *wavefunction representation* (WR)—,

$$\begin{array}{cccc} \mathscr{W}_{\mathfrak{R}} : \bigotimes_{k\in\mathbb{N}} L^2(\mathbb{R},dx) & = \bigoplus\limits_{\mathfrak{C}\in\Gamma}\left(\otimes_{k\in\mathbb{N}}^{\mathfrak{C}} L^2(\mathbb{R},dx)\right) & \xrightarrow{\mathscr{W}_{\mathfrak{R}}=\oplus_{\mathfrak{C}\in\Gamma}W_{\mathfrak{C}}} & \bigoplus\limits_{\mathfrak{C}\in\Gamma} L^2(\mathbb{R}^\infty, d^\infty\mu_{\rho^{\mathfrak{C}}}) \\ \Psi & = (\Psi_{\mathfrak{C}})_{\mathfrak{C}\in\ \Gamma} & \longmapsto & (g_{\mathfrak{C}})_{\mathfrak{C}\in\Gamma} \end{array},$$

is exactly a *joint diagonalization of all the position operators* $(\overline{\widehat{q}_k})_{k\in\mathbb{N}}$ (as defined in Theorem 19). Hence, for an abstract vector $\Psi\in\bigotimes_{k\in\mathbb{N}} L^2(\mathbb{R},dx)$ one calls the tuple of wavefunctionals $\mathscr{W}_{\mathfrak{R}}\Psi\in\bigoplus\limits_{\mathfrak{C}\in\Gamma} L^2(\mathbb{R}^\infty,d^\infty\mu_{\rho^{\mathfrak{C}}})$ its *"configuration representation"*.

- The choice of a WR basis $\mathfrak{R} = (\otimes_{j\in\mathbb{N}}\ \rho_j^{\mathfrak{C}})_{\mathfrak{C}\in\Gamma}$ (as defined in Theorem 13) is a choice of *spectral basis* for the joint PVM of $(\overline{\widehat{q}_k})_{k\in\mathbb{N}}$, $\odot_{k\in\mathbb{N}}Q_k$ (see Prop. 38), and $d^\infty\mu_{\rho^{\mathfrak{C}}}$ is exactly the *spectral measure* associated to the vector $\otimes_{j\in\mathbb{N}}\ \rho_j^{\mathfrak{C}}$ by that PVM.

Although there are different possible choices of spectral bases $(\psi_{\mathfrak{N}})_{\mathfrak{N}\in\mathfrak{X}}$ for $\odot_{k\in\mathbb{N}}Q_k$, as long as it is composed of elementary tensor products of different layers (no layer repetition), $\mathfrak{R} = (\psi_{\mathfrak{N}})_{\mathfrak{N}\in\mathfrak{X}}$ is a WR basis and the diagonal representation of $(\overline{\widehat{q}_k})_{k\in\mathbb{N}}$ is exactly that given by $\mathscr{W}_{\mathfrak{R}}$. That is, the *WR bases* $\mathfrak{R}$ of Theorem 13 *exhaust* the *spectral bases* made of elementary tensor products with no layer repetition.

- With all, the natural notion of measure in configuration-space associated to $\Psi = (\psi^{\mathfrak{C}})_{\mathfrak{C}\in\Gamma}\in\bigotimes_{k\in\mathbb{N}} L^2(\mathbb{R},dx)$ is its spectral measure $d\mu_\Psi(\cdot) = \langle\Psi,\ (\odot_{k\in\mathbb{N}}Q_k)(\cdot)\Psi\rangle$, and it satisfies:

$$d\mu_\Psi = \sum_{\mathfrak{C}\in\Gamma} d\mu_{\psi^{\mathfrak{C}}} = \sum_{\mathfrak{C}\in\Gamma}|W_{\mathfrak{C}}\Psi^{\mathfrak{C}}|^2(x_1,x_2,...)d^\infty\mu_{\rho^{\mathfrak{C}}}. \tag{5.75}$$

This is exactly the measure $\mathbb{P}^{\Psi}$ defined in Theorem 13. (Hence, this is another way to prove that it is a well-defined measure on configuration-space $\mathbb{R}^{\infty}$ associated to each vector $\Psi \in \bigotimes_{k\in\mathbb{N}} L^2(\mathbb{R}, dx)$, and that it is independent of the choice of WR basis $\mathfrak{R}$ employed to compute it.) In particular, it is a *probability measure* for all unit Ψ and the natural[5] generalization of the Born rule to $\mathbb{R}^{\infty}$.

Proof: First note that by Prop. 23, a choice of generator per layer, $(\bigotimes_{j\in\mathbb{N}} \rho_j^{\mathfrak{C}})_{\mathfrak{C}\in\Gamma}$, makes an orthonormal family of vectors. Next, fix some $\mathfrak{C} \in \Gamma$. Note that by Prop. 38.(iv) the spectral measure of $\bigotimes_{k\in\mathbb{N}} \rho_k^{\mathfrak{C}}$ according to the joint spectral PVM of $(\widehat{q}_k,\ D_0(\widehat{q}_k))_{k\in\mathbb{N}}$ is the infinite product measure $d^{\infty}\mu_{\rho^{\mathfrak{C}}} := \odot_{j\in\mathbb{N}}|\rho_j^{\mathfrak{C}}|^2 dx$. Hence, the spectral subspace it generates (by its definition in Theorem 16) is

$$\Big\{ \Xi(f)\Big(\bigotimes_{j\in\mathbb{N}} \rho_j^{\mathfrak{C}}\Big) \mid f \in L^2(\mathbb{R}^{\infty}, d^{\infty}\mu_{\rho^{\mathfrak{C}}})\Big\} \overset{\left(\otimes_{k\in\mathbb{N}}\rho_k^{\mathfrak{C}} \in \mathfrak{C}\right)}{=} \Big\{ W_{\mathfrak{C}}^{-1}\mathcal{M}_f W_{\mathfrak{C}}\Big(\bigotimes_{j\in\mathbb{N}} \rho_j^{\mathfrak{C}}\Big) \mid f \in L^2(\mathbb{R}^{\infty}, d^{\infty}\mu_{\rho^{\mathfrak{C}}})\Big\} =$$

$$\overset{\left(W_{\mathfrak{C}}(\otimes_{j\in\mathbb{N}}\rho_j^{\mathfrak{C}})(x)\equiv 1\right)}{=} \Big\{ W_{\mathfrak{C}}^{-1}\Big[(x_1, x_2, ...) \mapsto f(x_1, x_2, ...)\Big] \mid f \in L^2(\mathbb{R}^{\infty}, d^{\infty}\mu_{\rho^{\mathfrak{C}}})\Big\} =$$

$$= W_{\mathfrak{C}}^{-1} L^2(\mathbb{R}^{\infty}, d^{\infty}\mu_{\rho^{\mathfrak{C}}}) = \otimes_{j\in\mathbb{N}}^{\mathfrak{C}} L^2(\mathbb{R}, dx).$$

That is, the spectral subspace and the layer of the ITP that $\bigotimes_{k\in\mathbb{N}} \rho_k^{\mathfrak{C}}$ generates are exactly the same space. Finally, we know by Theorem 9 that all such layers are orthogonal to each other and they determine the full ITP, i.e., $\bigotimes_{k\in\mathbb{N}} L^2(\mathbb{R}, dx) = \oplus_{\mathfrak{C}\in\Gamma}\Big(\otimes_{k\in\mathbb{N}}^{\mathfrak{C}} L^2(\mathbb{R}, dx)\Big)$. With all, by definition (see Theorem 16) $(\bigotimes_{k\in\mathbb{N}} \rho_j^{\mathfrak{C}})_{\mathfrak{C}\in\Gamma}$ constitutes a spectral basis of the joint diagonalization PVM. As such, following Theorem 16, one can build a unitary (as done in Theorem 19) with

$$\begin{array}{ccccc} \bigotimes_{k\in\mathbb{N}} L^2(\mathbb{R}, dx) & = \bigoplus\limits_{\mathfrak{C}\in\Gamma}\Big(\otimes_{k\in\mathbb{N}}^{\mathfrak{C}} L^2(\mathbb{R}, dx)\Big) & \longrightarrow & \bigoplus\limits_{\mathfrak{C}\in\Gamma} L^2(\mathbb{R}^{\infty}, d^{\infty}\mu_{\rho^{\mathfrak{C}}}) \\ \Psi & = \Big(\Xi(f_{\mathfrak{C}}) \bigotimes_{k\in\mathbb{N}} \rho_k^{\mathfrak{C}}\Big)_{\mathfrak{C}\in\ \Gamma} & \longmapsto & (f_{\mathfrak{C}})_{\mathfrak{C}\in\Gamma} \end{array}. \tag{5.76}$$

- We now want to prove that this is exactly the unitary of Theorem 13. Choose an arbitrary $\mathfrak{C}_0 \in \Gamma$ and choose an arbitrary $\Psi = \psi_1 \otimes \cdots \otimes \psi_n \otimes \rho_{n+1}^{\mathfrak{C}} \otimes \cdots \in \otimes_{k\in\mathbb{N}}^{\mathfrak{C}} L^2(\mathbb{R}, dx)$. The map $f_{\mathfrak{C}} : (x_1, x_2, ...) \longmapsto (\psi_1(x_1)\cdots\psi_2(x_2))/(\rho_1^{\mathfrak{C}}(x_1)\cdots\rho_2^{\mathfrak{C}}(x_2))$ is in $L^2(\mathbb{R}^{\infty}, d^{\infty}\mu_{\rho^{\mathfrak{C}}})$ because

$$\int_{x\in\mathbb{R}^{\infty}} \frac{|\psi_1|^2(x_1)\cdots|\psi_2|^2(x_2)}{|\rho_1^{\mathfrak{C}}|^2(x_1)\cdots|\rho_2^{\mathfrak{C}}|^2(x_2)} d^{\infty}\mu_{\rho^{\mathfrak{C}}} \overset{(\text{Prop. }20)}{=} \prod_{j=1}^{n}\int_{x_j\in\mathbb{R}} \frac{|\psi_j|^2(x_j)}{|\rho_j^{\mathfrak{C}}|^2(x_j)} d\mu_{\rho_j^{\mathfrak{C}}} = \prod_{j=1}^{n}\int_{x_j\in\mathbb{R}} |\psi_j|^2(x_j) dx_j \overset{(\psi\in L^2)}{<} +\infty.$$

As such, by definition, $\otimes_{k\in\mathbb{N}}\rho_k^{\mathfrak{C}} \in D_{f_{\mathfrak{C}}}$, so, $\Xi(f_{\mathfrak{C}})(\bigotimes_{k\in\mathbb{N}} \rho_k^{\mathfrak{C}})$ is well-defined and yields

$$\Xi(f_{\mathfrak{C}})\Big(\bigotimes_{k\in\mathbb{N}} \rho_k^{\mathfrak{C}}\Big) = W_{\mathfrak{C}}^{-1}\mathcal{M}_{f_{\mathfrak{C}}} W_{\mathfrak{C}}\Big(\bigotimes_{k\in\mathbb{N}} \rho_k^{\mathfrak{C}}\Big) \overset{(W_{\mathfrak{C}}(\otimes_{k\in\mathbb{N}}\rho^{\mathfrak{C}})(x)\equiv 1)}{=} W_{\mathfrak{C}}^{-1}(f_{\mathfrak{C}}) = \psi_1\otimes\cdots\otimes\psi_n\otimes\rho_{n+1}^{\mathfrak{C}}\otimes\cdots = \Psi.$$

That is, the unitary of the joint diagonalization (5.76) identifies $\psi_1\otimes\cdots\otimes\psi_n\otimes\rho_{n+1}^{\mathfrak{C}}\otimes\cdots$ with $f_{\mathfrak{C}}$. But we proved in Theorem 10 that there is a unique unitary operator making this identification and that was $W_{\mathfrak{C}}$. Therefore, the unitary for the joint diagonalization of the position operators in (5.76) is exactly the unitary that we found in Theorem 13, $\mathscr{W}_{\mathfrak{R}} = \oplus_{\mathfrak{C}\in\Gamma} W_{\mathfrak{C}}$.

o.ε.δ.

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[5] After all, the Born rule of $\mathbb{R}^n$ is exactly the spectral measure of the joint position PVM.

6

THE SCHRÖDINGER QUANTUM THEORY OF COUNTABLY MANY DISTINGUISHABLE DEGREES OF FREEDOM

6.1 The Rigorous Archetype of a Pilot-Wave Theory over $\mathbb{R}^\infty$

The machinery exposed in the previous chapters immediately suggests a generalization of Definition 4 to an archetype of a pilot-wave theory over $\mathbb{R}^\infty$.

Definition 20. The *archetype of a pilot-wave theory* of $n = |\mathbb{N}|$ degrees of freedom with global time parameter $t \in \mathbb{R}$ is the following set of items.

(i) The postulate that $\mathbb{R}^\infty := \prod_{k\in\mathbb{N}} \mathbb{R}$ with the product topology parametrizes the possible configurations of some system (i.e., each point of $\mathbb{R}^\infty$ corresponds to a different ontological arrangement of the system at each fixed *time* $t \in \mathbb{R}$).

(ii) A *Schrödinger picture model on* $\mathbb{R}^\infty$, by which we mean what follows.

(a) As a mathematical structure,

- the specification of a unit vector $\Psi_0 = (\Psi_0^{\mathfrak{c}})_{\mathfrak{c}\in\Gamma} \in \bigotimes_{k\in\mathbb{N}} L^2(\mathbb{R}, dx)$ called "initial" *wavefunction* and
- the specification of a self-adjoint densely defined operator $(D(H), H)$ acting on $\bigotimes_{k\in\mathbb{N}} L^2(\mathbb{R}, dx)$, called *Hamiltonian.* Note that via the functional calculus it generates for each $t \in \mathbb{R}$ an operator $U_t := exp(-\frac{i}{\hbar}Ht)$, constituting a SCOPUG acting on $\bigotimes_{k\in\mathbb{N}} L^2(\mathbb{R}, dx)$.[1] $\hbar$ is a positive constant called *Planck constant.*

(b) As a "law of physics",

- the postulate that this SCOPUG yields the *dynamical law* for Ψ, which is a time dependent vector in $\bigotimes_{k\in\mathbb{N}} L^2(\mathbb{R}, dx)$ called *wavefunction.* Namely, that $\Psi : t \in \mathbb{R} \longmapsto \Psi_t := U_t\Psi_0$. Equivalently (by Proposition 6.5 in (Schmüdgen, 2012)), if

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[1] Equivalently, by Stone's theorem (Theorem 6.2 in (Schmüdgen, 2012)), one could specify a SCOPUG $\{U_t\}_{t\in\mathbb{R}}$ on $\bigotimes_{k\in\mathbb{N}} L^2(\mathbb{R}, dx)$ and uniquely recover a Hamiltonian $(H, D(H))$ generating it as $U_t = exp(-i\frac{i}{\hbar}Ht)$.

$\Psi_0 \in D(H)$, Ψ is the unique solution to the differential equation $i\hbar\frac{\mathrm{d}}{\mathrm{d}t}\Psi_t = H\Psi_t$, which is called the *Schrödinger equation (of Hamiltonian H).*

(iii) A trajectory-based *primitive ontology,* by which we mean what follows.

(a) As a mathematical structure,

- the specification of a *guidance law*, i.e., the mapping of each wavefunction $\Psi : t \mapsto U_t\psi_0$ to some "flow"

$$\begin{array}{rccc} Q^{\Psi}: & \mathbb{R}\times\mathbb{R}^{\infty} & \longrightarrow & \mathbb{R}^{\infty} \\ & (t, q_0) & \longmapsto & Q^{\Psi}_t(q_0) \end{array}$$

satisfying that each $\{Q^{\Psi}_t(\cdot)\}_{t\in\mathbb{R}}$ is a strongly continuous[2] family of homeomorphisms with $Q^{\Psi}_0 = Id_{\mathbb{R}^{\infty}}$ and is *equivariant* with the measure $\mathbb{P}^{\Psi}$, i.e., $\mathbb{P}^{\Psi_0}\circ Q_0 = \mathbb{P}^{\Psi_t}\circ Q_t$, or symbolically,

$$\text{“}\int_{x\in B}|\Psi_0|^2(x)d^{\infty}x\text{”} = \text{“}\int_{x\in Q_t(B)}|\Psi_t|^2(x)d^{\infty}x\text{”} \qquad \forall B\in\mathfrak{B}(\mathbb{R}^{\infty}),\ \forall t\in\mathbb{R}. \tag{6.1}$$

More explicitly, $\{Q^{\psi}_t(\cdot)\}_{t\in\mathbb{R}}$ is such that for *one and hence any* joint diagonalization spectral basis $\mathfrak{R} = (\bigotimes_{k\in\mathbb{N}}\rho^{\mathfrak{C}}_k)_{\mathfrak{C}\in\Gamma}$ of the lifted position operators $\{Id\otimes\cdots\otimes Id\otimes\hat{q}\otimes Id\otimes\cdots\}_{k\in\mathbb{N}}$,

$$\sum_{\mathfrak{C}\in\Gamma}\int_{x\in B}|\widetilde{\Psi^{\mathfrak{C}}_0}|^2(x)\ d^{\infty}\mu_{\rho^{\mathfrak{C}}} = \sum_{\mathfrak{C}\in\Gamma}\int_{x\in Q_t(B)}|\widetilde{\Psi^{\mathfrak{C}}_t}|^2(x)\ d^{\infty}\mu_{\rho^{\mathfrak{C}}} \qquad \forall B\in\mathfrak{B}(\mathbb{R}^{\infty}),\ \forall t\in\mathbb{R}, \tag{6.2}$$

where $\widetilde{\Psi}_t := (\widetilde{\Psi^{\mathfrak{C}}_t})_{\mathfrak{C}\in\Gamma} := \mathscr{W}_{\mathfrak{R}}\Psi_t$ is the joint position representation of Ψ_t,

(b) As a "law of physics",

- the postulate that the system has an actual configuration $q_0 \in \mathbb{R}^{\infty}$ at $t = 0$ which is unknown to us but is "sampled" from a $\mathbb{P}^{\Psi_0}$-distribution, and the postulate that it follows the deterministic trajectory $t \mapsto Q^{\Psi}_t(q_0)$ at all times. Consequently, by equivariance, the configuration of the system is $\mathbb{P}^{\Psi_t}$-distributed at all $t \in \mathbb{R}$. As such, Q^{Ψ} is the *ensemble of possible trajectories of the system.*

(iv) As a corollary, item (iii) explains the main predictive backbone of items (i) and (ii): the so-called *Born Rule.* Namely, that the "probability" that at time $t \in \mathbb{R}$ the system is found in the configuration $q \in B$, for some $B \in \mathfrak{B}(\mathbb{R}^{\infty})$, is:

$$\mathbb{P}(q\in B \text{ at } t) = \mathbb{P}^{\Psi_t}(B) = \text{“}\int_{x\in B}|\Psi_t|^2(x)d^{\infty}x\text{”}.$$

◆

The clarification of what "sampled" means is exactly identical to that provided following Definition 4.

It is worth mentioning that in this work we are dealing with *distinguishable degrees of freedom.* By this we mean that for a generic permutation $\sigma \in S(\mathbb{N})$, the tuple $(x_1, x_2, ...) \in \mathbb{R}^{\infty}$ and $(x_{\sigma(1)}, x_{\sigma(2)}, ...) \in \mathbb{R}^{\infty}$ represent *different* ontological configurations of the system. The "indistinguishable degree of freedom" case will be discussed elsewhere.

At this point, the natural path to follow would be to provide a generalization of Bohmian mechanics to $N = |\mathbb{N}|$ distinguishable point-like particles in $\mathbb{R}^3$. For that, one would need to find a reasonable generalization of the Hamiltonian of Example 1. There are several ways in which one could proceed to build such Hamiltonians on the full ITP:

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[2] Meaning that for any choice $q_0 \in \mathbb{R}^{\infty}$, the path $t \mapsto Q^{\Psi}_t(q_0)$ is continuous in $\mathbb{R}^{\infty}$.

- Lift operators acting on each factor via Proposition 32 and then consider strong limits of their infinite sums (in line with Reed's (1970) proposal).
- Find interesting SCOPUGs in the full ITP, say, by taking products of SCOPUGs acting on each factor (in line with Streit's (1967) proposal).
- Provide a self-adjoint operator per improper ITP layer (where things are simpler to work with) and merge them using Proposition 33.

Presumably, the simplest models would provide dynamics that get reduced by the improper ITPs, while realistic models of interacting particles would require superpositions of different layers and transitions between them.

We will develop these ideas further in a future publication.

6.2 When the Configuration-Space is itself a Hilbert Space

Many interesting *field configuration-spaces*[3] are described by real separable Hilbert spaces. For instance, the fields that are determined by prescribing a real value on each point of $\mathbb{R}^3$ make a well-known Hilbert space if we impose that they decay square-integrably towards infinity: namely $L^2(\mathbb{R}^3, \mathbb{R}, d^3x)$. If we require that the fields possess k (generalized) derivatives with the same decay, the resulting space of fields is also a Hilbert space: the *Sobolev space* $H^k(\mathbb{R}^3, \mathbb{R}, d^3x)$. All such real-valued fields are generically called *scalar fields*. But many other fields, say, vector-valued fields, can also be accommodated within a Hilbert space. The heuristic example we provided in §1.3 embodies this idea. Note that even general tensor-valued fields can also be treated this way.[4]

Now, a configuration-space with separable Hilbert space structure has distinguishable and countably many degrees of freedom. To see this, recall that for any separable Hilbert space $\mathcal{H}$ there exists a *countable* ONB $\{e_k\}_{k\in\mathbb{N}} \subseteq \mathcal{H}$ with the natural unitary isomorphism associated to it

$$\begin{array}{rccc} \mathcal{U}: & \mathcal{H} & \longrightarrow & \ell^2(\mathbb{N},\mathbb{R}) \\ & \psi = \sum\limits_{k\in\mathbb{N}} \alpha_k \; e_k & \longmapsto & (\alpha_k)_{k\in\mathbb{N}} \end{array} . \tag{6.3}$$

That is, each vector $\psi \in \mathcal{H}$ can be naturally identified with a square summable sequence of real numbers $(\alpha_1, \alpha_2, ...) \in \ell^2(\mathbb{N}, \mathbb{R})$.[5] Now, $\ell^2(\mathbb{N}, \mathbb{R})$ is a vector subspace of $\mathbb{R}^\infty$. Hence, one can fully parametrize the Hilbert space $\mathcal{H}$ (representing the ontological configurations of some field) using $\mathbb{R}^\infty$. Following this line of thought, we will now suggest what one could call "$L^2(\ \ell^2(\mathbb{N},\mathbb{R}),\ d^\infty x)$".

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[3] Namely, spaces parameterizing possible arrangements for fields —as opposed to arrangements of particle positions.

[4] As a very raw idea, one could consider a tensor field representing the spacetime metric's restriction to the slices of a foliation by compact hypersurfaces on some manifold. Or, in order to work with foliations by non-compact hypersurfaces, one could consider for instance, a tensor field representing a perturbation of the Minkowski (or some other background) metric, such that each configuration decaying asymptotically corresponds to an asymptotically flat metric.

[5] Usually, one interprets this as "working in coordinates" (as when one uses $\mathbb{R}^n$ as a coordinate space for abstract vector spaces of dimension $n \in \mathbb{N}$).

6.2.1 The Space "$L^2(\ \ell^2(\mathbb{N},\mathbb{R}),\ d^\infty x\ )$"

We have seen that "$L^2(\ \mathbb{R}^\infty,\ d^\infty x\ )$" $= \bigotimes_{k\in\mathbb{N}} L^2(\mathbb{R},dx) \overset{\mathfrak{R}}{\cong} \oplus_{\mathfrak{C}\in\Gamma} L^2(\mathbb{R}^\infty,\ d^\infty\mu_{\mathfrak{C}})$. Then, how naive would be to replace $\mathbb{R}^\infty$ by its proper subspace $\ell^2(\mathbb{N},\mathbb{R})$ in that equation? There are several issues that could spoil such a restriction in general. However, as we will explain now, in our case (yet again) everything orchestrates itself so that we can do it. (Hereafter, we denote $\ell^2 := \ell^2(\mathbb{N},\mathbb{R})$.)

Assume for a second that $\ell^2 \in \mathfrak{B}(\mathbb{R}^\infty)$. On the one hand, if the measure of ℓ^2 according to some $d^\infty\mu_{\mathfrak{C}}$ were 1 (i.e., if it had *full measure* —equal to that of the full space $\mathbb{R}^\infty$), then one could canonically identify $L^2(\mathbb{R}^\infty, d^\infty\mu_{\mathfrak{C}})$ with $L^2(\ell^2, d^\infty\mu_{\mathfrak{C}})$ by just "forgetting the null sets". In fact, we found an example of such a $d^\infty\mu_{\mathfrak{C}}$ in the proof of Lemma 10. Importantly, in such a case, any other measure $d^\infty\nu_{\mathfrak{C}}$ of the same layer $\mathfrak{C}\in\Gamma$ would also yield measure 1 for ℓ^2.[6] On the other hand, if the measure of ℓ^2 according to some $d^\infty\mu_{\mathfrak{C}}$ were 0, then one could simply ignore that layer and its corresponding L^2-space. As for the case of full measure, if $d^\infty\mu_{\mathfrak{C}}$ yields 0 for ℓ^2, then every other measure in the same $\mathfrak{C}\in\Gamma$ will attribute 0 to ℓ^2. Also for this case, we found an example $d\mu_{\mathfrak{C}}$ in the proof of Lemma 10.

Yet, there is another possibility. If the measure of ℓ^2 was a number between 0 and 1 for some $d^\infty\mu_{\mathfrak{C}}$, then (i) other measures of the same $\mathfrak{C}\in\Gamma$ would not be forced to attribute the same measure to ℓ^2 and (ii) it would be unclear how to "restrict" meaningfully the space $L^2(\mathbb{R}^\infty, d^\infty\mu_{\mathfrak{C}})$ to some L^2-space over ℓ^2. Miraculously (or not), this third possibility never occurs —as we will prove now.

First, we check that ℓ^2 is in $\mathfrak{B}(\mathbb{R}^\infty)$, such that our concern is meaningful to begin with:

Lemma 28. $\ell^2(\mathbb{N},\mathbb{R}) \in \mathfrak{B}(\mathbb{R}^\infty)$. ♦

Proof: By definition, all projections $\pi_k : \mathbb{R}^\infty \to \mathbb{R}$ are measurable maps in the product σ-algebra $\odot_{j\in\mathbb{N}}\mathfrak{B}(\mathbb{R})$ —which equals $\mathfrak{B}(\mathbb{R}^\infty)$ by Prop. 1. As such, the maps $S_N := \sum_{k=1}^N \pi_k^2$ are measurable because finite sums and products of measurable functions are measurable (see Prop. 2.6 in (Folland, 1999)). But then, by Prop. 2.7 in (Folland, 1999), the mapping

$$\begin{array}{cccc} S_\infty : & \mathbb{R}^\infty & \longrightarrow & [0,+\infty] \\ & \alpha := (\alpha_1,\alpha_2,\ldots) & \longmapsto & \lim\limits_{N\to\infty} S_N(\alpha) = \sum\limits_{k=1}^{\infty} \alpha_k^2 \end{array} \tag{6.4}$$

is measurable as well. As such, since $[0,+\infty) \subset \mathbb{R}$ is measurable, $S_\infty^{-1}([0,+\infty)) = \ell^2(\mathbb{N},\mathbb{R})$ is measurable. But that is to say that ℓ^2 is an element of $\mathfrak{B}(\mathbb{R}^\infty)$. ***o.ε.δ.***

Definition 21. • Given a sequence of measurable spaces $(X_j,\Sigma_j)_{j\in\mathbb{N}}$ and the projections $\pi_k : \prod_{j\in\mathbb{N}} X_j \to X_k$ for $k\in\mathbb{N}$, define for each $J\subseteq\mathbb{N}$

$$\mathfrak{c}_J := \left\{ \pi_k^{-1}(E_k) \;\middle|\; k\in J,\ E_k\in\Sigma_k \right\} \subseteq \mathfrak{c}_0, \tag{6.5}$$

with $\mathfrak{c}_0$ as in Def. 7, such that $\sigma(\mathfrak{c}_0) = \odot_{j\in\mathbb{N}}\Sigma_j$. In particular, $\mathfrak{c}_{\mathbb{N}} = \mathfrak{c}_0$. As such, we will name $\sigma(\mathfrak{c}_J)$ the *J-partial product σ-algebra* on $\prod_{j\in\mathbb{N}} X_j$. Note that, $\mathfrak{c}_J \subseteq \mathfrak{c}_I$ for all $J\subseteq I$, which implies

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[6] Since different measures $d^\infty\mu_{\mathfrak{C}}, d^\infty d\nu_{\mathfrak{C}}$ in the same $\mathfrak{C}\in\Gamma$ are mut. a.c. by Theorem 11, they must agree on measure zero sets and thus, $d^\infty\mu_{\mathfrak{C}}(\mathbb{R}^\infty\setminus\ell^2) = 0 \iff d^\infty\nu_{\mathfrak{C}}(\mathbb{R}^\infty\setminus\ell^2) = 0$, which is equivalent (by finite additivity of disjoint sets) to $d^\infty\mu_{\mathfrak{C}}(\ell^2) = d^\infty\mu_{\mathfrak{C}}(\mathbb{R}^\infty) \iff d^\infty\nu_{\mathfrak{C}}(\ell^2) = d^\infty\nu_{\mathfrak{C}}(\mathbb{R}^\infty)$.

that $\sigma(\mathfrak{c}_J) \subseteq \sigma(\mathfrak{c}_I)$ —hence, all such partial product σ-algebras are sub-algebras of $\odot_{j\in\mathbb{N}}\Sigma_j$.[7]

- The σ-algebra on $\prod_{j\in\mathbb{N}} X_j$ defined by

$$\Sigma_{tail} := \bigcap_{n=2}^{\infty} \sigma(\mathfrak{c}_{\{n,n+1,\ldots\}})$$

is called the *tail* or *asymptotic σ-algebra.* It is the intersection of all infinite partial product σ-algebras.

(Informally, *Σ_{tail} contains all sets that are mapped to themselves by transformations of $\prod_{j\in\mathbb{N}} X_j$ that only modify finitely many coordinates.*) ♦

Theorem 21 (*Kolmogorov's 0-1 Law).* Given a family of probability spaces $(X_j, \Sigma_j, d\mu_j)_{j\in\mathbb{N}}$,

$$\text{for every} \quad E \in \Sigma_{tail} \begin{cases} \bullet \ \text{ either} & (\odot_{j\in\mathbb{N}} d\mu_j)(E) = 1 \\ \bullet \ \text{ or} & (\odot_{j\in\mathbb{N}} d\mu_j)(E) = 0 \end{cases}.$$

♦

Proof: See Theorem 10.10.17 in (Bogachev, 2007). ☐

Corollary 18. Given a sequence of probability measures $(d\mu_j)_{j\in\mathbb{N}}$ on $(\mathbb{R}, \mathfrak{B}(\mathbb{R}))$, denoting their infinite product measure by $d^\infty\mu := \odot_{k\in\mathbb{N}} d\mu_j$, then

$$\text{either} \quad d^\infty\mu(\ \ell^2(\mathbb{N},\mathbb{R})\ ) = 1 \quad \text{or} \quad d^\infty\mu(\ \ell^2(\mathbb{N},\mathbb{R})\ ) = 0.$$

(Informally, *since the membership of a sequence to ℓ^2 is not altered if we change finitely many of the elements in the sequence, Kolmogorov's 0-1 law applies.*) ♦

Proof: Following the notation in the proof of Lemma 28, consider the map $S_{n,N} := \sum_{k=n}^{N} \pi_k^2$. Each $\pi_k : \mathbb{R}^\infty \to \mathbb{R}$ with $k \in \{n, n+1, \ldots, N\}$ is $\sigma(\mathfrak{c}_{\{n,n+1,\ldots,N\}})$-measurable by definition, so, given that finite sums and products of measurable functions are measurable, then $S_{n,N}$ is $\sigma(\mathfrak{c}_{\{n,n+1,\ldots,N\}})$-measurable. Now, we saw that $\sigma(\mathfrak{c}_{\{n,n+1,\ldots,N\}}) \subsetneq \sigma(\mathfrak{c}_{\{n,n+1,\ldots\}})$, so, given a fixed $n \in \mathbb{N}$, $S_{n,N}$ is $\sigma(\mathfrak{c}_{\{n,n+1,\ldots\}})$-measurable for every $N \in \mathbb{N}$. But then, by Prop. 2.7 in (Folland, 1999), for each $n \in \mathbb{N}$, the point-wise limit map $S_{n,\infty} := \lim_{N\to\infty} S_{n,N}$ (on the extended real line) is also $\sigma(\mathfrak{c}_{\{n,n+1,\ldots\}})$-measurable. Therefore, because $[0, \infty)$ is measurable, $S_{n,\infty}^{-1}([0, +\infty)) \in \sigma(\mathfrak{c}_{\{n,n+1,\ldots\}})$ for each and every $n \in \mathbb{N}$. But trivially, if $(\alpha_1, \alpha_2, \ldots) \in \mathbb{R}^\infty$,

$$\left(\sum_{k=1}^{\infty} \alpha_k^2 < +\infty \right) \iff \left(\sum_{k=n}^{\infty} \alpha_k^2 < +\infty \quad \text{for some } n \in \mathbb{N} \right). \tag{6.6}$$

Thus, for all $n \in \mathbb{N}$ $S_{n,\infty}^{-1}([0, +\infty)) = \ell^2(\mathbb{N},\mathbb{R})$. This proves that $\ell^2(\mathbb{N},\mathbb{R}) \in \sigma(\mathfrak{c}_{\{n,n+1,\ldots\}})$ for all $n \in \mathbb{N}$. But by definition of Σ_{tail}, this implies that $\ell^2(\mathbb{N},\mathbb{R}) \in \Sigma_{tail}$. Finally, by Theorem 21, this causes that either $d^\infty\mu(\ell^2(\mathbb{N},\mathbb{R})) = 0$ or $d^\infty\mu(\ell^2(\mathbb{N},\mathbb{R})) = 1$. ***o.ε.δ.***

With all, it is indeed the case that for each $\mathfrak{C} \in \Gamma$ one can either ignore its layer or canonically do $L^2(\mathbb{R}^\infty, d^\infty\mu_{\mathfrak{C}}) = L^2(\ell^2, d^\infty\mu_{\mathfrak{C}})$. Let us make the latter precise.

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[7] Observe that the sets in $\sigma(\mathfrak{c}_J)$ are precisely $E_J \in \odot_{j\in J} X_j$ but "lifted" to the full space: i.e., sets like $E_J \times \prod_{j\in\mathbb{N}\setminus J} X_j \in \sigma(\mathfrak{c}_J)$.

To identify $L^2(\mathbb{R}^\infty, d^\infty\mu_{\mathfrak{C}}) = L^2(\ell^2, d^\infty\mu_{\mathfrak{C}})$ means that (i) the restriction of $\mathfrak{B}(\mathbb{R}^\infty)$ to ℓ^2 is taken as the σ-algebra of ℓ^2, namely,

$$\mathfrak{B}(\mathbb{R}^\infty) \restriction_{\ell^2} := \{\ M \cap \ell^2 \mid M \in \mathfrak{B}(\mathbb{R}^\infty)\ \}, \tag{6.7}$$

and (ii) that the measure $d^\infty\mu_{\mathfrak{C}}$ gets its domain restricted to those measurable sets. Now, we claim that $\mathfrak{B}(\mathbb{R}^\infty) \restriction_{\ell^2}$ equals the Borel σ-algebra of the subset topology on ℓ^2 induced by the product topology of $\mathbb{R}^\infty$. If we denote this induced topology by $\tau_{\ell^2}^{prod}$, what we are claiming is that $\mathfrak{B}(\mathbb{R}^\infty) \restriction_{\ell^2} = \sigma(\tau_{\ell^2}^{prod})$.

Lemma 29. Given a topological space (X, τ_X) and a set $A \subseteq X$, the *restriction of X's Borel σ-algebra*,

$$\sigma(\tau_X) \restriction_A := \{\ M \cap A \mid M \in \sigma(\tau_X)\ \}, \tag{6.8}$$

equals the Borel σ-algebra of A's subspace topology $(\tau_X) \restriction_A := \{U \bigcap A \mid U \in \tau_X\}$. That is, $\sigma(\tau_X) \restriction_A = \sigma(\tau_X \restriction_A)$. ♦

Proof: • Let $i_A : A \to X$ denote the inclusion map. Since $i_A^{-1}(B) = B \cap A$ for all $B \subseteq X$, then $(\tau_X) \restriction_A = i_A^{-1}(\tau_X)$ and $\sigma(\tau_X) \restriction_A = i_A^{-1}(\sigma(\tau_X))$. Therefore,

$$\sigma(\tau_X) \restriction_A = i_A^{-1}(\sigma(\tau_X)) \overset{\text{(Lem. 3)}}{=} \sigma(i_A^{-1}(\tau_X)) = \sigma((\tau_X) \restriction_A).$$

o.ε.δ.

Now, one could anticipate a delicate issue: ℓ^2, as a Hilbert space, already comes with its own norm-topology $\tau_{\ell^2}^{norm}$, and as such, the obvious σ-algebra to consider is its associated Borel σ-algebra $\sigma(\tau_{\ell^2}^{norm})$ —and not $\sigma(\tau_{\ell^2}^{prod})$. If they were different σ-algebras, we would be left with no obvious way to attribute a measure to every open set of $\tau_{\ell^2}^{norm}$ —which is presumably necessary for a reasonable calculus. Remarkably, the two σ-algebras coincide.

Proposition 39. $\tau_{\ell^2}^{prod} \subsetneqq \tau_{\ell^2}^{norm}$. ♦

Proof: Given $\mathfrak{G}_0$ denotes the base of the product topology of $\mathbb{R}^\infty$ as in Def. 6, each $E \in \mathfrak{G}_0$, is such that $E = \prod_{j\in J} U_j \times \prod_{j\in\mathbb{N}\setminus J} \mathbb{R}$ for some $U_j \in \mathfrak{B}(\mathbb{R})$ and $J \subseteq \mathbb{N}$ finite. In particular, by definition, $E \bigcap \ell^2(\mathbb{N}, \mathbb{R}) \in \tau_{\ell^2}^{prod}$.

- **Claim 1:** Furthermore, $E \bigcap \ell^2(\mathbb{N}, \mathbb{R}) \in \tau_{\ell^2}^{norm}$.

Check 1: We will prove that every point $\alpha := (\alpha_1, \alpha_2, ...) \in E \bigcap \ell^2$ is an interior point of $E \cap \ell^2$ in the $\tau_{\ell^2}^{norm}$ topology. In particular, we will prove that there exists an open ℓ^2-norm ball around any such α that is contained in $E \bigcap \ell^2$.

For each $j \in J$, since $U_j \subseteq \mathbb{R}$ is open, there exists an open and bounded interval $I_j \subset U_j$ containing α_j. Hence, one can define

$$\varepsilon := \min_{j\in J} \Big\{\ \text{distance}(\alpha_j,\ \mathbb{R} \setminus I_j)\ \Big\}, \tag{6.9}$$

which is the minimum distance from the α_j to the boundary of their respective intervals I_j. Then, for an arbitrary radius $r \in (0, \varepsilon)$, consider the ball

$$B_r(\alpha) := \Big\{\ \beta \in \ell^2\ \Big|\ \|\alpha - \beta\|_{\ell^2} < r\ \Big\}.$$

On the one hand, every $\beta \in B_r(\alpha)$ is trivially in ℓ^2. On the other hand,

$$\|\alpha-\beta\|_{\ell^2} < r \iff \sum_{k=1}^{\infty}(\alpha_k-\beta_k)^2 < r^2 \implies (\alpha_k-\beta_k)^2 < r^2 \iff |\alpha_k-\beta_k| < r < \varepsilon.$$

Hence, for all $k \in J$, $|\alpha_k-\beta_k| =: \text{distance}(\alpha_k,\beta_k) < \varepsilon \leq \text{distance}(\alpha_k,\ \mathbb{R}\setminus I_k)$. Thus, $\forall k \in J$, $\beta_k \in I_k$, such that $\beta_k \in U_k$ and thereby, $\beta \in E$. With all, $B_r(\alpha) \subseteq E \cap \ell^2$ for all $r \in (0,\varepsilon)$.

- **Claim 2:** If B is a base for a topology on X, then $B\restriction_Y := \{A \cap Y \mid A \in B\}$ is a base for the relative topology induced on Y.

Check 2: Theorem 6.3.(f) in (Willard, 2012).

- **Corollary:** $\tau_{\ell^2}^{prod} \subseteq \tau_{\ell^2}^{norm}$.

Check: $\mathfrak{G}_0$ is a base of the product topology of $\mathbb{R}^\infty$, so, by Claim 2, $\mathfrak{G}_0\restriction_{\ell^2} := \{E \cap \ell^2 |\ E \in \mathfrak{G}_0\}$ is a base for $\tau_{\ell^2}^{prod}$. But by Claim 1, $\mathfrak{G}_0\restriction_{\ell^2} \subseteq \tau_{\ell^2}^{norm}$.

- **Claim 3:** $\tau_{\ell^2}^{prod} \not\supseteq \tau_{\ell^2}^{norm}$, i.e., $\exists U \in \tau_{\ell^2}^{norm}$ such that $U \notin \tau_{\ell^2}^{prod}$.

Check 3: Assume that the claim is false. Then, by the corollary above $\tau_{\ell^2}^{norm} = \tau_{\ell^2}^{prod}$. As such, the unit ℓ^2-norm ball around the origin $B_1(0) := \{\alpha \in \ell^2 \mid \|\alpha\|_{\ell^2} < 1\}$ is also an open set in $\tau_{\ell^2}^{prod}$. Furthermore, because $\mathfrak{G}_0\restriction_{\ell^2}$ is a topological base of $\tau_{\ell^2}^{prod}$, there exists some $E = \prod_{j\in J} U_j \times \prod_{j \in \mathbb{N}\setminus J}\mathbb{R} \in \mathfrak{G}_0$ (with some finite J and open $U_j \subseteq \mathbb{R}$) such that $E \bigcap \ell^2 \subseteq B_1(0)$. But then, for each $k \in \mathbb{N}\setminus J$, $x = (0,...,0,2,0,....)$ (with 2 in the k-th position) is an element of $E \cap \ell^2$ and clearly, $\|x\|_{\ell^2} = 2$. That is, there is a vector of norm 2 inside $B_1(0)$. Absurd!

(The key was to notice that the product topology of $\mathbb{R}^\infty$ is *not* normable.) ***o.ε.δ.***

And even still:

Proposition 40. $\sigma(\tau_{\ell^2}^{prod}) = \sigma(\tau_{\ell^2}^{norm})$. We will denote this σ-algebra by $\mathfrak{B}(\ell^2)$ and call it *the* Borel σ-algebra of $\ell^2(\mathbb{N},\mathbb{R})$. ♦

Proof: We found in Proposition 39 that $\tau_{\ell^2}^{prod} \subset \tau_{\ell^2}^{norm}$, and as such, that $\sigma(\tau_{\ell^2}^{prod}) \subseteq \sigma(\tau_{\ell^2}^{norm})$. To prove the reverse inclusion, we will find a subfamily of $\sigma(\tau_{\ell^2}^{norm})$ that generates $\sigma(\tau_{\ell^2}^{prod})$.

- **Claim :** Every ℓ^2-norm open ball (which together, constitute a base of the topology $\tau_{\ell^2}^{norm}$—and hence, make a subfamily of $\sigma(\tau_{\ell^2}^{norm})$) is in $\sigma(\tau_{\ell^2}^{prod})$.

Check : One can prove exactly as we did in the proof of Lemma 28 that for all $\beta \in \ell^2$, the mapping

$$\begin{array}{rccc} f_\beta : & \mathbb{R}^\infty & \longrightarrow & [0,+\infty] \\ & \alpha = (\alpha_1,\alpha_2,...) & \longmapsto & \sum_{k=1}^{\infty}(\alpha_k-\beta_k)^2 \end{array} \tag{6.10}$$

is $\mathfrak{B}(\mathbb{R}^\infty)$-measurable. Then, using that $[0,r^2) \subset \mathbb{R}$ is measurable for all $r \geq 0$ we get that $f_\beta^{-1}([0,r^2)) =: B_r(\beta)$ (i.e., the ball of radius r around β) is a measurable set of $\mathfrak{B}(\mathbb{R}^\infty)$. Finally, since $B_r(\beta) \cap \ell^2 = B_r(\beta)$, $B_r(\beta) \in \mathfrak{B}(\mathbb{R}^\infty)\restriction_{\ell^2}$ and thus, by Lemma 29, $B_r(\beta)$ is an element of $\sigma(\tau_{\ell^2}^{prod})$.

- Now, by Theorem 16.11 in Willard (2012), every separable space is second countable, so $\ell^2(\mathbb{N},\mathbb{R})$ is second countable. By 16B.2 in (Willard, 2012), any topological base of a second countable space has a countable sub-family that is also a base for the topology. Hence, the open ball base of ℓ^2 has a sub-family V made of countably many open balls that also make up a base.[a] This means that any open set of $\tau^{norm}_{\ell^2}$ is a countable union of open balls. But a σ-algebra is closed under countable unions, so, any σ-algebra containing V also contains $\tau^{norm}_{\ell^2}$. In particular, $\sigma(V) = \sigma(\tau^{norm}_{\ell^2})$. But by the claim above, $V \subseteq \sigma(\tau^{prod}_{\ell^2})$, so, $\sigma(\tau^{norm}_{\ell^2}) \subseteq \sigma(\tau^{prod}_{\ell^2})$.

o.ε.δ.

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[a] In particular, by separability, one can take a dense countable subset of ℓ^2, say, A and consider the open balls with rational radius centered in A.

All in all, as measurable spaces $\left(\ell^2(\mathbb{N},\mathbb{R}),\ \sigma(\tau^{prod}_{\ell^2})\right) = \left(\ell^2(\mathbb{N},\mathbb{R}),\ \sigma(\tau^{norm}_{\ell^2})\right)$ and hence, if $d^\infty\mu_{\mathfrak{C}}(\ell^2) = 1$ for some layer $\mathfrak{C} \in \Gamma$ in *one (and hence any)* WR basis $\mathfrak{R}$, canonically, $L^2(\mathbb{R}^\infty, d^\infty\mu_{\mathfrak{C}}) = L^2(\ell^2, d^\infty\mu_{\mathfrak{C}})$ —irrespective of ℓ^2 having the extra structure and extra open sets given by the inner product.[8] In any other case, $d^\infty\mu_{\mathfrak{C}}(\ell^2) = 0$ and the layer naturally vanishes from our consideration. With all, our idea is indeed rigorously meaningful.

Definition 22. The restriction of the configuration-space $\mathbb{R}^\infty$ in "$L^2(\ \mathbb{R}^\infty,\ d^\infty x\ )$" to $\ell^2(\mathbb{N},\mathbb{R})$ can be naturally defined to be

$$\text{“}L^2(\ \ell^2(\mathbb{N},\mathbb{R}),\ d^\infty x\ )\text{”} := \bigoplus_{\mathfrak{C}\in\Gamma_1} \left(\otimes^{\mathfrak{C}}_{k\in\mathbb{N}} L^2(\mathbb{R}, dx)\right) \overset{\mathfrak{R}}{\cong} \bigoplus_{\mathfrak{C}\in\Gamma_1} L^2(\ \ell^2(\mathbb{N},\mathbb{R}),\ d^\infty\mu_{\mathfrak{C}}\ ), \tag{6.11}$$

where $\Gamma_1 \subset \Gamma$ is the set of $\approx$-classes for which $d^\infty\mu_{\mathfrak{C}}(\ell^2) = 1$ in *one (and hence any)* WR basis $\mathfrak{R}$. In particular, for $\mathfrak{C} \in \Gamma \setminus \Gamma_1$, $d^\infty\mu_{\mathfrak{C}}(\ell^2) = 0$ in *one (and hence any)* WR basis $\mathfrak{R}$.

In words, "$L^2(\ \ell^2(\mathbb{N},\mathbb{R}),\ d^\infty x\ )$" is the closed vector subspace of $\bigotimes_{k\in\mathbb{N}} L^2(\mathbb{R}, dx)$ constituted by the layers $\mathfrak{C} \in \Gamma_1$. ♦

As such, $\mathbb{P}^\Psi$ still gives an invariant notion of Born rule (as in Theorems 13 and 20), together with a well-defined notion of equivariant trajectories. In particular, all the results of the present document also apply to the restricted space (e.g., the considerations about PVMs, the lifts of operators, etc.).

With that, clearly, by merely exchanging $\mathbb{R}^\infty$ for $\ell^2(\mathbb{N},\mathbb{R})$ in Definition 20, one can easily formulate the archetype of a rigorous pilot-wave theory for systems with a configuration-space that is parametrizable by ℓ^2.

Note that in this setting, the "position" operators $(\widehat{q}_k,\ D(\widehat{q}_k))$ would no longer represent quantized *position* per se, but the quantization of the *coefficients of an expansion* in some ONB. That is, a wavefunctional $\Psi = (\Psi_{\mathfrak{C}})_{\mathfrak{C}\in\Gamma_1} \in \bigoplus_{\mathfrak{C}\in\Gamma_1} L^2(\ \ell^2(\mathbb{N},\mathbb{R}),\ d^\infty\mu_{\mathfrak{C}}\ )$ is a tuple of functions $(\alpha_1, \alpha_2, ...) \in \ell^2(\mathbb{N},\mathbb{R}) \longmapsto \Psi^{\mathfrak{C}}(\alpha_1, \alpha_2, ...)$, for which the Born rule dictates the probability to find a specific value for each expansion coefficient —namely, the probability to find a specific configuration of the field parametrized by those coefficients. Likewise, for each $q_0 \in \ell^2$, the trajectory $t \mapsto Q^\Psi_t(q_0) = (\alpha^{q_0}_1(t), \alpha^{q_0}_2(t), ...)$ of the associated pilot-wave theory (see Def. 20) would tell us that it is perfectly reasonable to postulate the existence of a well-defined field $\varphi(t) := \sum_{n=1}^\infty \alpha^{q_0}_n(t) e_n$ that exists at all times and is "piloted by the wave-vector Ψ". If so, one could now rigorously substantiate the believe that when one measures a field in a laboratory "it is there" because it was already there before we measured it. This would bring *field* "flavored"[9]

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[8] In particular, for purely measure-theoretic purposes, once we provide a σ-algebra, the topology and the rest of structures —such as an inner product— play no role.

[9] As opposed to the *particle* ontology proposed in say, (Dürr et al., 2005).

understandability and speak-ability to the world of quantum fields —as heuristically proposed by Struyve (2010) or Bohm (1952b), among others.

As we proceed to see now, in this setting, the operators $\widehat{q}_k$ and their linear combinations are exactly what physicists call the "field operators".

6.2.2 Outlook: The Pullback to $\mathcal{H}$ —is this "$L^2(\mathcal{H},\ d^\infty x)$" ?

If one wanted a wavefunction space over $\mathcal{H}$ instead of over ℓ^2, the procedure seems simple: pullback the measure space structure by the unitary $\mathcal{U} : \mathcal{H} \to \ell^2(\mathbb{N}, \mathbb{R})$ of (6.3) that acts as coordinate chart for $\mathcal{H}$. That is, consider $\mathcal{H}$ with the pullback measure and the pullback σ-algebra —which, following the results of the last section and the fact that $\mathcal{U}$ is an homeomorphism, trivially coincides with the Borel σ-algebra of $\mathcal{H}$'s norm-topology. That is, given a WR basis $\mathfrak{R}$, for each $\mathfrak{C} \in \Gamma$ and each $E \in \mathfrak{B}(\mathcal{H})$, define

$$d\mu_{\mathfrak{C}}^{\mathcal{H}}(E) := \mathcal{U}^* d^\infty \mu_{\mathfrak{C}}(E) := d^\infty \mu_{\mathfrak{C}}(\ \mathcal{U}(E)\ ). \tag{6.12}$$

(*"The measure of a set E of fields is the measure of the corresponding set of coordinates"*.)

With that we get the probability spaces $(\mathcal{H},\ \mathfrak{B}(\mathcal{H}),\ d^\infty \mu_{\mathfrak{C}}^{\mathcal{H}})$ and associated to them the Hilbert spaces $L^2(\mathcal{H},\ d\mu_{\mathfrak{C}}^{\mathcal{H}})$. Each element of $L^2(\mathcal{H}, d\mu_{\mathfrak{C}}^{\mathcal{H}})$ is a wavefunction that takes as arguments the fields themselves: some $\psi \in \mathcal{H} \longmapsto \Psi(\psi)$. This is often called a *wavefunctional.* In addition, $L^2(\mathcal{H}, d\mu_{\mathfrak{C}}^{\mathcal{H}})$ comes with an obvious identification with $L^2(\ell^2,\ d^\infty \mu_{\mathfrak{C}})$: just map the arguments using $\mathcal{U}$ (as a sort of "coordinate transformation") as we propose after the following lemma.

Lemma 30. Given (X, Σ_X), and (Y, Σ_Y) are measurable spaces, let $F : X \to Y$ be a measurable bijection, and let $d\mu$ be a measure on Y. Then,

(i) $F^* d\mu := d\mu \circ F$ is a measure on (X, Σ_X). The so-called, *pullback* measure.

(ii) For all integrable $f : Y \to [0, \infty]$ and $B \in \Sigma_X$,

$$\int_{y \in F(B)} f(y)\ d\mu = \int_{x \in B} (f \circ F)(x)\ (d\mu \circ F).$$

♦

Proof: **Item (i) :** First, $d\mu \circ F(\emptyset) = d\mu(\emptyset) = 0$. Second, since F is a bijection, given the inverse map F^{-1}, $F = (F^{-1})^{-1}$. Now, given arbitrary pairwise disjoint $(B_j)_{j\in\mathbb{N}} \subseteq \Sigma_X$ and using in $(\star)$ that the pre-image map commutes with a countable union,

$$(d\mu \circ F)\Big(\bigsqcup_{j\in\mathbb{N}} B_j\Big) = d\mu\Big((F^{-1})^{-1}\Big(\bigsqcup_{j\in\mathbb{N}} B_j\Big)\Big) \overset{(\star)}{=} d\mu\Big(\bigsqcup_{j\in\mathbb{N}} (F^{-1})^{-1}\Big(B_j\Big)\Big) \overset{(d\mu\ \sigma\text{-add.})}{=} \sum_{j=1}^{\infty} \underbrace{d\mu((F^{-1})^{-1}}_{(d\mu\circ F)}(B_j)).$$

Thus, $(d\mu \circ F)$ is σ-additive and as such, a measure.

- **Item (ii) :** By Them. 2.10 in (Folland, 1999), there is a monotonously increasing sequence of simple functions $s_n := \sum_{j=1}^{N_n} c_j^n\ \mathbb{1}_{B_j^n}$ with $N_n \in \mathbb{N}$, $c_n^j > 0$, $B_j^n \in \Sigma_Y$ such that point-wise almost everywhere $s_n(y) \xrightarrow[n\to\infty]{} f(y) \cdot \mathbb{1}_{F(B)}(y)$. Since $\mathbb{1}_{F(B)}(F(x)) = \mathbb{1}_B(x)$, we also have that $(s_n \circ F)(x) \xrightarrow[n\to\infty]{} (f \circ F)(x) \cdot \mathbb{1}_B(x)$ point-wise. Hence,

$$\int_{y\in F(B)} f(y)\ d\mu \overset{(\text{mont. conv})}{=} \lim_{n\to\infty} \sum_{j=1}^{N_n} c_j\ d\mu(B_j^n) = \lim_{n\to\infty} \sum_{j=1}^{N_n} c_j\ (d\mu \circ F)\Big(F^{-1}(B_j^n)\Big) =$$

$$= \lim_{n\to\infty} \int_{x\in X} \sum_{j=1}^{N_n} c_j \ \mathbb{1}_{F^{-1}(B_j^n)}(x) \ (d\mu \circ F) = \lim_{n\to\infty} \int_{x\in X} \underbrace{\sum_{j=1}^{N_n} c_j \ \mathbb{1}_{B_j^n}(F(x))}_{(s_n\circ F)(x)} \ (d\mu \circ F) =$$

$$= \lim_{n\to\infty} \int_{x\in X} (s_n \circ F)(x) \ (d\mu \circ F) \overset{\text{(mont. conv)}}{=} \int_{x\in B} (f \circ F)(x) \ (d\mu \circ F).$$

o.ε.δ.

Lemma 31. For each $\mathfrak{C} \in \Gamma_1$, given $\mathcal{U} : \mathcal{H} \to \ell^2(\mathbb{N}, \mathbb{R})$ as in (6.3), the pullback measure space $(\mathcal{H}, \ \mathfrak{B}(\mathcal{H}), \ d\mu_{\mathfrak{C}}^{\mathcal{H}})$ with $d\mu_{\mathfrak{C}}^{\mathcal{H}} := d^\infty\mu_{\mathfrak{C}} \circ \mathcal{U}$ is well-defined. Then, the map

$$\begin{array}{rccc} \mathcal{W}_{\mathcal{U}} : & L^2(\ \ell^2(\mathbb{N},\mathbb{R}), \ d^\infty\mu_{\mathfrak{C}}\ ) & \longrightarrow & L^2(\mathcal{H}, d\mu_{\mathfrak{C}}^{\mathcal{H}}) \\ & \Psi & \longrightarrow & \Psi \circ \mathcal{U}, \end{array} \tag{6.13}$$

is a unitary map. Explicitly, $(\mathcal{W}_{\mathcal{U}}\Psi)(\psi) := \Psi\Big(\alpha_1 = \langle e_1, \psi\rangle, \quad \alpha_2 = \langle e_2, \psi\rangle, \cdots\Big)$. ♦

Proof: Well-definition follows from Lemma 30 and the fact that U is an homeomorphism. To prove that $\mathcal{W}_{\mathcal{U}}$ is an isometry, let $\Psi \in L^2(\ell^2, d^\infty\mu_{\mathfrak{C}})$. Then,

$$\|\mathcal{W}_{\mathcal{U}}\Psi\|^2_{L^2(\mathcal{H})} = \int_{\psi\in\mathcal{H}} \Big|\mathcal{W}_{\mathcal{U}}\Psi\Big|^2(\psi) \ d\mu_{\mathfrak{C}}^{\mathcal{H}} = \int_{\psi\in\mathcal{H}} |\Psi|^2 \circ \mathcal{U}\,(\psi) \ (d^\infty\mu_{\mathfrak{C}} \circ \mathcal{U}) \overset{\text{(Lem. 30)}}{=} \int_{\alpha\in\ell^2} |\Psi|^2(\alpha) d^\infty\mu_{\mathfrak{C}} = \|\Psi\|^2_{L^2(\ell^2)}.$$

To prove surjectivity, let $\Phi \in L^2(\mathcal{H}, d\mu_{\mathfrak{C}}^{\mathcal{H}})$ arbitrary. Then, $\Phi \circ (\mathfrak{U}^{-1}) \in L^2(\ell^2, d^\infty\mu_{\mathfrak{C}})$ because

$$\big\|\Phi \circ (\mathfrak{U}^{-1})\big\|^2_{L^2(\ell^2)} = \int_{\alpha\in\ell^2} (|\Phi|^2 \circ \mathcal{U}^{-1})(\alpha) d^\infty\mu_{\mathfrak{C}} \overset{\left(\substack{d^\infty\mu = (d\mu^{\mathcal{H}} \circ \mathcal{U}^{-1}) \\ + \text{ Lem. } 30}\right)}{=} \int_{\psi\in\mathcal{H}} |\Phi|^2(\psi) d\mu_{\mathfrak{C}}^{\mathcal{H}} = \|\Phi\|^2_{L^2(\mathcal{H})} < \infty.$$

But obviously, $\mathcal{W}_{\mathcal{U}}\Big(\Phi \circ (\mathfrak{U}^{-1})\Big) = \Phi$. Hence, $\mathcal{W}_{\mathcal{U}}$ is unitary. ***o.ε.δ.***

Why This Seems to be On the Right Track

It is often heuristically emphasized in physics textbooks that for Schrödinger picture wavefunctionals $\Psi(\psi)$, a *field operator* should act as “multiplication by the field” ψ. Namely, that such a WF space should be a representation space where the “field operators are diagonal.” Our construction naturally brings up rigorous field operators that indeed act as multiplication by the fields in the argument.

Proposition 41. Fix some WR basis $\mathfrak{R}$ and let $\mathfrak{C} \in \Gamma_1$. For each fixed $\beta \in \ell^2(\mathbb{N}, \mathbb{R})$ (that we call a *coordinate test function*) define

$$\begin{array}{rccc} x_\beta : & \ell^2(\mathbb{N},\mathbb{R}) & \longrightarrow & \mathbb{R} \\ & \alpha & \longmapsto & \langle \beta, \alpha\rangle_{\ell^2} \end{array}. \tag{6.14}$$

It is everywhere finite, so, one can define an associated multiplication operator $(\ \widehat{x_\beta}, D\Big(\widehat{x_\beta}\Big)\ ) := (\mathcal{M}_{x_\beta}, \ D(\mathcal{M}_{x_\beta}))$ acting on $L^2(\ell^2, d^\infty\mu_{\mathfrak{C}})$ as in Lemma 23. By the same lemma, it is a densely defined self-adjoint operator. We call the family of self-adjoint operators $\{\widehat{x_\beta}\}_{\beta\in\ell^2}$, the *coordinate field operators.*

- Note that for $\tilde{e}_k := (\overbrace{0, ..., 0, 1}^{k}, 0, ...)$, $\widehat{x_{\tilde{e}_k}} = \hat{x}_k$, i.e., it is the k-th “position” operator (Def. 19). Hence, heuristically, $\widehat{x_\beta} = \sum_{n=1}^{\infty} \beta_n \hat{x}_n$ —namely, the coordinate field operators are linear combinations of the lifted position operators.

- Given a real separable Hilbert space $\mathcal{H}$ with ONB $\{e_n\}_{n\in\mathbb{N}} \subseteq \mathcal{H}$, and given $\mathcal{U} : \mathcal{H} \to \ell^2$ is its coordinate chart (6.3), the family of self-adjoint operators $\{\widehat{x_\beta}\}_{\beta\in\ell^2}$ induces for each (so-called, *test function*) $\varphi \in \mathcal{H}$, the operator on $L^2(\mathcal{H}, d\mu_{\mathfrak{C}}^{\mathcal{H}})$ given by

$$\widehat{\psi_\varphi} := \mathcal{W}_{\mathcal{U}}\ \widehat{x_{\mathcal{U}\psi}}\ \ \mathcal{W}_{\mathcal{U}}^{-1} \quad \text{with domain} \quad \mathcal{W}_{\mathcal{U}} D(\widehat{x_{\mathcal{U}\psi}}). \tag{6.15}$$

For each test function $\varphi \in \mathcal{H}$, $\widehat{\psi_\varphi}$ acts on $\Psi \in L^2(\mathcal{H}, d\mu^{\mathcal{H}})$ in its domain as

$$\widehat{\psi_\varphi}\ \Psi(\psi) = \langle\varphi, \psi\rangle\ \Psi(\psi). \tag{6.16}$$

We call the family of self-adjoint operators $\{\widehat{\psi_\varphi}\}_{\psi\in\mathcal{H}}$ on $L^2(\mathcal{H},\ d\mu^{\mathcal{H}})$, the *field operators.*

♦

Proof: By Lemma 3.88 in (Teufel, 2021), the conjugation of a self-adjoint operator by a unitary map leads to another self-adjoint operator. Hence, the only thing left to be proven is that (6.16) follows from (6.15). Let $\eta \in D(\widehat{x_\beta})$ with $\Psi := \mathcal{W}_{\mathcal{U}}\ \eta$. Then,

$$\widehat{\psi_\varphi}\ \Psi(\psi) = \Big(\mathcal{W}_{\mathcal{U}}\ \widehat{x_{\mathcal{U}\varphi}}\ \mathcal{W}_{\mathcal{U}}^{-1} \circ \mathcal{W}_{\mathcal{U}}\ \eta\Big)(\psi) = \Big(\mathcal{W}_{\mathcal{U}} \circ \widehat{x_{\mathcal{U}\varphi}}\ \eta\Big)(\psi) \overset{\text{(by def)}}{=} \Big(\widehat{x_{\mathcal{U}\varphi}}\ \eta\Big)(\mathcal{U}\,\psi) =$$

$$\overset{\text{(by def)}}{=} \langle\ \mathcal{U}\varphi,\ \mathcal{U}\psi\ \rangle_{\ell^2}\ \eta(\mathcal{U}\psi) \overset{\left(\substack{\mathcal{U}\text{ isometry}\\ +\text{ def. of } W_{\mathcal{U}}}\right)}{=} \langle\ \varphi,\ \psi\ \rangle_{\mathcal{H}}\ \ (\mathcal{W}_{\mathcal{U}}\eta)(\psi) = \langle\varphi, \psi\rangle_{\mathcal{H}}\ \Psi(\psi).$$

o.ε.δ.

As an example, if $\mathcal{H} = L^2(\mathbb{R}^3, \mathbb{R}, d^3x)$, for $\Psi \in L^2(\ L^2(\mathbb{R}^3, \mathbb{R}, d^3x),\ d\mu^{\mathcal{H}})$,

$$\widehat{\psi_\varphi}\ \Psi(\psi) = \int_{x\in\mathbb{R}^3} \varphi(x)\psi(x)\ \Psi(\psi)\ d^3x. \tag{6.17}$$

In physics, this is called an *operator valued distribution*, because if we define the formal symbol "$\widehat{\psi_{\delta_x}}$" $\equiv \widehat{\psi_x}$ to act as $\widehat{\psi_x}\Psi(\psi) := \psi(x)\Psi(\psi)$ (just like a position operator in $\mathbb{R}^n$ did $\widehat{q_k}\psi(x) := x_k\psi(x)$), then,

$$\widehat{\psi_\varphi}\ \Psi(\psi) = \int_{x\in\mathbb{R}^3} \varphi(x)(\widehat{\psi_x}\ \Psi)(\psi)\ d^3x = \Bigg(\underbrace{\text{“}\int_{x\in\mathbb{R}^3} \varphi(x)\widehat{\psi_x}\ d^3x\text{”}}_{=:\widehat{\psi_\varphi}}\Bigg)\ \Psi(\psi).$$

In fact, if one considers that physical space is a discrete lattice, such as $\mathbb{Z}^3$, then the correct field configuration-space for a scalar field is $\mathcal{H} = L^2(\mathbb{Z}^3, \mathbb{R}, d\nu)$ instead of the one above. In such a setting, for each $x_0 \in \mathbb{Z}^3$, the delta $\delta_{x_0}(x) :=$ "1 *if* $x = x_0$ *and* 0 *else*" is a rigorous vector in $L^2(\mathbb{Z}^3, \mathbb{R}, d\nu)$ —they even make up an ONB. Hence, in such a case, for an arbitrary $\Psi \in L^2(\ L^2(\mathbb{Z}^3, \mathbb{R}, d\nu),\ d\mu^{\mathcal{H}}\ )$ the map

$$\widehat{\psi_{\delta_x}}\Psi(\psi) := \psi(x)\Psi(\psi) \tag{6.18}$$

is rigorously a densely defined self-adjoint operator $\forall x \in \mathbb{Z}^3$.

In a very similar way, by pulling back linear combinations of lifted "momentum operators" from $L^2(\ell^2, d\mu_{\mathfrak{C}})$ to $L^2(\mathcal{H}, d\mu_{\mathfrak{C}}^{\mathcal{H}})$, one can define families of operators that act as directional (functional) derivatives on the wavefunctionals. This allows to define *"canonical momentum field operators"* that satisfy the canonical commutation relations with the field operators above (in common dense domains).

We will provide an exhaustive account of all this elsewhere —for now, in Appendix B we offer an analysis of the coordinate field operators and their commutation relations.

With all, the reason why this is promising is that it hints towards an explicit rigorization of what is arguably the most successful heuristic tool that physicists employ in QFT: what Wallace (2006) calls *"Lagrangian QFT"*. There, one generalizes (for now, completely non-rigorously) the Dirac heuristic of "canonical quantization" to Hamiltonian formulations of field theory. Namely, one "puts hats" over the field and canonical conjugate momentum fields and in doing so, one obtains Hamiltonian operators for the quantum analogue of the field theory in question.

Still a lot to be Cleared Out

With all, given a separable real Hilbert space $\mathcal{H}$ and an ONB $\{e_k\}_{k\in\mathbb{N}} \subseteq \mathcal{H}$, for each WR basis $\mathfrak{R} = \{\bigotimes_{j\in\mathbb{N}} \rho_j^{\mathfrak{C}}\}_{\mathfrak{C}\in\Gamma}$, the ℓ^2 coordinates induce the Hilbert space of wavefunctionals

$$\bigoplus_{\mathfrak{C}\in\Gamma_1} L^2(\mathcal{H},\ d\mu_{\mathfrak{C}}^{\mathcal{H}}), \tag{6.19}$$

where $d\mu_{\mathfrak{C}}^{\mathcal{H}} := d^{\infty}\mu_{\rho_{\mathfrak{C}}}$. In particular, it induces a unitary identification $\mathbb{W}_{\mathcal{U}} := \oplus_{\mathfrak{C}\in\Gamma_1}\mathcal{W}_{\mathcal{U}}$,

$$\begin{array}{rccc} \mathbb{W}_{\mathcal{U}}: & \bigoplus_{\mathfrak{C}\in\Gamma_1} L^2(\,\ell^2(\mathbb{N},\mathbb{R}),\ d\mu_{\mathfrak{C}}^{\mathcal{H}}) & \longrightarrow & \bigoplus_{\mathfrak{C}\in\Gamma_1} L^2(\mathcal{H},\ d\mu_{\mathfrak{C}}^{\mathcal{H}}) \\ & (\Psi^{\mathfrak{C}})_{\mathfrak{C}\in\Gamma_1} & \longmapsto & (\mathcal{W}_{\mathcal{U}}\Psi^{\mathfrak{C}})_{\mathfrak{C}\in\Gamma_1} := (\Psi^{\mathfrak{C}}\circ\mathcal{U})_{\mathfrak{C}\in\Gamma_1} \end{array}. \tag{6.20}$$

Thus, one can embed the wavefunctionals over $\mathcal{H}$ as a subspace of $\bigotimes_{k\in\mathbb{N}} L^2(\mathbb{R}, dx)$ by coupling $\mathbb{W}_{\mathcal{U}}$ with the WR representation map $\mathscr{W}_{\mathfrak{R}}$ of Theorems 13 and 20.

Now, one could be tempted to call this construction "$L^2(\mathcal{H}, d^{\infty}x)$". However, there are two different notions of "basis choice" made to define it and both require notable care (and further research):

1. There is a dependence on the representation basis $\mathfrak{R}$ that is not at all transparent. In the case of ℓ^2 this dependence was only apparent, because by (6.11) the whole thing was a subspace of $\bigotimes_{k\in\mathbb{N}} L^2(\mathbb{R}, dx)$, which is a space that does not depend on any choice (the abstract "invariant" space —much like an abstract manifold is to its coordinate chart spaces). If we still want to use an invariant notion of space for (6.19), we should:
    - **(i)** either keep the link to the abstract $\bigotimes_{k\in\mathbb{N}} L^2(\mathbb{R}, dx)$ now extended until $\mathcal{H}$, as suggested after (6.19). That is, treat (6.19) as the "coordinate representation" of the abstract subspace $\oplus_{\mathfrak{C}\in\Gamma_1}(\otimes_{k\in\mathbb{N}}^{\mathfrak{C}} L^2(\mathbb{R}, dx))$ of $\bigotimes_{k\in\mathbb{N}} L^2(\mathbb{R}, dx)$ —say, as the "joint diagonalization space for field operators".
    - **(ii)** or put $\bigotimes_{k\in\mathbb{N}} L^2(\mathbb{R}, dx)$ under the rug by defining equivalence classes[a]

$$\left(\Psi \in \bigoplus_{\mathfrak{C}\in\Gamma_1} L^2(\mathcal{H},\ d\mu_{\rho_{\mathfrak{C}}}^{\mathcal{H}}),\ \mathfrak{R} = \Big\{\bigotimes_{j\in\mathbb{N}} \rho_j^{\mathfrak{C}}\Big\}_{\mathfrak{C}\in\Gamma}\right) \sim \left(\Phi \in \bigoplus_{\mathfrak{C}\in\Gamma_1} L^2(\mathcal{H},\ d\mu_{\eta_{\mathfrak{C}}}^{\mathcal{H}}),\ \tilde{\mathfrak{R}} = \Big\{\bigotimes_{j\in\mathbb{N}} \eta_j^{\mathfrak{C}}\Big\}_{\mathfrak{C}\in\Gamma}\right)$$

$$\text{when} \quad (\mathscr{W}_{\mathfrak{R}}^{-1}\circ\mathbb{W}_{\mathcal{U}}^{-1})\,\Psi \ = \ (\mathscr{W}_{\tilde{\mathfrak{R}}}^{-1}\circ\mathbb{W}_{\mathcal{U}}^{-1})\,\widetilde{\Psi},$$

    namely, when they refer to the same vector of $\bigotimes_{k\in\mathbb{N}} L^2(\mathbb{R}, dx)$ that "lurks" behind.

    Any of the two constructions could deserve the name "$L^2(\mathcal{H}, d^{\infty}x)$", since now, there would be an $\mathfrak{R}$-choice-independent notion of "Born rule" directly in $\mathcal{H}$ (without needing

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[a] Much like how physicists often define a tangent space in the settings of manifolds as classes of equivalence of vectors in $\mathbb{R}^n$ associated to Jacobians of chart-changes.

to pullback from ℓ^2). Namely, one would have a "Born rule"-measure associated to either (i) each "field operator representation" of $\Psi \in \oplus_{\mathfrak{C}\in\Gamma_1}(\otimes_{k\in\mathbb{N}} L^2(\mathbb{R}, dx))$ or (ii) to each class of vectors $[\Psi]$. Subordinate to that we would have an equivariance notion for trajectory ensembles in $\mathcal{H}$. With all, what this suggests is to look for the analogue of an abstract construction like $\bigotimes_{k\in\mathbb{N}} L^2(\mathbb{R}, dx)$ perhaps closer to $\mathcal{H}$, and/or to make precise sense of the "diagonalization of field operators".

2. There seems to be a dependence on the ONB $\{e_k\}_{k\in\mathbb{N}} \subseteq \mathcal{H}$ that we chose to build $\mathcal{U} : \mathcal{H} \to \ell^2$. For instance, given another ONB $\{\tilde{e}_k\}_{k\in\mathbb{N}} \subseteq \mathcal{H}$ and its associated $\widetilde{\mathcal{U}} : \mathcal{H} \to \ell^2$, the pullbacks of the same measure $d^\infty\mu_{\mathfrak{C}}$ on $(\ell^2, \mathfrak{B}(\ell^2))$, namely, $d^\infty\mu_{\mathfrak{C}} \circ \mathcal{U}$ and $d^\infty\mu_{\mathfrak{C}} \circ \widetilde{\mathcal{U}}$ can assign different measures to the same subsets of $\mathcal{H}$. And yet, it could happen that the resulting Born rule (built by putting all layers together) is invariant. Our concern here is due to a phenomenon that also happens in manifold theory, the closest analogue being an abstract finite dimensional vector space with an inner product (V^n, g). Just as here we have a chart $\mathcal{U}$ from $\mathcal{H}$ to $(\ell^2, \langle\cdot,\cdot\rangle_{\ell^2})$ for each ONB, there, one has a chart mapping (V^n, g) to $(\mathbb{R}^n, \langle\cdot,\cdot\rangle_{Eucl})$ for each ONB (and even for each non-orthonormal basis). Also in (V^n, g), if we had a measure in the image of some coordinate chart, its pullback to V could yield a different measure as a function of the employed chart.[a] That is why, in order to formulate a theory of integration on manifolds (where one "secretly" employs integrals in coordinates), it is necessary to introduce some additional structure in the manifold that selects a fixed measure in each coordinate chart —representing the measure in the abstract space. That is, one needs to introduce a consensus between the chart images. In the case of (V, g) the usual way is to build the measure using g, as the so-called Riemann volume form $d\mu_g$ on V, which in coordinates looks like "$\sqrt{det\ g_{ij}}\ d^n x$". Now that we have a reasonable notion of "$d^\infty x$" in $\ell^2(\mathbb{N})$, if needed, one could look for a similar idea. The solution of this issue would provide a notion of integration over field configurations that could also be interesting in other contexts.

In order to face both obstacles, the promising direction seems to be to view the results we found in the previous chapters as the coordinate representation of structures living on abstract Hilbert manifolds —the infinite dimensional generalization of a Riemannian manifold (Abraham et al., 2001). We will provide a detailed solution to the issues explained in this box in another publication.

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[a] Although it must be said that the coordinate charts built from V^n's ONBs are isometries from (V^n, g) to $(\mathbb{R}^n, \langle\cdot,\cdot\rangle_{Eucl})$. As such, if one pullbacks the Lebesgue measure (which is the Riemann volume form of Euclidean space), all ONB charts will agree with the resulting measure induced in the manifold V^n —it will be the Riemann volume form of V^n. It is this why we suggested the possibility that in the ITP setting, the measure built by assembling the pullbacks of all the layers is independent of the chosen ONB for the construction of the "coordinate chart".

6.3 The Connection to Fock space and the CCR representations

If $\otimes_{k\in\mathbb{N}} L^2(\mathbb{R}, dx)$ (or a closed subspace therein) really corresponds to a rigorization of an L^2-space over the expansion coefficients, following our derivation of the Fock space in §1.3, one would expect to be able to develop something close to Fock space out of $\otimes_{k\in\mathbb{N}} L^2(\mathbb{R}, dx)$. Furthermore, since Fock space is the canonical space where models of algebraic QFT are implemented (Arai, 2018), in view of the previous section, one would expect that there is indeed a connection to our structure. And indeed, as we prove in detail in Appendix B, the generalization of our construction of §1.3 yields the so-called "bosonic" Fock space —among others. In Appendix B, we explain how Fock space corresponds to only one of the (quasi-)layers of our ITP: the one that has as generator the product of infinitely many harmonic oscillator ground states. Interestingly, even in the infinite case, the "bosonic particles" of Fock space will turn out to be "excitations" of expansion coefficients. In particular, the coordinate field operators as defined above will be representations of the canonical commutation relations (CCR) that when mapped to Fock space, will correspond to the usual Fock representation field operators —the ones obtained from the Segal field operators. Finally, we will prove that what the rest of (quasi-)layers describe are "exotic Fock spaces", representations of the CCR inequivalent to the Fock representation, where the "vacuum" has infinitely many bosons —as a sort of bosonic "Dirac sea"—, the erstwhile so-called, "myriotic representations".

6.4 Bonus: A Way to Parametrize Non-Decaying Fields by $\mathbb{R}^\infty$

So far, we have only seen a way to parametrize fields that *decay* at infinity using $\mathbb{R}^\infty$. As the following example suggests, there could be ways to go beyond.

For any separable topological space, such as $\mathbb{R}^3$, there exists a dense countable subset. The lattice made of rational numbers, $\mathbb{Q}^3 \subset \mathbb{R}^3$, is one such example. Then, note that:

Lemma 32. Given topological spaces X, Y with Y Hausdorff, if two continuous maps $f, g : X \to Y$ agree in a dense subset $\mathfrak{V} \subseteq X$ (i.e., $f \restriction_{\mathfrak{V}} = g \restriction_{\mathfrak{V}}$), then, they agree everywhere: $f = g$. ♦

Proof: See Corollary 13.14 in (Willard, 2012). □

As such, any continuous field $\varphi \in \mathcal{C}^0(\mathbb{R}^3, \mathbb{R})$ —which has no *decay* constraint— is uniquely characterized by its values over the countable set $\mathbb{Q}^3$. That is, given a fixed bijection $q : \mathbb{N} \to \mathbb{Q}^3$, $n \mapsto q_n$, the sequence $(f(q_1), f(q_2), ...) \in \mathbb{R}^\infty$, fully determines the whole continuous field f.

Now, not all sequences $(g_1, g_2, ...) \in \mathbb{R}^\infty$ necessarily have a corresponding continuous field $g \in \mathcal{C}^0(\mathbb{R}^3, \mathbb{R})$ such that $g(x_n) = g_n$ for all $n \in \mathbb{N}$. Presumably, we could find a wildly oscillating sequence such that when putting its values over $q_n \in \mathbb{Q}^3$, in every open ball of $\mathbb{R}^3$, there is some q_k with $q_k = 1$ and some q_ℓ with $q_\ell = -1$. The same issue also occurred with $\ell^2(\mathbb{N}, \mathbb{R})$, namely, it was also a proper subset of $\mathbb{R}^\infty$, and still, we found a natural solution.

However, as we did with $\ell^2(\mathbb{N}, \mathbb{R})$, in order to assert any meaningful statement, one would first need to study the measure-theoretic properties of the subsets of $\mathbb{R}^\infty$ corresponding to continuous functions. Be that as it may, the fact is that $\mathcal{C}^0(\mathbb{R}^3, \mathbb{R})$ —or in general, $\mathcal{C}^0(X, \mathbb{R})$ for any separable topological space X—, has countably many distinguishable degrees of freedom and thus, it can be parametrized by $\mathbb{R}^\infty$.

7

HISTORICAL REMARKS: AN OVERLOOKED PATH IN MATHEMATICAL PHYSICS?

7.1 von Neumann's Motives to Develop the ITP

When von Neumann introduced and "dissected" the ITP of Hilbert spaces in his 1939 article, he did it with two clear objectives. In the introduction (point 6), he shared those two projections and the first one reads as follows:[1]

> *"An essential result of our theory is, that the ring $\mathcal{L}^q(\otimes_{j\in I}\mathcal{H}_j)$ of all those bounded operators of $\bigotimes_{j\in I}\mathcal{H}_j$ which are generated (algebraically or by limiting-processes) by operators of the $\mathcal{H}_j$, $j \in I$, does not contain all bounded operators of $\bigotimes_{j\in I}\mathcal{H}_j$.* [...] *What happens could be described in the quantum-mechanical terminology as a 'splitting up' of $\bigotimes_{j\in I}\mathcal{H}_j$ into 'non-intercombining systems of states', corresponding to the 'incomplete' direct products $\otimes^{\mathfrak{C}}_{j\in I}\mathcal{H}_j$. This viewpoint, as well as its connection with the theory of 'hyperquantisation' will be discussed elsewhere."*
>
> (von Neumann, 1939)

"Hyperquantisation" was the name with which the quantization of infinitely many degrees of freedom was referred to in those days —when physicists just started to talk about the quantization of fields. So, seemingly von Neumann's idea was to make a precise account of QFT using the theory of ITPs. As a remark for later, note how he underlined the fact that there exist bounded operators that cannot be obtained in terms of operators "lifted" from individual factor spaces.

On the other hand, the second motivation he outlined reads as follows:

> *"Another application of our theory could be made to the theory of measure in infinite products of spaces, which is the basis for the modern theory of probabilities. (Cf. (2), (3), (5).) Here a certain 'incomplete'* [in our terminology, 'improper'] *direct product $\otimes^{\mathfrak{C}}_{j\in I}\mathcal{H}_j$ is fundamental. This application too, will be discussed in another publication."*
>
> (von Neumann, 1939)

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[1] We took the liberty to change the mathematical symbols to the ones we employ in the present document in all the quotes. Also, note that we have written comments inside the quotes using "$[\cdots]$".

After this, it should not come as a surprise that the "incredibly opportune" theorem by Kakutani (1948) —which was *exactly* what we needed to classify infinite product measures— was precisely obtained employing von Neumann's ITP. In fact, in the paper where Kakutani exposed this theorem in 1948 he explicitly said:

> *"The results of this paper were much amplified and the arguments used below much simplified, thanks to certain suggestions kindly made by Professor John von Neumann. In particular, the introduction of inner product and isometric embedding of* [...] *are due to Professor J. von Neumann. For all these I wish to express my heartiest thank."* (Kakutani, 1948)

With all, the reader will agree that our discoveries point precisely to the intersection von Neumann was talking about. But then, were not these topics already covered in the "discussions elsewhere" that he promised? The sad story is that such "discussions" were never published. Let us understand why. From 1939 (when he published the paper on the ITP) until 1949 (when Kakutani found some of the results anticipated in the ITP paper) von Neumann was occupied attending the human condition: as his foreword to a publication in 1949 exposes,

> *"This paper was written in 1937-38. Various other commitments prevented the author from effecting some changes, which he had intended to carry out before publishing the paper. This delayed the publication until the present time."*
>
> (Von Neumann, 1949)

When analyzing that period of his life (which encompasses the Second World War and the beginning of the Cold War), one can find that he joined many different scientific and governmental organizations related to the US military. Indeed, in Ulam's words:

> *"An incomplete list of scientific and organizational activities contains the following positions: From 1940-1957, he was a member of the Scientific Advisory Committee, Ballistic Research Laboratories, Aberdeen Proving Ground, Maryland; the Navy Bureau of Ordnance, Washington, D. C. from 1941-1955; consultant to Los Alamos Scientific Laboratory 1943-1955; also the Naval Ordnance Laboratory, Silver Spring, Maryland from 1947-1955; member of the Research and Development Board, Washington, D. C. 1949-1953; a consultant to the Oak Ridge National Laboratory, Oak Ridge, Tennessee 1949- 1954; member from 1950 to 1955 of the Armed Forces Special Weapons Project, Washington, D. C; also in Washington a member of the Scientific Advisory Board, U. S. Air Force, Washington, D. C. 1951-1957; a member of the General Advisory Committee by presidential appointment 1952-1954; and on the Technical Advisory Panel on Atomic Energy, Washington, D. C. 1953-1957; Chairman of the Advisory Committee on Guided Missiles (1954-1957 with Clark Millikan as acting chairman in 1956)." (Ulam, 1958)*

All this is reflected in his scientific publications, which "all of a sudden", after 1940 concentrate in numerical methods, shock waves, hydrodynamics, partial differential equations, electronic computation, the theory of automata and game theory —Beebe (2013) has provided a remarkably complete list. With all, in the end, he could not write down his thoughts on "hyperquantisation" (nor its connection to probability theory) even if he actually did have plans to do so,[2] since, in 1955 he was diagnosed cancer, which made him pass away two years later (at the very young age of 53).

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[2] His intentions are clear when one reads the posthumous words of (Ulam, 1958) quoted in §7.4.

7.2 What Became of the ITP?

As it is obvious, eventually, the mathematical quantum physics community —which one could say, von Neumann was the founder of— took over his endeavor and employed the theory of ITPs to describe certain aspects of QFT (as we will comment in a moment). However, oddly enough, in modern treatments of mathematical QM, it is often stated that there is no proper definition of ITP for Hilbert spaces. As an example, one can read the following in a textbook about "quantum spin systems on *infinite* lattices":

> *"Hence, a reasonable assumption of what the Hilbert space of infinitely many copies of the spin system (say, a spin chain) would be is*
>
> $$\mathscr{H} := \bigotimes_{n=-\infty}^{\infty} \mathbb{C}^2.$$
>
> *This, unfortunately, is not well defined. In particular, the natural choice of an inner product on this space does not converge, and hence is ill-defined."*
>
> (Naaijkens, 2017)

And yet, there are a few modern textbooks where the ITP is indeed introduced, such as (Weaver, 2001) (see §2.5), Baez et al. (2014) (see §4.5), (Thirring, 1979) (see §1.4) or (Klauder, 1999) (see §6.4). But even in those, the way ITPs are introduced is mainly as what we called *improper* ITPs, as though a single layer of the ITP *was* the ITP. They call them ITPs of *"grounded Hilbert spaces"*: they assume that a family of Hilbert spaces $(\mathcal{H}_j)_{j\in\mathbb{N}}$ is given, each with a distinguished unit vector $\psi_j^o \in \mathcal{H}_j$ called "ground state". Then, they make a construction where instead of taking all $\mathscr{C}$-sequences, one takes only those in the $\approx$-class of $\psi_1^o \otimes \psi_2^o \otimes \cdots$ —a vector they call, the "infinite ground state" or "vacuum state". That is, they consider the layer of the ITP generated by $\bigotimes_{j\in\mathbb{N}} \psi_j^o$, independently of the rest of layers and independently of any ambient space they may be part of. Of course, this makes the ITPs unappealing because then it looks like there is no unique construction!

Fortunately, a couple of those textbooks, such as both (Weaver, 2001) and (Baez et al., 2014), do state the fact that there is a more general notion of ITP in which each "grounded" ITP is a natural subspace, but they also point out its "uselessness" for physics:

> *"In the case of infinitely many factors there are two competing definitions of tensor products, both of which reduce to the preceding construction when the number of factors is finite.* [Briefly explains the concept of the proper ITP...] *This tensor product has had little application, probably because it is essentially never separable. The other definition, which we adopt, requires the choice of a distinguished unit vector u_k in each H_k; this vector plays the role of a sort of zero or ground state."*
>
> (Weaver, 2001)

Finally, it must be said that other authors like Arai (2018) truthfully mention in passing the existence of the ITP construction, but insist on the same lack of use for QM:

> *"[...] it is a non-trivial task to define* [the ITP] *rigorously, however. For details, see the original [...] But, nowadays, infinite tensor product Hilbert spaces are not so often used, at least in the field of quantum mathematical physics."* (Arai, 2018)

With all, the feeling that the reader must have (as does the author) is that sometime in between von Neumann and us, the community realized that there was something wrong with the full ITP construction. The author has not yet been able to find exactly what, but in this chapter they present the main hypotheses of their (admittedly not yet exhaustive) bibliographic research.

7.3 What Went Wrong with von Neumann's Project?

On the one hand, it is true that the notion of *quasi*-convergence used to define the inner product of the ITP seems ad-hoc and that one can resort to individual ITP layers to avoid it (after all, quasi-convergence is only used to define orthogonality between the different layers inside a same quasi-layer —see Prop. 58). However, von Neumann already provided a way to dodge the issue within the full ITP construction (even in the advantage of a quantum description), as explained in §A.2.2 and A.3. Moreover, as sketched in §A.1.b, one could employ extensions of the notion of net limit (akin to Banach limits) to justify von Neumann's quasi-convergence, as he himself pointed out —see the quote after Thm. 28 in Appendix A.

The issue seems to be more subtle (or perhaps more human). To comprehend it, one needs to understand first why essentially the only approach to QFT that mathematical physicists worked out until its last conclusions has been an algebraic or Heisenberg-picture approach —with *abstract algebras of "observables"* as the main characters. As such, let us travel back in time to the birth of the mathematical school of algebraic QM in the 1930-1950's.

7.3.1 The Ascent of Algebraic QM

Consider the standard quantum theory over a finite dimensional configuration-space (such as, $\mathbb{R}^n$). Within the formulation famously made rigorous by VonNeumann (1932), one way to specify the theory would be to choose a Hilbert space $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ and a set of self-adjoint operators on $\mathcal{H}$, $\{\hat{O}_j\}_j$, labeled by names manifesting their "experimental" or "observable" correlatives: e.g., "position of j-th degree of freedom" or "canonical momentum of k-th degree of freedom". Then, as the instrumental formulation it is, one declares that vectors $\psi \in \mathcal{H}$ describe laboratory preparations such that $\langle \psi, \hat{O}_j \psi \rangle$ provides the expected value for the respective "observable correlative". Next, one singles out a function of those operators, $\hat{H}$, called "Hamiltonian", declares it generates the time evolution of vectors ψ in $\mathcal{H}$ via $U_t = e^{-it\hat{H}}$ and makes each "observable" operator convey the effect of time evolution: $\hat{O}_j(t) := U_t \hat{O}_j U_t^{-1}$. According to the standard view, the set of time evolved operators encodes *all the predictions of the theory*, so one could now forget about the U_t. That is, providing only the action of the "observable" operators in time on $\mathcal{H}$, namely, $\hat{O}_j(t)$, together with their experimental correlatives, one has all standard QM needs. Now, what if we forgot about the action of $\hat{O}_j(t)$ on $\mathcal{H}$ and even forgot the fact they are operators on $\mathcal{H}$? That is, what if we only kept the abstract algebraic relations between them[3] and their experimental labels? Could we recover the initial theory from there? Remarkably, the answer is affirmative *for finitely many* "position-momentum operator" pairs. This is the content of the famous Stone-von Neumann theorem from 1931 —see also Theorem VIII.14 in (Reed and Simon, 1981). The result was celebrated for making the algebraic or Heisenberg approach and Schrödinger's $L^2(\mathbb{R}^n)$-space approach mathematically "equivalent". Slightly more precisely, what one abstracts is the algebra of *bounded* operators distinguishing those that satisfy the *canonical commutation relations* (CCR) in the *Weyl exponential form* —useful to dodge unbounded operators. Then, the theorem loosely states that there is a

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[3] Or more precisely, between the "bounded functions" built from those position and momentum operators.

"unique"[4] *irreducible representation* (irrep) of the algebraic structure as operators on a Hilbert space and that it is precisely the Schrödinger formulation of QM.

This discovery led to the abstract $*$-algebra theory and its ramifications during the 1930's and 40's —again, mainly by the hands of von Neumann. Arguably, this was a "dream come true" for many mathematicians: it was now possible to work on QM in terms of abstract algebras for which the Schrödinger theory on $L^2(\mathbb{R}^n, d^n x)$ was a mere representation, a mere "coordinate system". As a step towards the generalization of the Stone-von Neumann theorem to any quantum theory (such as the incipient QFT) —and as a reinforcement of the "mantra" about the *irreducibility* of representations—, in 1947, Segal offered a polished theory of irreps of operator algebras (C^*-algebras), with a theorem that, according to him:

> *"[...] implies that for the computational purposes of quantum theory it is sufficient to consider only those representations of a C^*-algebra (which in quantum theory is the algebra generated by the observables) which are irreducible. [...] In practice only the irreducible representations are used, in general. Our result shows that this limitation to irreducible representations is valid for an arbitrary physical system which is describable by an algebra of operators on a Hubert space (in the case of the Schrödinger-Heisenberg non-relativistic theory this fact had been established by special considerations by von Neumann* [the Stone-von Neumann theorem]."
>
> (Segal, 1947)

However, things became less elegant when it was discovered at the turn of decade that the Stone-von Neumann theorem was critically false for the infinitely many "position and momentum" operators needed for QFT.

> *"Von Neumann's theorem* [...]*states that up to multiplicity, there is only one representation (the Schrödinger representation) of the relations (X.95)* [the Weyl CCR] *if n ranges over a finite set of integers. It was thought for a long time that von Neumann's theorem held for the case of infinitely many n, but examples of inequivalent representations occurred in the work of Friedrichs in the late 1940's and were emphasized in the later work* [during the 1950's] *of Segal and Garding-Wightman."*
>
> (Reed and Simon, 1975)

It is worth noting the following chronology to get a sense of how far heuristic QFT had progressed by 1950. Quantum electrodynamics (hence, QFT) and the occupation number representation —essentially the bosonic Fock space— were first introduced by Dirac (1927) and then extended by Heisenberg and Pauli (1929); Fock space as such, was introduced in (Fock, 1932) —although it would not be rigorized until (Cook, 1953); the S-matrix formalism for scattering was introduced by Wheeler (1937) and Heisenberg (1943); and finally, the first relativistically invariant (and perturbative) QFT formulations were introduced by Tomonaga (1946), Schwinger (1948) and Feynman (1949), being the work of Dyson (1949a,b) the first one unifying them and introducing the S-matrix for a perturbative account of QED.

At this turn of decade, the aforementioned violation of the Stone-von Neumann theorem for QFT was manifestly discovered by Segal and van Hove, constituting a serious "scandal" for the

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[4] Unique up to unitaries that identify the "position and momentum operators" between representations —or rather their generated SCOPUGs—, among other technicalities we avoided in our prosaic explanation.

mathematical physicists attempting the first steps towards a non-perturbative formalization of QFT: an asymptotic S-matrix treatment, even for the most trivial interacting dynamics (and even for some coordinate transformations) could only be avoided using non-Fock (meaning "inequivalent to Fock") representations of the CCR. According to Segal (1963), he had already proven the violation privately[5] in 1950, when he communicated it orally to van Hove. But it was van Hove's 1952 paper what publicly announced that the "folklore extrapolation" of the Stone-von Neumann theorem to QFT was false —leaving the "Schrödinger-Heisenberg equivalence" in check within QFT. It is important to say that both Segal and van Hove employed layers of the ITP as model spaces to prove their announcements. In particular, —although it was not phrased in these words— van Hove revealed that to make a non-perturbative sense of the "free QFT" Hamiltonian (made of countably many harmonic oscillators) when interaction terms are added (linear anharmonicity terms for each oscillator), one needs to use (quasi-)layers of the ITP other than that generated by the tensor product of infinitely many harmonic oscillator ground states —which we will informally refer to as the "Gaussian layer" and is equivalent to the Fock representation (see Appendix B). In a very literal sense, what he found can be expressed from the perspective of the full ITP as the ground state of the interacting Hamiltonian being no longer in the Gaussian layer, instead belonging to a layer with an inequivalent CCR irrep (as we prove in Appendix B, the obvious CCR representation of any layer in $\otimes_{j\in\mathbb{N}}L^2(\mathbb{R},dx)$ that is in a different quasi-equivalence class to the Gaussian layer's, is inequivalent to the Fock representation). In van Hove's own words:

> *"The origin of the divergences is found to lie in the fact that the stationary states of the field interacting with the sources* [namely, the eigenstates of the interacting Hamiltonian] *are no linear combinations of the stationary states of the free field.* [Those of the infinitely many free harmonic oscillators, which belong to the Gaussian layer.] *The former are not contained in the Hilbert space spanned by the latter (they even turn out to be orthogonal to this space)."* [The claim on orthogonality sounded awkward in his presentation, because it seemed like the state vector, say, the ground state, was 'getting out of its Hilbert space'. And yet, it is literally true from the perspective of the full ITP —the new ground state becomes orthogonal to the whole layer identified with the Fock space because it transitions to a different layer.]
>
> (Van Hove, 1952)

This announcement was followed by the (still mathematically "partially heuristic"[6]) examples in (Friedrichs, 1953), (Haag, 1955) and (Gårding and Wightman, 1954; Wightman and Schweber, 1955), that established what today is known as "Haag's theorem". As a side note, these examples were now given in Fock-like spaces (instead of an ITP): in Segal (1963)'s words *"These authors* [says, after mentioning the three publications just cited] *employ an 'occupation number' representation which has a somewhat obscure physical interpretation and is dependent on the choice of a basis in the single-particle space."* As nicely exposed by Earman and Fraser

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[5] *"Our first observation of the existence of inequivalent representations of the commutation relations (1950)* [year that corresponds with no reference (to Segal) in the bibliography] *appears to provide one of the simplest rigorous demonstrations of the phenomenon in relevant concrete terms.* [... (Explains the remark)] *The foregoing results were communicated orally to van Hove, among others, who not long afterwards formulated his well-known paradox (1952), to the effect that the eigenstates of the 'free' hamiltonian* [... (shows the formal Hamiltonian of countably many harmonic oscillators)] *are orthogonal to those of the 'total' hamiltonian* [... (he adds a linear anharmonicity to each oscillator of the free Hamiltonian)]."

[6] Segal (1963) deems them *"particular examples [...] of a mathematically partially heuristic character."*

(2006), these authors described the issue in line with how Wightman's axioms of QFT invite one to formulate QM —which is not in terms of an ITP, but in terms of a space where there is a distinguished state to be called "vacuum" even in the absence of a Hamiltonian.[7]

The discovery that the abstract algebraic picture does not tell everything about the quantum system could have seriously "checkmated" the dream of algebraic QM. But remarkably, it caused the opposite. Since the issue of irreps was proven to be physics and not a mere "mathematical curiosity", it urged more than ever to work as abstractly as possible, favoring results that were as representation-independent as possible. The spirit of algebraic QM was boiling and one can clearly read this fervor for instance in (Segal, 1963). As explained there, they were even ready to abandon the framework of Hilbert spaces altogether before abandoning the algebras —and all with a fervent instrumentalist perspective.[8]

(If the reader is interested in having a more explicit understanding of how the inequivalent representations, the Haag theorem and even "renormalizaiton theory" fit within the ITP language, §6.4 and 6.5 in (Klauder, 1999) provide a very pedagogical explanation.)

7.3.2 Rise and Fall of ITP Research in QM: a Feint to von Neumann's Aim?

Under the above prism, it is not a surprise that even the research that was conducted on the ITP was framed within an algebraic perspective. Namely, the main objective was not the study of QM on an ITP, but the study of CCR irreps (hence, at best of QM on particular layers of the ITP), together with how the physical situation (say, the Hamiltonian) could force a choice of a single irrep. Some exemplar references of the "boom" that ITP research experienced during the late 60's (with their citation numbers according to *Google Scholar*) are:

- (Klauder and McKenna, 1965): [51 citations] they discovered a way to employ particular layers of the ITP as a representation space for bosonic fields.
- (Klauder et al., 1966): [82 citations] they discovered that the condition for the position and momentum operators of different $\approx$-layers to be equivalent irreps is exactly that they belong to the same $\overset{q}{\approx}$-equivalence class.
- (Chaiken, 1968) [70 citations] they study the meaning of a number operator in non-Fock representations employing the layers of the ITP as a case study. They also provide one of the first proofs that the only layers allowing a number operator in the style of a Fock space are those $\overset{q}{\approx}$-equivalent to the "Gaussian layer".
- (Streit, 1967) [36 citations] they provide one of the first mathematically rigorous studies of ITP spaces as irreps of the Weyl CCR, where they provide a simpler an clearer proof of how different $\overset{q}{\approx}$-layers correspond precisely to inequivalent irreps.

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[7] As explained in Appendix B, even within the same $\approx$-layer of an ITP, there are multiple generators that could deserve the name "vacuum". Hence, even in a single layer of the ITP there is no single distinguished vector analogous to the vacuum vector of Fock space —that is, unless one explicitly introduces a Hamiltonian with a ground state.

[8] The instrumentalist perspective should not come as a surprise in an age when it was (mistakenly) well-known that even non-relativistic QM was "incomprehensible", admitting no "hidden variable" explanation.

- (Reed, 1970) [20 citations] and (Reed, 1969) [8 citations] they provide rigorous techniques to define infinite sums of self-adjoint operators in layers of the ITP and how their well-definition could "force" the use of one class of layers over the other ones. They also clarify and generalize results of previous authors.
- (Reents, 1974) [4 citations] and (Kraus et al., 1977) [12 citations] they extend Streit's (1967) work determining when infinite products of SCOPUGs are still strongly continuous when restricted to a single layer of the ITP —usually such products cause transitions between the layers making them non-continuous from the perspective of a single layer.
- (Polley, 1980) [0 citations] they prove that only certain layers of the ITP make the restriction of the infinite product of unitary representations of the Lorentz group still strongly continuous.

Interestingly, all of them approached the ITP as a collection of possible irreps of the CCR to be employed independently of each other. No author seems to consider that the ambient space where the layers are natural to could be an arena for QM. Could it be due to its non-separability? Or rather, its reducibility? "Why use something non-separable and reducible if the irreps on separable Hilbert spaces are working well to explain QFT?" Or was it perhaps due to the categorical words (that we will review in a moment) of the eminences in the field like Streater and Wightman? Be that as it may, this might explain why today, if anything, it is the improper ITP that is explained in mathematical physics textbooks —it was how these researchers explained it to their students.

Now, it is visible from the citation chain that by the beginning of the 1970's there is a sudden decay in the interest on ITPs, essentially killing the field by the end of the decade. The most convincing proof of this deflation is that despite the quality and the foundationality of the results in (Reed, 1970) about dynamics in ITPs —not to talk about the relevance of the author in the mathematical QM community–, there exist only 20 citations of the work and none of them is particularly "magnetic" for later potential citations. At least, the reason why Reed himself stopped working on the ITP (which should be an alarm due to the relevance of that author) has a more mundane explanation.[9] The true mystery is that the *whole* community seemed to loose their interest. It might be that we still have to find the paper that dissuaded the community with a dramatic result, but for now there are three other reasons the author is mulling over.

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[9] M. Reed earned the Ph.D. in 1969 with a thesis on self-adjointness of operators in the layers of the ITP —which is not available in any library the author has access to. Yet, as Reed himself points out in (the "Acknowledgments" sections of) both papers cited above, they consist of results obtained in his doctoral thesis, so one can gain a good picture of his thesis from the two papers. About a year after their publication, he was already writing with B. Simon the series of books that still today are the standard reference of mathematical QM. The first one was published in 1972. During the 8 years they spent writing the series of textbooks, Reed's other publications centered on hyperbolic differential equation theory (with no other result on the ITP), anticipating what was about to come: after the last QM book in the series was published in 1979, he went into mathematical biology —where still today he is one of the Giants. Hence, not only did he loose interest in the ITP, but apparently also in QM in general. One could then wonder why Simon did not try the ITP road in QFT, but then, one would realize that as one of the founders of Euclidean QFT, he developed his own school of mathematical QFT. Noticeably, he even jokes in (Simon, 1979) about using non-separable Hilbert spaces in QM: *"Throughout, all our Hilbert spaces will be complex and separable (are there any others?) and our inner product is linear in [...]."*

7.3.3 The Authorities “Disliked” it: there were more Productive Approaches

The first possible reason isthat: the full ITP implies a *non-separable Hilbert space*. So, while there existed alternatives employing separable Hilbert spaces, “why even bother?” One should only try the hard endeavor once the obvious simpler alternatives are exhausted (perhaps now?). Most importantly, the Giants of the community, such as Streater and Wightman, explicitly discouraged it during those first critical days.[10] It turns out that in their famous 1964 book *“PCT, Spin and All That”* (fore-wording the “boom” we mentioned), Streater and Wightman explicitly (and very categorically) warned against non-separable Hilbert spaces and in particular, against the full ITP construction. Due to its capital relevance, we include the whole passage with a few inline comments.

> *“In non-relativistic quantum mechanics it is natural to consider only separable Hilbert spaces because one is usually dealing with a finite number of particles and can realize the states as vectors in* $L^2(\mathbb{R}^n)$, [... (they explain the space)] *this Hilbert space is separable (Ref. 20). It is sometimes argued that in quantum field theory one is dealing with a system of an infinite number of degrees of freedom and so must use a non-separable Hilbert space. Roughly, the idea is:*

$$\begin{pmatrix}\text{system of a finite}\\ \text{number of degrees}\\ \text{of freedom}\end{pmatrix} \longleftrightarrow \begin{pmatrix}\text{separable}\\ \text{Hilbert space}\end{pmatrix}; \quad \begin{pmatrix}\text{system of an}\\ \text{infinite number of}\\ \text{degrees of freedom}\\ \text{or something}\end{pmatrix} \longleftrightarrow \begin{pmatrix}\text{non-separable}\\ \text{Hilbert space}\end{pmatrix}$$

> *Our next task is to explain why this is wrong, or at best grossly misleading.*
>
> *In the first place note that if* $\mathcal{H}_1, \mathcal{H}_2, \ldots$ *is a sequence of separable Hilbert spaces then their direct sum* $\oplus_j \mathcal{H}_j$ *is a separable Hilbert space.* [... (They explain what a countable direct sum is)] *The direct sum can be used to construct state vectors which are superpositions of states for an arbitrarily large number of particles.* [...] *Just this Hilbert space will be used in Chapter 3 to give explicit formulas for the theory of a free field. This is a clearcut counter example to the above assertion; a free field is a system of an infinite number of degrees of freedom.* [
>
> It is worth recalling that a vector in Fock space can describe arbitrarily many particles but *not* infinitely many. After all, given $\Psi = (\psi^{(0)}, \psi^{(1)}, \ldots)$,

$$\text{“}\mathbb{P}\Big(\#\textit{particles} \geq N\Big)\text{”} = \sum_{n\geq N} ||\psi^{(n)}||^2 \xrightarrow[N\to+\infty]{} 0.$$

> So, in a sense, it does not genuinely employ infinitely many degrees of freedom — although, in the representation-theoretic sense it certainly does: there are infinitely many “position and momentum” operators (see Appendix B).
>
>] *Of course, this example does not describe interacting particles, but, to the extent that the arguments of Chapter 1 are correct, the presence of interaction makes no difference.* [Well, today we know that it does. After all, there is still no realistic interacting particle QFT in 4-dimensional spacetime.] *The collision states of an asymptotically complete theory span a separable Hilbert space, because they are again*

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[10] In fact, still today, when one queries the internet for the usage of non-separable Hilbert spaces in QM, the quote from (Streater and Wightman, 1964) is a common answer found in academic forums like “stackexchange” (e.g., see https://physics.stackexchange.com/questions/90004/separability-axiom-really-necessary).

just of the form $\oplus_n \mathcal{H}_n$ *(Here,* $\mathcal{H}_n$ *n is the subspace spanned by the ingoing n (or outgoing) collision states with n colliding particles.) All these arguments make it clear that there is no evidence that separable Hilbert spaces are not the natural state spaces for quantum field theory.* [

Summary: "there is an alternative approach in which one does not need to employ a non-separable Hilbert space". Now, note that this is not a no-go result against non-separable spaces, but just an Occam's razor-style argument.

] *When, in fact, do non-separable Hilbert spaces appear in quantum mechanics? There are two cases which deserve mention. The first arises when one takes an infinite tensor product of separable Hilbert spaces. We shall not give the rather technical definition of infinite tensor product here but only remark that it is a natural generalization of the ordinary tensor product which is used to describe a composite system. Infinite tensor products of Hilbert spaces (of dimension greater than 1!) are always non-separable. Since a (Bose) field can be thought as a system composed of an infinity of oscillators, one might think that such an infinite tensor product is the natural state space. However, it is characteristic of field theory that some of its observables involve all the oscillators at once and it turns out that such observables can be naturally defined only on vectors belonging to a tiny separable subset of the infinite tensor product.* [

Probably, the observables they are referring to are exemplified by the number operator of Fock space, which involves all the sectors at once. As Chaiken (1968) rigorously proved, such a number operator is only definable in the layers that are quasi-equivalent to the Gaussian layer mentioned above. However, as Chaiken (1967, 1968) also proved, the more general notion of "number of excitations relative to the vacuum", does have a properly definable number operator in the other layers.

] *It is the subspace spanned by such a subset which is the natural state space rather than the whole infinite tensor product itself. Thus, while it may be a matter of convenience to regard the state space as part of the infinite tensor product, it is not necessary.* [So again, not a *no-go theorem*.]

A second example of the occurrence of non-separable Hilbert spaces appears in statistical mechanics when one passes to the limit in which the conventional box containing the system becomes arbitrarily large, the density being maintained constant. Two states of the limiting system which have different densities actually differ by the presence of an infinite number of particles. One might expect them to be orthogonal and in fact that is the case in all examples worked out so far. Thus there is an orthonormal system labeled by a continuous parameter, the density, and the Hilbert space is non-separable. [

They are likely referring to Araki and Wood's (1963) model for non-relativistic infinite free Bose gases, which was well-known at the time.

] *It seems to us that such phenomena are consequences of considering systems in which every state contains an actual infinity of physical particles and provide no argument for analogous phenomena in relativistic quantum field theory. This completes our discussion of separability."* [Not a *no-go theorem* either.]

(Streater and Wightman, 1964)

Similar warnings can be found elsewhere in the literature of the time. For instance:

> *"A certain amount of confusion exists in the literature regarding the interpretation and properties of representations pertaining to an infinite number of degrees of freedom. The convenient characterization of certain of such representations in* [infinite] *tensor product spaces has perhaps fostered the false impression that non-separable Hilbert spaces are required, and that in certain models 'the interaction maps every state out of Hilbert space.'* [This was how people used to frame van Hove's (1952) result.[11]] *Firstly, field representations are completely definable in separable Hilbert spaces as is illustrated by the usual direct sum formulation of the Fock representation. Secondly, any sequence of operators all of whose matrix elements vanish in the limit simply converges (weakly) to the zero operator.*[[12]] *This is just the set of circumstances that so often prevails for the S matrix in the interaction representation when a cutoff form factor is removed. The vanishing of the matrix elements should not be regarded as 'mapping the states out of Hilbert space', but is indicative of an ill-chosen representation of field operators, one for which the sequence of unitary S matrices with form factors does not converge to a unitary operator as the form factor is removed. It is for this reason that the selection of the right CCR representation for the application becomes crucial."*
>
> (Klauder et al., 1966)

But again, no *no-go theorem*, only an appeal to simpler models.

Recently, Earman (2020) published a very detailed (and unfortunately the only serious) exploration of the possible obstacles in the path of using non-separable Hilbert spaces for QM. What the author draws from Earman's survey is that there is only one potential hindrance other than "what we are used to", "where there are more results to be used" or "who preferred what" (e.g., what von Neumann, Streater or Wightman preferred to use for QM). The more serious potential obstacle seems to be what one calls "super-selection rules".

7.3.4 The Latent "Mantras"

Quite grossly (but also quite explicitly):[13] if one can write the state space of a quantum system as a direct sum (which is our case) and there is some reason in the physics being modeled that demands superpositions of the summed subspaces to be forbidden (either by postulate or as a consequence of the known-to-be-correct dynamical flows), then, it is enough to use the subspace "super-selected" by "physics". Namely, one can forget about the rest of subspaces and their joint space and hence —in our case— recover a separable Hilbert space (a single layer of the ITP). A heuristic example would be how Streater and Wightman mentioned that some important operators are only well-defined in one of the layers of the ITP. Earman (2020) exemplifies more rigorously how super-selection rules could obstruct the use of a full ITP employing infinite spin chains for that.

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[11] What they meant is the fact that when certain interactions are "switched on", one needs to employ a different irrep of the CCR, so, a different Hilbert space —say, a different subspace of the ITP. That is, a vector starting in the free-from-interaction asymptotic end of a scattering setting (given as a vector in Fock space —hence the Gaussian layer), needs to be mapped to "another" space (say, another layer) mid-evolution.

[12] See page 24 in (Segal, 1963) or §6.5 in (Klauder, 1999) for explicit examples in this direction.

[13] There are more rigorous ways to define super-selection rules in terms of abstract algebras and abstract states on them, but the author decided to put it more "down-to-earthly" in order to manifest the arbitrariness that such rules may suppose.

Another example relevant to the ITP occurs in the generalization of those spin chains: *lattice quantum theories* (Thirring, 1979; Bratteli and Robinson, 2012). In such theories, one considers that physical space is a discrete lattice I and fixes a Hilbert space $\mathcal{H}_j$ at each lattice site $j \in I$, such that technically, the Hilbert space of the whole system —if the lattice were infinite— would be the ITP $\otimes_{j\in I}\mathcal{H}_j$. Now, in the usual approach, one avoids to mention an ITP by using the following (absolutely "Heisenberg-ian") trick. Take the algebras of bounded operators $\mathcal{L}(\otimes_{j\in J}\mathcal{H}_j)$ acting on the finite sub-lattice $J \subset I$ and call it *the algebra of operators acting "locally" on the sub-lattice* J. Then, embed the local algebra $\mathcal{L}(\otimes_{j\in J}\mathcal{H}_j)$ as a sub-algebra of any $\mathcal{L}(\otimes_{j\in K}\mathcal{H}_j)$ (with $K \supset J$ being a larger finite portion of the lattice) by "putting identities" acting on the Hilbert spaces of the extra lattice sites. Finally, take the union of all such "local algebras of observables" and take their abstract *completion* (not closure) in the relevant operator topology. Doing so, one obtains what they call the algebra of *"quasi-local observables"*. It represents observables acting on the actual infinite lattice I —despite usually there is no explicit ITP defined for them to act on.[14] Note that the quasi-local observables are not supposed to be all the bounded operators on the infinite lattice, but only those that can be written as "limits" of operators that act on finite sub-lattices (i.e., locally). Now, from the names that we intentionally employed in §A.3, the reader that already went through that section will agree that the above mentioned completion must precisely yield what we denoted in §A.3 by $\mathcal{L}^q(\otimes_{j\in I}\mathcal{H}_j)$.[15]

Heuristically, the following super-selection rule could emerge here. By Theorem 28, every operator in $\mathcal{L}^q(\otimes_{j\in I}\mathcal{H}_j)$ is reduced by each layer of the ITP. Hence, under the *postulate* that in reality only "local observables" can be measured on the lattice, one could claim that we should super-select the $\approx$-layers and hence forget about the full ITP. In doing so, one would be claiming that the operators in $\mathcal{L}(\bigotimes_{j\in I}\mathcal{H}_j) \setminus \mathcal{L}^q(\bigotimes_{j\in I}\mathcal{H})$ are somewhat "unphysical". Such a consideration seems reasonable, but beyond the obvious limitations of human experiments, why would also the fundamental quantum laws (say, the fundamental dynamical maps) be restricted to "local operators" when (at least some form of) non-locality is an experimentally verified fact (after the violation of Bell's inequalities)? On a different note, one could claim that the state of an infinitely extended system has parameters such as the "average magnetization" or the "density", that are macroscopically meaningful. One could then argue that there should be no superposition of infinite states differing in these quantities, because that would contradict that "the macroscopic world is classical". Hence, one could decide to impose the super-selection of a layer of the ITP because the vectors there "share a common tail" and hence, agree on the average magnetization or density of the infinite system. As Bohmian mechanics and many other theories show, such a claim is flawed: there exist rigorous ways to consider macroscopic superpositions and still properly explain why we see no "simultaneously dead and alive cat".

For this author (perhaps naively, let the doubt be clear), super-selection and similar concerns are likely the reason that made the community leave the ITP in the drawer. Undoubtedly, such concepts are often useful and necessary, but it feels like they are often self-imposed and self-restricting, and can end up leading to "mysteries" where there were none. The key example is that of the van Hove model mentioned above. If one decides to postulate that dynamics in QFT must be implemented in an *irreducible* representation of the CCR (so, a minimal reducing space

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[14] At best, one employs the (so-called) GNS representation theory to get a Hilbert space representation of the algebra. We will review the relation between the GNS and the ITP constructions elsewhere.

[15] Although technically, in §A.3 we took the closure in the *weak* operator topology, while in lattice systems the practice is to take the completion in the *operator norm* topology.

for the "position and momentum operators", but perhaps not for the dynamical SCOPUG), then certainly, a Hamiltonian that requires several such representations to even be described will seem to cause a paradox. A single irrep of the CCR, which was postulated to be enough, will simply not be enough. This aura of mystery based on some self-imposed algebraic "mantras" is clearly manifest in the series of results we itemized above. Reents (1974); Kraus et al. (1977) and Streit (1967) discovered that a typical tensor product of infinitely many SCOPUGs is reduced (or admits an implementation) at best in a few layers of the ITP. They found that the restriction to some other layers is not even continuous nor unitary (cannot be implemented as a unitary). Now, one could take this as a warning that generic interactions in systems with infinitely many degrees of freedom should be described in larger spaces. Spaces that possibly possess no obvious *irreducible* representations of the CCR (although the dynamically relevant operators might be irreducible there). Instead, the message they took under the self-imposed "super-selection of the CCR algebra irreps" was that *only some* spaces —say, some layers of the ITP— admitted infinite products of those SCOPUGs.

It is fair to recall that the belief in *irreducible* representations is not as unfounded as it may sound. There were results like that of Segal's 1947 paper quoted above, pointing in the direction that irreps are enough for QFT. But still, how could one make such an assertion when even today, there is still no definitive rigorous framework for QFT.

With all, it does not seem wild to say that the community left the ITP out of their consideration due to "beliefs" against non-separable and reducible representations of the CCR. At least there is no clear no-go theorem in that direction.

7.3.5 Tensions with Relativity?

Our final and most speculative hypothesis is that there were difficulties when trying to use the ITP representations to make a relativistic QFT rigorous. Whoever tried it failed and "returned the idea to the drawer". Fortunately, we have no big evidence of this. Our main hint is that the Giants of the field only employed ITP representations as toy-models or examples. For instance, in Klauder's 1999 QFT book —which is one of the only modern QFT books mentioning ITPs—, the ITP is employed to give a precise understanding of the divergences and representation issues that emerge when dealing with infinitely many degrees of freedom. And despite the clarity it seems to provide, the whole thing is left as an example of mere academic interest before going to the "serious theories".

> [...] *"most problems of physical interest do not involve product representations of the canonical operators."*
>
> (Klauder, 1999)

A similar thing happens in (Segal, 1963)[16] and in (Reed and Simon, 1975).[17] And yet, the issue is never explicitly stated. Perhaps the most explicit divide we have found so-far between product representations and relativistic theories is the following one:[18]

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[16] He employs the ITP as a pedagogic example to conceptualize the ideas before explaining the more "relativistically covariant" techniques.

[17] They introduce the *Q-space* (namely, $L^2(\mathbb{R}^\infty, d^\infty\mu)$ with a Gaussian measure $d^\infty\mu$) only as an auxiliary representation of the CCR for technical proofs —see Appendix B.

[18] The underlines are not present in the original text.

> *"But* [after talking about representations of the CCR in layers of the ITP] *we must emphasize that only in product cases is it clear that an appropriate CCR representation can be found; in relativistic cases it is by no means clear that a correct representation even exists."*
>
> (Klauder et al., 1966)

Nevertheless, if there was really a weighty reason not to use ITPs in relativistic QFT (other than because some other structures were more appropriate, or they fitted better the preferred axioms of the time), one would expect that such an issue would be clearly stated in more than one place. And yet, we found none. We will continue digging the literature looking for the issue, in case we just missed those statements. However, for now it simply looks like the community chose to go in certain directions that ended up leaving the non-separable ITP approach too far away or too "incommensurate" with the developed techniques, to even consider it once the chosen strategies proved to be unsuccessful.

7.4 But, which are the Reasons to Explore QM on a Full ITP?

Why should we pursue such a delicate endeavor to begin with?

- For the author the clearest reason is that in order to attempt the construction of a rigorous quantum theory with a *field ontology* and, in particular, within a pilot-wave framework, the ITP seems to be the natural arena —as the present work foretells. Namely, the presented seems to be the math for a quantum theory where there are rigorous "Schrödinger picture" wavefunctionals that evolve over a field configuration-space with well-defined associated primitive ontology.
- As we explained in this chapter, it seems to be an unexplored path to attempt the rigorization of QFT. In particular, we happen to be in a period where the subject is stagnant due to issues with interacting theories. Since the full ITP naturally includes many of the inequivalent irreps —which, as we saw, seem to be intricately tied to interactions— the ITP promises to help with that key issue of QFT. In fact, some eminent physicists have historically advocated for a Schrödinger picture in QFT, the best example being Jackiw (1995), who actually came very close to the ITP (willingly or not) as the following passage shows. In a section titled *"Schrödinger representation for quantized fields",* he first explains how to build a Schrödinger representation wavefunctional $\Psi(\psi)$ and considers a Hamiltonian describing infinitely many harmonic oscillators with some 'covariance parameters' ω (reflecting the 'spring constants'). He denotes its ground state by $\Psi_{\omega,0}(\psi)$ and calls it a 'Fock vacuum'. Defining some creation and annihilation operators (using the field operators) and acting on the Fock vacuum, he then builds a 'Fock basis' subordinate to $\Psi_{0,\omega}$. Finally, he says:

  > *"Consider two Fock vacua, with translationally invariant covariance ω_1 and ω_2. These are ground states of two Hamiltonians H_1, and H_2,* [...]*the overlap between the two Fock vacua is* [Eq.1.47] *Since* [...] *the overlap (1.47a) vanishes. Then* [...] *the two Fock bases, as well as the two Fock spaces built on these bases are mutually orthogonal. In other words, our functional space contains within it inequivalent Fock spaces."*
  >
  > (Jackiw, 1995)

  Clearly, he was talking about something close to the structure we expose in Appendix B.

- As the present document reveals, there are many unknown but reasonably provable facts surrounding the ITP, and the mathematical aspects of them alone are already relevant.

- As we will show in a forthcoming publication, it is possible to generalize the notion of symmetrized and anti-symmetrized tensor product to the infinite setting in a way that the resulting space is a subspace of von Neumann's ITP. In doing so, one must employ "linear combinations" of different layers of the ITP, such that if one wants to build a rigorous *Dirac sea* for instance, the "super-selection" into proper ITP layers does not seem to be the right direction.

- von Neumann had something in mind and the mathematical physics community has not fully found yet what. Recall his emphasis in the "hyperquantisation" passage on the "extra" (the non-quasi-local) operators. He seemed aware of the fact that a description of QFT in terms of individual ITP layers could miss some relevant operators that are only conceivable in their ambient space. Such operators have not been seriously studied in QM yet. But perhaps the most evocative reason showing that physics has not yet found his goal with the ITP, is that he also developed *uncountable* ITPs —for which there is not even a characterization of the CCR representations yet (to our knowledge). Revealingly, while commenting the main publications of his recently passed away friend von Neumann, Ulam (1958) wrote the following words:[18]

  > *"Von Neumann intended to discuss the analogy of this elaborate system* [the ITP] *with the theory of hyperquantization in quantum theory, and considered the paper in particular* [the one on ITPs] *as a mathematical preparation for dealing with non-enumerable products."*
  >
  > (Ulam, 1958)

  It is time for us to discover what he could have been making the "preparations" for.

A

APPENDIX: THE INFINITE TENSOR PRODUCT OF HILBERT SPACES

Given an arbitrary index set I and a set of Hilbert spaces $\{\mathcal{H}_j\}_{j\in I}$, one could ask whether there is a meaningful notion of tensor product $\otimes_{j\in\mathbb{N}}\mathcal{H}_j$ that generalizes the usual tensor product of (finitely many) Hilbert spaces. von Neumann (1939) found an essentially unique way to define such an ITP. In this chapter we motivate and explain his construction.

A.1 Generalizing a "Finite Construction" to an "Infinite" One

From von Neumann's 1939 paper, one can infer which could have been his mental "checklist" to ensure a proper generalization of the *finite* tensor product. For a clearer exposition, we abstract first this checklist and then exemplify each point with the tensor product construction:

a. Study the possible definitions of the finite case and formulate the *desiderata* that a generalization is expected to satisfy.

b. Extend the mathematical operations employed in the finite case to an infinite setting (possibly with ad-hoc tweaks to satisfy the desiderata).

c. Among the alternative definitions of the finite case, generalize the one employing structures with most well-known related theorems (so they can be employed). Important: be ready to abandon some desiderata.

d. Prove that alternative generalizations that also satisfy (at least some of) the desiderata would yield an equivalent construction in a suitable sense.

A.1.a Desiderata for an Infinite Tensor Product

Before writing his treatise, von Neumann could have been mulling over the following standard[1] way to define the tensor product $\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n$ of Hilbert spaces $\mathcal{H}_j$.

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[1] Or at least a basis independent reformulations of a standard definition —convenient to avoid making any choice.

Definition 23. Given that $\widetilde{\mathcal{L}}$ denotes the vector space of all conjugate-multilinear forms like $\Phi : \mathcal{H}_1 \times \cdots \times \mathcal{H}_n \to \mathbb{C}$,[2]

1. Define $f_1 \otimes \cdots \otimes f_n$, $f_j \in \mathcal{H}_j$ to be the vector of $\widetilde{\mathcal{L}}$ satisfying
$$\Big(f_1 \otimes \cdots \otimes f_n\Big)\,(g_1, ..., g_n) = \prod_{j=1}^{n} \langle g_j,\, f_j\rangle. \tag{A.1}$$
for each $g_j \in \mathcal{H}_j$. We call it an *elementary tensor product.*
2. Define $V := span\{f_1 \otimes \cdots \otimes f_n \,|\, f_j \in \mathcal{H}_j\}$ —it is a complex vector subspace of $\widetilde{\mathcal{L}}$.
3. Define the map $\langle\cdot,\cdot\rangle : V \times V \to \mathbb{C}$
$$\Big\langle \sum_{j=1}^{r} d_j\; g_1^j \otimes\cdots\otimes g_n^j,\; \sum_{k=1}^{m} c_k\; f_1^k \otimes\cdots\otimes f_n^k \Big\rangle := \sum_{j=1}^{r}\sum_{k=1}^{m} \overline{d_j}\, c_k\, \Big(f_1^k \otimes\cdots\otimes f_n^k\Big)\,(g_1^j, ..., g_n^j). \tag{A.2}$$
It is well-defined (independent of the chosen representative expansion),[3] satisfies $\langle g_1 \otimes \cdots \otimes g_n, v\rangle = v(g_1, ..., g_n)\ \forall v \in V, g_j \in \mathcal{H}_j$ and it is an inner product on V: it is sesquilinear and conjugate symmetric by definition, while positive definiteness ($\langle v, v\rangle > 0 \quad \forall v \neq \vec{0}$) follows non-trivially.[4] As such, $\|\cdot\| := \sqrt{\langle\cdot,\cdot\rangle}$ is a norm on V.
4. Observe that the point-wise (i.e., strong) limit of every $\|\cdot\|$-Cauchy sequence $(\phi_j)_{j\in\mathbb{N}} \subseteq V$ is again a form in $\widetilde{\mathcal{L}}$ —i.e., for every choice of $f_j \in \mathcal{H}_j$, $\lim_{j\to} \phi_j(f_1, ..., f_n)$ exists and $(f_1, ..., f_n) \mapsto \lim_{j\to\infty} \phi_j(f_1, ..., f_n)$ is a conjugate multilinear form.[5] Define $\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n$ as the subspace of $\widetilde{\mathcal{L}}$ composed by point-wise limits of $||\cdot||$−Cauchy sequences in V. Explicitly, $\Psi \in \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n \subseteq \widetilde{\mathcal{L}}$ *if and only if* $\exists(\phi_j)_{j\in\mathbb{N}} \subset V$ with
$$(i)\ \Phi(f_1, ..., f_n) = \lim_{j\to\infty} \phi_j(f_1, ..., f_n)\ \forall f_j \in \mathcal{H}_j, \qquad (ii)\ \lim_{j,k\to\infty}\ \sup_{j,k\geq N} \|\phi_j - \phi_k\| = 0. \tag{A.3}$$

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[2] That is, those forms such that $\Phi(f_1, ..., \lambda f_j + \mu g_j, ..., f_n) = \overline{\lambda}\Phi(f_1, ..., f_n) + \overline{\mu}\Phi(f_1, ..., g_j, ..., f_n)$ for all $j \in \{1, ..., n\}$, $f_k, g_k \in \mathcal{H}_k$ and $\mu, \lambda \in \mathbb{C}$. The canonical vector space structure of $\tilde{\mathcal{L}}$ is defined by: $(\lambda\Phi + \mu\Psi)(f_1, ..., f_n) := \lambda(\Phi(f_1, ..., f_n)) + \mu(\Psi(f_1, ..., f_n))$ for $\Phi, \Psi \in \widetilde{\mathcal{L}}$ and $\mu, \lambda \in \mathbb{C}$.

[3] Let $\sum_{k=1}^{m} c_k f_1^k \otimes \cdots \otimes f_n^k = \sum_{k=1}^{\tilde{m}} \tilde{c}_k \tilde{f}_1^j \otimes \cdots \otimes \tilde{f}_n^j$ and $\sum_{j=1}^{r} d_j g_1^j \otimes \cdots \otimes g_n^j = \sum_{j=1}^{\tilde{r}} \tilde{d}_j \tilde{g}_1^j \otimes \cdots \otimes \tilde{g}_n^j$. By definition, in $\sum_{j=1}^{r}\sum_{k=1}^{m} \overline{d_j}\, c_k\ f_1^k \otimes \cdots \otimes f_n^k\,(g_1^j, ..., g_n^j)$ we can substitute the f_k, c_k of each j by the tilde version. If we develop it in terms of the inner products using (A.1) and we use the conjugate symmetry of each $\mathcal{H}_j$'s inner product, the result is $\overline{\sum_{j=1}^{r}\sum_{k=1}^{\tilde{m}} \overline{\widetilde{c}_k}\, d_j\ g_1^j \otimes \cdots \otimes g_n^j(\widetilde{f}_1^k, ..., \widetilde{f}_n^k)}$. But there, by definition, we can substitute the g_j, d_j sum for each k by the tilde version. Then reverting the conjugate symmetry one gets: $\sum_{j=1}^{\tilde{r}}\sum_{k=1}^{\tilde{m}} \overline{\tilde{d}_j}\, \tilde{c}_k\ \widetilde{f}_1^k \otimes \cdots \otimes \widetilde{f}_n^k\,(\widetilde{g}_1^j, ..., \widetilde{g}_n^j)$.

[4] For arbitrary $v \in V \setminus \{\vec{0}\}$, by definition, there is a representative expansion $v = \sum_{j=1}^{m} f_1^j \otimes \cdots \otimes f_n^j$ (absorb the coefficients in some factor). **Claim:** $\langle v, v\rangle = \sum_{j,k=1}^{m}\prod_{l=1}^{n}\langle f_l^j, f_l^k\rangle > 0$. *Proof:*

- We check that $\langle v, v\rangle \geq 0$. Fix $l \in \{1, ..., n\}$. The Gram matrix $(\langle f_l^j, f_l^k\rangle)_{j,k=1}^{m}$ is positive-semi-definite because for all $(x_j)_j \in \mathbb{C}^m$, $\sum_{j,k=1}^{m} x^j \langle f_l^j, f_l^k\rangle x^k = ||\sum_{j=1}^{m} x^j f_l^j||^2 \geq 0$. Hence, it can be diagonalized in an ONB $(u^{lr})_{r=1}^{m} \subset \mathbb{C}^m$ with eigenvalues $\lambda_r \geq 0$, i.e., $(\langle f_l^j, f_l^k\rangle)_{j,k} = \sum_{r=1}^{m} \lambda_r\,(u_j^{lr} \quad \overline{u_k^{lr}})_{j,k}$. Thus, $\sum_{j,k=1}^{m}\prod_{l=1}^{n}\langle f_l^j, f_l^k\rangle = \sum_{j,k=1}^{m}\prod_{l=1}^{n}\sum_{r=1}^{m}\lambda_r\, u_j^{lr} \quad \overline{u_k^{lr}} = \sum_{j,k=1}^{m}\sum_{r_1,...,r_n=1}^{m}\lambda_{r_1}\cdots\lambda_{r_n} u_j^{1r_1}\overline{u_k^{1r_1}}\cdots u_j^{nr_n}\overline{u_k^{nr_n}} = \sum_{r_1,...,r_n=1}^{m}\lambda_{r_1}\cdots\lambda_{r_n}(\sum_{j=1}^{m} u_j^{1r_1}\cdots u_j^{nr_n})(\sum_{k=1}^{m}\overline{u_k^{1r_1}}\cdots\overline{u_k^{nr_n}}) = \sum_{r_1,...,r_n=1}^{m}\lambda_{r_1}\cdots\lambda_{r_n}|\sum_{j=1}^{m} u_j^{1r_1}\cdots u_j^{nr_n}|^2$, which is ≥ 0.
- Now, note that $|\langle v, u\rangle| \leq \sqrt{\langle v, v\rangle\langle u, u\rangle}$ (the Cauchy-Schwarz inequality) follows exactly after the same arguments as those in the proof of Lemma 3.4.2 in (von Neumann, 1939).
- Finally, $\langle v, v\rangle = 0$ implies by the last item that for all $u \in V$, $|\langle v, u\rangle| \leq 0$. Thus, $\langle u, v\rangle = 0$ for all $u \in V$ and in particular, $\langle f_1 \otimes\cdots\otimes f_n,\ v\rangle = 0$ for all elementary tensor products. But this implies that $v(f_1, ..., f_n) = 0$ for all $f_j \in \mathcal{H}$. Hence, $v = 0$. □

[5] The proof is the same as the one given for Lemma 3.5.2 in (von Neumann, 1939).

5. Equip $\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n$ with the inner product that is the natural extension of $\langle\cdot,\cdot\rangle$ (defined via Cauchy sequences in V).[6] It makes $\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n$ a Hilbert space and V a dense subspace of $\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n$.[7] ♦

Given any permutation $\sigma \in S(n)$, the space $\mathcal{H}_{\sigma(1)} \otimes \cdots \otimes \mathcal{H}_{\sigma(n)}$ is canonically unitarily identified with $\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n$ —by identifying $f_{\sigma(1)} \otimes \cdots \otimes f_{\sigma(n)}$ with $f_1 \otimes \cdots \otimes f_n$. This makes the constructions, in a sense, independent of the order of the factors (although, once chosen an order it must be preserved — this has nothing to do with the "symmetrization" of the elements).

There are several alternative definitions, the most famous one being based on a Cantor-style construction (using an abstract completion of "linear combinations of abstract symbolic tensor product strings"). However, it is well-known that identifying the elementary tensor products of each construction with ours, the alternative constructions end up being unitarily isomorphic to the one we have given here. In this sense, there is a *unique* tensor product construction.

In view of all the above, for a generalization to infinitely many factor spaces $\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots$, one can formulate the *desiderata* (and consequent restrictions) that follow.

(a) We desire $\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots$ to be a Hilbert space and there to be a "unique" such construction up to unitary isomorphisms that identify the "elementary tensor products".

(b) The inner product should be such that

$$\Big\langle f_1 \otimes f_2 \otimes \cdots \,,\; g_1 \otimes g_2 \otimes \dots \Big\rangle = \text{“}\prod_{j\in\mathbb{N}} \langle f_j, g_j\rangle\text{”}, \tag{A.4}$$

for some notion of infinite product of complex numbers to be defined.

(c) In particular, desiderata (b) implies a norm such that: $\|f_1 \otimes f_2 \otimes \cdots\| = \text{“}\prod_{j\in\mathbb{N}} \|f_j\|\text{”}$. This imposes an important *restriction*: not all $f_1 \otimes f_2 \otimes \cdots$ can be in the resulting space, only those for which "$\prod_{j\in\mathbb{N}} \|f_j\|$" converges can. von Neumann called them *$\mathscr{C}$-sequences*. Furthermore, all those $f_1 \otimes f_2 \otimes \cdots$ for which "$\prod_{j\in\mathbb{N}} \|f_j\|$" $= 0$ must be identical to the zero vector (otherwise $\|\cdot\|$ could not be a norm —violating (a)). In the absence of any a priori reason to exclude other $\mathscr{C}$-sequences, we wish that all the remaining $\mathscr{C}$-sequences have a non-zero representative "elementary tensor product" $f_1 \otimes f_2 \otimes \cdots$ in the space.

(d) We wish that if $f_1 \otimes f_2 \otimes \cdots$ and $g_1 \otimes g_2 \otimes \cdots$ are part of the space, and hence, "$\prod_{j\in\mathbb{N}} \|f_j\|$" and "$\prod_{j\in\mathbb{N}} \|g_j\|$" converge to a non-zero value, their inner product (A.4), namely, "$\prod_{j\in\mathbb{N}} \langle f_j, g_j\rangle$", also converges. Hence, our definition of infinite product should be so that this is guaranteed.[8]

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[6] For $\Psi, \Phi \in \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n$, we define $\langle\Psi,\Phi\rangle := \lim_{j\to\infty}\langle\psi_j,\phi_j\rangle$, where $(\psi_j)_j, (\phi_j)_{\subseteq} V$ are Cauchy sequences whose strong limits, respectively, yield Ψ, Φ. It is well-defined and it coincides with $\langle\cdot,\cdot\rangle$ in V. For a proof of both statements see Lemma 3.5.3 in (von Neumann, 1939).

[7] The proofs of the claims in this last sentence are exactly the same as the proofs given for Lemmas 3.5.4, 3.5.8 and 3.5.9 in (von Neumann, 1939).

[8] One could propose to define the belonging to $\otimes_{j\in\mathbb{N}}\mathcal{H}_j$ by checking the sequences of vectors for which the inner product (A.4) is convergent (instead of prioritizing that the product of norms is convergent). However, this might be undesirable because it would employ a *relation between sequences* $(f_j)_{j\in\mathbb{N}}, (g_j)_{j\in\mathbb{N}}$ instead of a property of *each sequence* as the principle to define the *belonging* to the space. If the result was an equivalence relation, one could suggest to define $\otimes_{j\in\mathbb{N}}\mathcal{H}_j$ in one of the equivalence classes. However, this would lead to a non-unique construction unless there was only one equivalence class. And indeed, we will find that there are multiple classes of equivalence. That is, it will indeed happen that the inner product of two elementary tensor products is not convergent even if they are both $\mathscr{C}$-sequences. von Neumann added a tweak to the notion of convergence in view of (a) to amend this issue.

(e) We expect that the span of elementary tensor products $f_1 \otimes f_2 \otimes \cdots$ is *dense*, i.e.,

$$\overline{span\Big\{f_1 \otimes f_2 \otimes \cdots \in \bigotimes_{j\in\mathbb{N}} \mathcal{H}_j\Big\}} = \bigotimes_{j\in\mathbb{N}} \mathcal{H}_j. \tag{A.5}$$

(f) Ideally, we wish that the construction has no intrinsic dependence on the order of the factor Hilbert spaces, so the notion of convergence for (A.4) should not depend on the order of the factors.

Finally, if we find a generalization that also allows uncountably many factor Hilbert spaces by only adding slightly more to the structure, we will develop it as well.

A.1.b Make sense of an Infinite Product of Complex Numbers

von Neumann realized that in formulating an order independent notion of convergence for the product of countably many complex numbers (to satisfy (f)), a definition that is also independent of the cardinality of the index set emerges.

Definition 24 (*Arbitrary Product*)**.** Let I be an index set of arbitrary cardinality and $(z_j)_{j\in I} \subset \mathbb{C}$. We say that $\prod_{j\in I} z_j$ *converges to* $\alpha \in \mathbb{C}$ when

$$\forall \varepsilon > 0\ \exists \text{ a finite subset } I^\varepsilon := I^\varepsilon[(z_j)_{j\in I}] \subseteq I \text{ such that } \forall \text{ finite super-set } J \supseteq I^\varepsilon,\ J \subseteq I,$$

$$\Big|\prod_{j\in J} z_j - \alpha\Big| \leq \varepsilon. \tag{A.6}$$

(Informally, *any (finite) partial product after* I^ε *is* ε *away from* α). ♦

Lemma 33. If such an α exists, it is *unique*. ♦

Proof: See the Corollary after Definition 2.2.1 in (von Neumann, 1939). ☐

Example 2 (*Trivial case*)**.** Given $(z_j)_{j\in I} \subseteq \mathbb{C}$, if $\exists j \in I : z_j = 0 \implies \prod_{j\in I} z_j$ converges to 0. ♦

Today, this definition would have been formulated in terms of *nets*.

Definition 25. A *directed set* $(D, \leq)$ is a non-empty set D equipped with a reflexive and transitive relation $\leq$:

$$\forall d_1, d_2 \in D,\ \exists d \in D \ \text{ with } \ d_1 \leq d \ \&\ d_2 \leq d. \tag{A.7}$$

(Informally, *every pair of elements has a common "bigger" element.*) ♦

For example, $\mathbb{N}$ with the usual number order is a directed set. So is any family of sets I with the set-inclusion $\subseteq$.

Definition 26. Given a topological space X and a directed set $(D, \leq)$, a *net in* X *indexed by* D is an element $x \in \prod_{d\in D} X$, namely, a function $x : D \to X,\ d \mapsto x_d$. We will abuse notation and write $(x_d)_{d\in D} \subseteq X$. ♦

It is the generalization of a sequence (which is the case $D = \mathbb{N}$ with the usual number order as the relation $\leq$).

Definition 27 (*Convergence of a Net*)**.** The net $(x_d)_{d\in D}$ converges to a point $x \in X$, called a *limit point*, when[a]

$$\forall \text{ neighborhood } \mathscr{U} \text{ of } x,\ \exists d_0 \in D \text{ s.th. } \forall d \geq d_0,\ x_d \in \mathscr{U}. \tag{A.8}$$

(In words, *eventually the x_d "live" in arbitrarily "small" neighborhoods of x.*)

If so, we denote $x_d \xrightarrow[d\to D]{} x$, or when x is unique, also, $\lim_{d\to D} x_d = x$. ♦

Proposition 42. If X is a Hausdorff topological space, a convergent net $(x_d)_{d\in D} \subseteq X$ has a *unique* limit point.

Proof. Assume $\alpha_1, \alpha_2 \in X$ are two different limit points of $(x_d)_{d\in D} \subseteq X$. By the Hausdorff property, there exist two disjoint open neighborhoods $\mathscr{U}_1, \mathscr{U}_2 \subset X$ of α_1, α_2 respectively. But then, by $x_d \xrightarrow[d\to D]{} \alpha_1$, $\exists d_1 \in D$ such that $x_d \in \mathscr{U}_1$ for all $d \geq d_1$. Likewise, by $x_d \xrightarrow[d\to D]{} \alpha_2$, $\exists d_2 \in D$ such that $x_d \in \mathscr{U}_2$ for all $d \geq d_2$. Because $(D, \leq)$ is a directed set, $\exists d_0 \in D$ with $d_1 \leq d_0$ and $d_2 \leq d_0$. Hence, $x_d \in \mathscr{U}_1 \cap \mathscr{U}_2$ for all $d \geq d_0$, which is absurd because $\mathscr{U}_1 \cap \mathscr{U}_2 = \emptyset$. ***o.ε.δ.***

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[a] For notational convenience we introduce $\geq$, defined to be the relation $\leq$ with swapped arguments

Definition 28. Given an arbitrary set I, we denote by $\mathcal{P}_0(I)$ the *family of finite subsets of I*, directed by the set-inclusion $\subseteq$.

Then, one can formulate Definition 24 equivalently as:

Definition 24'. Given $(z_j)_{j\in I} \subset \mathbb{C}$,

$$\prod_{j\in I} z_j \text{ denotes the limit point (if it exists) of the net } \Big(\prod_{j\in J} z_j\Big)_{J\in\mathcal{P}_0(I)} \subseteq \mathbb{C}.$$

♦

Proof (Equivalence of Definitions 24 and 24'): $\prod_{j\in J} z_j \xrightarrow[J\to\mathcal{P}_0(I)]{} \alpha \in \mathbb{C}$ in the net sense holds (by definition) *if and only if* for all neighborhood $\mathscr{U}$ of α, condition (A.8) holds. But all neighborhoods of some α in a topological space contain an open neighborhood of α, and in the case of a metric space like $\mathbb{C}$, all such open neighborhoods contain a metric ball around α, which are also neighborhoods of α. Hence, the above statement holds *if and only if* for all open ball around α the condition in (A.8) holds. But using that the direction operation of $\mathcal{P}_0(I)$ is subset $\subseteq$, this is precisely the statement in Def. 24. ***o.ε.δ.***

It is also convenient to define a notion of arbitrary sum in order to provide characterizations of the convergence of arbitrary products in more practical terms.

Definition 30 (*Arbitrary Sum*)**.** Given $(z_j)_{j\in I} \subset \mathbb{C}$,

$$\sum_{j\in I} z_j \text{ denotes the limit point (if it exists) of the net } \Big(\sum_{j\in J} z_j\Big)_{J\in\mathcal{P}_0(I)} \subseteq \mathbb{C}.$$

Equivalently, we write $\sum_{j\in I} z_j = \alpha$ when

$$\forall \varepsilon > 0\ \exists \text{ a finite subset } I^\varepsilon := I^\varepsilon[(z_j)_{j\in I}] \subseteq I \text{ such that } \forall \text{ finite super-set } J \supseteq I^\varepsilon,\ J \subseteq I,$$

$$\Big|\sum_{j\in J} z_j - \alpha\Big| \leq \varepsilon.$$

♦

One can prove that such a notion of sum reduces to the usual notion of absolute summability:

Lemma 34. Given $(z_j)_{j\in I} \subseteq [0,+\infty)$,

$$\left(\sum_{j\in I} z_j \text{ exists}\right) \iff \left(\sup_{J\in\mathcal{P}_0(I)}\left\{\sum_{j\in J} z_j\right\} < +\infty \text{ i.e., } \textit{the set of finite partial sums is bounded}\right).$$

If so, then $\sum_{j\in I} z_j$ equals the supremum of those finite partial sums. ♦

Proof: See Lemma 2.3.1 in (Von Neumann, 1949). ☐

Proposition 43. Given $(z_j)_{j\in I} \subset \mathbb{C}$, the following three conditions are equivalent:

- $\sum_{j\in I} z_j$ exists.
- $\sum_{j\in I} |z_j|$ exists.
- $$\begin{cases} z_j \neq 0 \text{ for at most countably many } j\in I, \textit{ and } \text{in the countable case,} \\ \text{if } z_j \neq 0 \text{ only for } j\in\{j_k\}_{k\in\mathbb{N}}, \text{ then } \displaystyle\sum_{k=1}^{\infty} |z_{j_k}| \text{ is finite in the usual}^{[9]} \text{ sense.} \end{cases}$$

  If so, the value of $\sum\limits_{j\in I} z_j$ is the usual (finite or countable) sum of the non-zero terms. ♦

Proof: See Lemmas 2.3.3 and 2.3.4 in (von Neumann, 1939). ☐

At this point, von Neumann provides a very convenient lemma for arbitrary products of positive numbers, which can only fail to converge if the partial products become unbounded:

Proposition 44. Given $(z_j)_{j\in I} \subseteq (0,+\infty)$,

(i) $$\left(\prod_{j\in I} z_j \text{ exists}\right) \iff \left(\sum_{j\in I} \log(z_j) \text{ exists}\right) \iff \left(\sum_{j\in I} \max(z_j-1,\ 0) \text{ exists}\right).$$

(Informally, *the excesses of* z_j *over 1 must be summable.*)

If so, then, $\log\left(\prod_{j\in I} z_j\right) = \sum_{j\in I}(\log(z_j))$.

(ii) $$\left(\prod_{j\in I} z_j \text{ exists and is} \neq 0\right) \iff \left(\sum_{j\in I} |\log(z_j)| \text{ exists}\right) \iff \left(\sum_{j\in I} |z_j-1| \text{ exists}\right).$$

(Informally, *the distances of* z_j *to 1 must be summable.*) ♦

Proof: See Lemma 2.4.1 in (von Neumann, 1939). ☐

Next, he characterizes the convergence of arbitrary products as follows:

Proposition 45. Given $(z_j)_{j\in I} \subseteq \mathbb{C}$ and denoting the argument of z_j by $arg(z_j) \in (-\pi,\pi]$,

$$\left(\begin{array}{c}\prod\limits_{j\in I} z_j \\ \text{exists}\end{array}\right) \iff \begin{cases} \text{either}: \displaystyle\prod_{j\in I} |z_j| = 0 & \implies \displaystyle\prod_{j\in I} z_j = 0 \\ \text{or}: \left(\displaystyle\prod_{j\in I} |z_j| \in (0,\infty) \ \ \& \ \begin{array}{c}\displaystyle\sum_{j\in I} |arg(z_j)| \\ \text{is finite}\end{array}\right) & \implies \displaystyle\prod_{j\in I} z_j = e^{i\sum_{j\in I} arg(z_j)} \cdot \prod_{j\in I} |z_j| \end{cases}$$

♦

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[9] Recall that we denote $\sum\limits_{k=1}^{\infty} z_k := \lim\limits_{N\to\infty} \sum\limits_{k=1}^{N} z_k = \alpha$.

Proof: See Lemma 2.4.2 in (von Neumann, 1939). ☐

Example 3. Given $(\theta_j)_{j\in I} \subseteq (-\pi, \pi]$,

$$\left(\prod_{j\in I} e^{i\theta_j} \quad \text{exists}\right) \quad \overset{(\text{Prop. } 45)}{\Longleftrightarrow} \quad \left(\sum_{j\in I} |\theta_j| \quad \text{exists}\right) \quad \overset{(\text{Lem. } 43)}{\Longleftrightarrow} \quad \left(\sum_{j\in I} \theta_j \quad \text{exists}\right).$$

If so, by Prop. 45, $\prod_{j\in I} e^{i\theta_j} = e^{i\sum_{j\in I}\theta_j}$. ◆

Corollary 19. Given $(z_j)_{j\in I} \subseteq \mathbb{C}\setminus\{0\}$,

$$\left(\prod_{j\in I} z_j \text{ exists } \textit{and} \quad \text{is} \neq 0 \right) \quad \Longleftrightarrow \quad \left(\sum_{j\in I} |z_j - 1| \text{ exists} \right).$$

(Informally, *the distances of* z_j *to 1 must be summable.*) ◆

Proof: See Lemma 2.5.1 in (von Neumann, 1939). ☐

Proposition 45 shows that absolute value convergence is not enough: non-convergent phases can spoil the existence of a limit. Identifying complex numbers with vectors in the $\mathbb{R}^2$ plane, one can imagine that a sequence of such vectors may converge in norm, but fail to have a convergent sequence of directing unit vectors (say, they approach some sphere of fixed radius in spirals).

This is precisely the issue that explains the following:

Proposition 46. Using the notion of convergence for products given in Definition 24,

(i) It can happen that $\prod_{j\in I}\|f_j\|$, $\prod_{j\in I}\|g_j\|$ exist and take non-zero values for some $f_j, g_j \in \mathcal{H}_j$, and at the same time, that $\prod_{j\in I}\langle f_j, g_j\rangle$ does not exist.

(ii) This can only be for such $(f_j)_{j\in I}, (g_j)_{j\in I}$ if $\prod\limits_{j\in I} |\langle f_j, g_j\rangle|$ exists but $\sum\limits_{j\in I} arg(z_j)$ does not. ◆

To prove it, we use the following lemma and corollary:

Lemma 35. Given $(a_j)_{j\in I}, (b_j)_{j\in I} \subseteq \mathbb{C}$, if $\exists A, B \in \mathbb{C}$ such that

$$\left(\prod_{j\in I} a_j = A \quad \text{and} \quad \prod_{j\in I} b_j = B \right) \quad \Longrightarrow \quad \left(\prod_{j\in I} a_j b_j = AB \right).$$

◆

Proof: For an arbitrary $J \in \mathcal{P}_0(I)$ we have:

$$\Big| \prod_{j\in J} a_j b_j - AB \Big| \quad \underbrace{\leq}_{\left(\substack{\pm A\cdot\prod_{j\in J} b_j \\ \text{\& triangl. ineq.}}\right)} \quad \Big|\prod_{j\in J} a_j - A\Big| \prod_{j\in J} |b_j| + |A| \Big|\prod_{j\in J} b_j - B\Big|. \tag{A.9}$$

Since $\prod_{j\in I} b_j = B$, by Prop. 45, $\prod_{j\in I}|b_j|$ exists and equals $|B|$. Hence, fixing an arbitrary $\delta > 0$,

$$\exists I^\delta[(|b_j|)_j] \in \mathcal{P}_0(I) \quad \text{with} \quad \prod_{j\in J} |b_j| \leq |B| + \delta, \quad \forall J \supseteq I^\delta.$$

Now, for any given $\varepsilon > 0$ take

$$\tilde{\varepsilon} := \frac{\varepsilon}{2\max\{1, |B|+\delta\}\cdot\max\{1, |A|\}}$$

and define $I^{\varepsilon} := I^{\tilde{\varepsilon}}[(a_j)_j] \cup I^{\tilde{\varepsilon}}[(|b_j|)_j] \cup I^{\delta}[(b_j)_j] \in \mathcal{P}_0(I)$. Then, $\forall J \in \mathcal{P}_0(I)$: $I^{\varepsilon} \subseteq J$, we have

$$\Big| \prod_{j\in J} a_j b_j - AB \Big| \overset{(A.9)}{\leq} \tilde{\varepsilon}(|B| + \delta) + |A|\tilde{\varepsilon} \leq \varepsilon. \quad \text{But, this is to say that } \prod_{j\in J} a_j b_j = AB.$$

o.ε.δ.

Corollary 20. Given $(a_j)_{j\in I}, (b_j)_{j\in I} \subseteq \mathbb{C}$ with $a_j \neq 0$ for all $j \in I$, if $\exists A, C \in \mathbb{C}$ such that

$$\left(\quad \prod_{j\in I} a_j = A \quad \text{and} \quad \prod_{j\in I} \frac{b_j}{a_j} = C \quad \right) \quad \Longrightarrow \quad \left(\quad \prod_{j\in I} b_j = AC \quad \right).$$

♦

Proof of Proposition 46:

(i) Let I be an infinite set and let $f_j \in \mathcal{H}_j$ be such that $\|f_j\| = 1$. Then, for $g_j := -f_j$, $\prod_{j\in I} \|f_j\| = 1$ and $\prod_{j\in I} \|g_j\| = 1$, but $\prod\limits_{j\in I} |\langle f_j, g_j\rangle| = \prod_{j\in I}(-1)$ which does not converge.

(ii) By the Cauchy-Schwarz inequality, for any $J \in \mathcal{P}_0(I)$, $\prod_{j\in J} |\langle f_j, g_j\rangle| \leq \prod_{j\in J} \|f_j\|\|g_j\|$. If $\prod_{j\in I} \|f_j\|$, $\prod_{j\in I} \|g_j\|$ exist, then by Lemma 35, $\prod_{j\in I} \|f_j\|\|g_j\|$ exists. If the latter is 0, then trivially, $\prod_{j\in J} \langle f_j, g_j\rangle = 0$ —so, it exists. But by Lemma 35, $\prod_{j\in I} \|f_j\|\|g_j\| = 0$ implies that $\prod_{j\in I} \|f_j\|$ or $\prod_{j\in I} \|g_j\|$ is zero, which we excluded from the claim. Thus, in particular, we are assuming that $\|f_j\|\|g_j\| \neq 0 \ \forall j \in I$.

By Prop. 44. (i), any product of non-negative real numbers bounded by 1 is convergen. Since $\prod_{j\in J} |\langle f_j, g_j\rangle|/\|f_j\|\|g_j\| \leq 1$, then $\prod_{j\in J} |\langle f_j, g_j\rangle|/\|f_j\|\|g_j\|$ converges. But by Cor. 20, this implies that $\prod_{j\in J} |\langle f_j, g_j\rangle|$ is convergent. Therefore, if $\prod_{j\in I} \|f_j\|$, $\prod_{j\in I} \|g_j\|$ converge but $\prod_{j\in J} \langle f_j, g_j\rangle$ does not converge, because $\prod_{j\in J} |\langle f_j, g_j\rangle|$ must converge, the only remaining possibility by Prop. 45 is that $\sum_{j\in J} |arg(z_j)|$ diverges. ***o.ε.δ.***

Therefore, when it comes to the norm, it could make sense that $\otimes_{j\in I} f_j, \otimes_{j\in I} g_j \in \otimes_{j\in I} \mathcal{H}_j$ for a yet-to-be-defined $\otimes_{j\in I}\mathcal{H}_j$, but then we could be unable to define their inner product following the desiderata. This is why, von Neumann finds the notion of convergence in Def. 24 too sparse and needs to extend it to ensure that $\prod_{j\in I}\langle f_j, g_j\rangle$ converges whenever $\prod_{j\in I} \|f_j\|$ and $\prod_{j\in I} \|g_j\|$ converge. In particular, he defines an extension of the notion of convergence Def. 24 that makes the discordant cases of Prop. 46 convergent to 0 by definition:

Definition 31 (*Quasi-convergent Arbitrary Product*)**.**

$$\left(\quad \Big(\prod_{j\in J} z_j\Big)_{J\in\mathcal{P}_0(\mathbb{N})} \text{ is } \textit{quasi-convergent} \quad \right) \quad :\Longleftrightarrow \quad \left(\quad \Big(\prod_{j\in J} |z_j|\Big)_{J\in\mathcal{P}_0(\mathbb{N})} \text{ is } \textit{convergent} \text{ (Def. 24)} \quad \right).$$

If so, we say that $\Big(\prod_{j\in I} z_j\Big)_{J\in\mathcal{P}_0(\mathbb{N})}$ *quasi-converges* to the value:

$${}^{q}\prod_{j\in I} z_j := \begin{cases} \displaystyle\prod_{j\in I} z_j & \text{if it exists} \\ 0 & \text{if } \displaystyle\prod_{j\in I} z_j \text{ is } \textit{quasi-convergent} \text{ but } \textit{not convergent} \text{ (i.e., if } \displaystyle\sum_{j\in I} |arg(z_j)| \text{ unbounded).} \end{cases}$$

♦

Despite its ad-hoc introduction, one could view quasi-convergence not as a rule on "products of numbers" per se, but a rule on "inner products of vectors". Namely, we introduce additional directions in the space to have all the meaningful elementary tensor products in a common space —even if some of them might seem to be "incommensurate" with each other at first glance.

If that argument still seems ad-hoc, von Neumann provided several justifications —in pages 4, 18 and 64 of his 1939 paper. The most convincing reasons for us are that

- In the last step of the construction, one naturally finds a way to "mod out" the tensor products that caused the issue (see Sections A.2.2 and A.3).[10]
- The W^*-algebra of "lifted" bounded operators that we will see in §A.3, naturally implements this "modding out".
- In a side-remark of page 64 von Neumann suggests that the use of quasi-convergence could be justified via *"generalized Banach limits"*, although he provides no additional detail. He just says that proceeding in that direction the overall construction would be *"highly artificial and arbitrary, and would have implied serious difficulties in the formulation of an associative law"* (the ultimate result of desiderata (f)).

In the following gray box, we will sketch what he meant by the last item.

Parenthesis: On von Neumann's Quasi-Convergence and Banach Limits

The space of $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$-valued bounded maps over an arbitrary directed set $(D, \leq)$ is $\ell^\infty(D, \mathbb{K}) := \{x : D \to \mathbb{K} \mid \sup_{J\in D} |x_J| < \infty\}$, which is also the vector space[a] of bounded $\mathbb{K}$-nets over D. If equipped with the supremum norm $\|x\|_\infty := \sup_{j\in D} |x_j|$, $(\ell^\infty, \|\cdot\|_\infty)$ is a Banach space.[b] The convergent (and hence bounded) nets $C := \{x \in \ell^\infty \mid \exists \alpha \in \mathbb{K} : x_n \xrightarrow[J\in D]{} \alpha\}$ are a vector subspace of ℓ^∞. In particular, one can consider the map $L_0 : C \to \mathbb{K}$ that attributes to each convergent net its limit value,[c] which happens to be a linear[d] functional on C with operator norm 1.[e] One could then wonder whether there is an extension of L_0 to the whole ℓ^∞. This would constitute a generalized notion of net limit. For the case of real sequences ($D = \mathbb{N}$) —where the net limits are just ordinary limits— this idea was conceived by Banach (1932), who answered it affirmatively as an immediate corollary of the Hahn-Banach theorem.

Theorem 22 (*Hahn-Banach Theorem*)**.** Let V be an arbitrary $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$-vector space with $\|\cdot\| : V \to \mathbb{R}$ a semi-norm and $W \subseteq V$ a vector subspace. Then, for any $\theta \in V'$ (where V' denotes the space of bounded $\mathbb{K}$-linear forms on V),

$$\Big(\, |\theta(v)| \leq \|v\|,\ \forall v \in W \,\Big) \quad \Longrightarrow \quad \left(\exists\, \tilde\theta \in V' : \begin{cases} \tilde\theta \restriction_W = \theta & \text{(extension of } \theta\text{)} \\ |\tilde\theta(v)| \leq \|v\|,\ \forall v \in V & \text{(same op. norm)} \end{cases}\right).$$

◆

Proof: See Theorem 2.2.12 in (Abraham et al., 2001). ☐

In fact, the theorem also proves the existence of generalized limits for arbitrary nets —just set $V = \ell^\infty$, $W = C$ and $\theta = L_0$. The issue is that it does not ensure a unique extension (un-

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[a] One defines $\alpha(x_J)_{J\in D} + \beta(y_J)_{J\in D} := (\alpha x_J + \beta y_J)_{J\in D}$ for $\alpha, \beta \in \mathbb{K}$, $x, y \in \ell^\infty(D, \mathbb{K})$.

[b] See for instance Example 2.1.12.A in (Abraham et al., 2001).

[c] $\mathbb{K}$ is a metric space and hence Hausdorff, so by Proposition 42, there is a unique limit point for each $x \in C$.

[d] By continuity of the sum and the product by scalar in $\mathbb{K}$, the limit of a linear combination of convergent nets is the linear combination of their limits.

[e] To prove it, note that $|L_0((x_J)_{J\in D})| = \lim_{J\to D} |x_J| \leq \sup_{J\in D} |x_J| = \|(x_J)_{J\in D}\|_\infty$. This upper bound is achieved by say, $L_0((1)_{J\in D}) = \lim_{J\to D} 1 = 1$. Hence, $\|L_0\|_{op} = 1$.

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[10] Although we have not defined them yet, will see later (in Prop. 58) that only vectors in different "layers" of the *same* "quasi-layer" of the ITP can have non-convergent but quasi-convergent inner product. Since for QM, one might identify the layers in a same quasi-layer, only "legally" convergent inner products will remain.

less say, C was dense in ℓ^∞, which is not the case even for $D = \mathbb{N}$). As such, a priori, there are multiple possible notions of generalized limit. In a clever attempt to reduce the landscape of possible extensions to a *nice enough* family, one can impose certain properties of L_0 (so, properties of limits for convergent nets) to the allowed generalizations L. For example, a *Banach limit* is a generalized limit of complex *sequences* (i.e., a unit norm linear functional $L : \ell^\infty(\mathbb{N}, \mathbb{C}) \to \mathbb{C}$ extending L_0) such that[a]

($\star$) $L \circ S = L$, where $S((x_1, x_2, ...)) := (x_2, x_3, ...)$, is the *shift operator* —capturing that convergence is a "tail property", i.e., that it is independent of finitely many modifications in the sequence.

With this single constraint, although there might still be multiple different extensions L, there is a family of classically non-convergent sequences that converge to a same value *independently of which extension* satisfying ($\star$) one takes.[b] For example, consider $x := (1, 0, 1, 0, 1, ...)$. We have that $S(x) + x = (1, 1, 1, ...) \in C$ with classical limit 1. Hence:

$$1 = L_0((1,1,...)) \overset{\text{(extens.)}}{=} L(S(x)+x) \overset{\text{(linearity)}}{=} L(S(x))+L(x) \overset{(\star)}{=} 2L(x) \Longrightarrow L((1,0,1,0,...)) = \frac{1}{2}.$$

A result that holds true for all Banach limits.

As such, sequences like x have a canonical notion of generalized limit. They are called *almost convergent* sequences. In particular, for x, the generalized limit yields the average of the cluster points of the sequence. In that direction, Lorentz (1948) gave an exact characterization of almost convergent sequences in terms of limits of averages:

Proposition 47.

$$\left(\begin{array}{c} x \in \ell^\infty(\mathbb{N}, \mathbb{C}) \text{ is } \textit{almost-convergent}, \text{ i.e.,} \\ L(x) = L'(x) \text{ for all Banach limits } L, L' \end{array} \right) \Longleftrightarrow \left(\begin{array}{c} \lim_{N\to\infty} \frac{1}{N}(x_k + x_{k+1} \cdots + x_{k+N-1}) \\ \text{converges uniformly in } k \in \mathbb{N} \end{array} \right).$$

♦

Proof: See Theorem 1 in (Lorentz, 1948). □

Following these ideas, we wonder, now for $D = \mathcal{P}_0(I)$, whether there is a set of properties of convergent nets $(\prod_{j\in J} z_j)_{J\in\mathcal{P}_0(I)}$ that when imposed on an extension L of $L_0 : C \subseteq \ell^\infty(D, \mathbb{C}) \to \mathbb{C}$, all non-convergent nets $(\prod_{j\in J} z_j)_{J\in\mathcal{P}_0(I)}$ for which $(\prod_{j\in J} |z_j|)_{J\in\mathcal{P}_0(I)}$ does converge (and hence, $(\prod_{j\in J} z_j)_{J\in\mathcal{P}_0(I)} \in \ell^\infty(D, \mathbb{C})$) are necessarily given value 0 by L. That is, whether there is a set of natural properties for convergent $\mathcal{P}_0(I)$-nets of products in $\mathbb{C}$ such that all net convergence extensions with those properties agree with von Neumann's quasi-convergence.

As we proceed to explain now, there are reasons to conjecture so. Firstly, by Definition 31, a net $(\prod_{j\in J} z_j)_{J\in\mathcal{P}_0(I)} \subseteq \mathbb{C}$ is quasi-convergent but not convergent when $(\prod_{j\in J} |z_j|)_{J\in\mathcal{P}_0(I)}$ converges to a non-zero value but $(\sum_{j\in I} |arg(z_j)|)_{J\in\mathcal{P}_0(I)}$ does not converge —i.e., by Lemma 34, when $\sup_{J\in\mathcal{P}_0(I)}\{\sum_{j\in I} |arg(z_j)|\} = +\infty$. By Corollary 19 and Proposition 44, this can only be if at most countably many $z_j \neq 1$ and thus, at most countably many $arg(z_j) \neq 0$. As such, the problem reduces to the case $I = \mathbb{N}$ for all practical purposes. But then, we have the following characterization:

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[a] Originally, an additional condition was also given: $L(x) \geq 0$ for all x with $x_j \in [0, \infty)$. And some authors alternatively also impose: $\liminf_{k\to\infty} x_k \leq L(x) \leq \limsup_{k\to\infty} x_k$ for all $x \in \ell^\infty(\mathbb{N}, \mathbb{R})$. However, Bennett and Kalton (1974) point out that these follow for all Hahn-Banach extensions.

[b] See (Bennett and Kalton, 1974) for a proof that at least one Banach limit exists.

Proposition 48. Given a topological space X,

$$\left(\begin{array}{c}(x_J)_{J\in\mathcal{P}_0(\mathbb{N})}\subseteq X\\ \text{converges to some } \alpha\in X\\ \text{i.e., } \lim\limits_{J\to\mathcal{P}_0(\mathbb{N})} x_J=\alpha\end{array}\right)\iff\left(\begin{array}{c}\text{for all nested exhaustions of } \mathbb{N} \text{ by finite sets,}\\ i.e., \forall (J_n)_{n\in\mathbb{N}}\subset\mathcal{P}_0(\mathbb{N}): J_n\subseteq J_{n+1}\ \&\ \bigcup\limits_{n\in\mathbb{N}} J_n=\mathbb{N},\\ \text{as a sequence, } \lim\limits_{n\to\infty} x_{J_n}=\alpha\end{array}\right).$$

Let us denote the family of *nested exhaustions of* $\mathbb{N}$ *by finite sets* as $\mathfrak{F}$. ◆

To prove this, we need to introduce some additional notions from net theory:

Definition 32 (*Cofinal Subset*)**.** Given a directed set $(D,\leq)$, a subset $A\subseteq D$ is said to be cofinal in D when $\forall d\in D,\ \exists a\in A: d\leq a$.

(Informally, A *contains arbitrarily "large" elements.*) ◆

Definition 33. Let X be a topological space and $(D,\leq)$ a directed set. Given a net $(x_d)_{d\in D}\subseteq X$, a *subnet* is a net $(x_{h(e)})_{e\in E}\subseteq X$ indexed by a directed set $(E,\preceq)$ with $h: E\to D$ an order preserving map, i.e., such that

$$\forall d\in D,\ \exists\ e_0\in E:\ \Big(e_0\preceq e\Longrightarrow d\leq h(e)\Big).\quad \textit{(Ensuring that } h(E)\subseteq D \textit{ is cofinal.)}$$ ◆

Example 4. If $A\subseteq D$ is a cofinal subset, then the restriction of a net $(x_d)_{d\in D}\subseteq X$ to A, namely $(x_d)_{d\in A}$, is a *subnet* indexed by A (with direction induced from $\leq$). ◆

Proposition 49. Given a net $(x_d)_{d\in D}\subseteq X$,

$$\Big((x_d)_{d\in D}\subseteq X \text{ converges to } \alpha\in X\Big)\Longrightarrow\Big(\text{every subnet } (x_{h(e)})_{e\in E} \text{ converges to } \alpha\in X\Big).$$ ◆

Proof: Let $\mathcal{U}$ be an arbitrary neighborhood of α. By convergence of $(x_d)_{d\in D}$, $\exists d_0\in D:\ x_d\in\mathcal{U}$ for all $d\geq d_0$. But then, by definition of subnet, $\exists e_0\in E$ such that for all $e\succeq e_0\Rightarrow h(e)\geq d_0$ and hence, by the previous sentence, $x_{h(e)}\in\mathcal{U}\ \forall e\succeq e_0$, proving the subnet converges to α. ***o.ε.δ.***

Proof **of Proposition 48:** ($\Rightarrow$) A nested exhaustion of $\mathbb{N}$ by finite sets $(J_n)_{n\in\mathbb{N}}\subseteq\mathcal{P}_0(\mathbb{N})$ is a cofinal subset of $\mathcal{P}_0(\mathbb{N})$ because any $J\in\mathcal{P}_0(\mathbb{N})$ will eventually be contained in some element of the exhaustion. Then, the restriction $(x_{J_n})_{n\in\mathbb{N}}$ is a subnet (as in Example 4) and by Proposition 49 it converges to α.

($\Leftarrow$) We prove the contrapositive statement: if $(x_J)_{J\in\mathcal{P}_0(\mathbb{N})}$ does not converge to α (i.e., $\exists\ \mathcal{U}$ neighborhood of x such that $\forall J\in\mathcal{P}_0(\mathbb{N})$, $\exists\tilde{J}\in\mathcal{P}_0(\mathbb{N}): J\subseteq\tilde{J}$ where $x_{\tilde{J}}\notin\mathcal{U}$ —*the net gets out of* $\mathcal{U}$ *arbitrarily often*), then, there exists a $(J_n)_{n\in\mathbb{N}}\in\mathfrak{F}$ such that $(x_{J_n})_{n\in\mathbb{N}}$ does not converge to α.

Let $\widetilde{J}_1:=\{1\}$. By hypothesis, $\exists J_1\supseteq\widetilde{J}_1$ finite with $x_{J_1}\notin\mathcal{U}$. Now, assume for an inductive construction that we have $J_{n-1}\in\mathcal{P}_0(\mathbb{N}): x_{J_{n-1}}\notin\mathcal{U}$ for some $n\in\mathbb{N}$. Define $\widetilde{J}_n:=J_{n-1}\cup\{n\}$. By assumption $\exists J_n\supseteq\widetilde{J}_n$ finite such that $x_{J_n}\notin\mathcal{U}$. By construction, $J_n\subseteq J_{n+1}$ and $\cup_{n\in\mathbb{N}}J_n=\mathbb{N}$ but $x_{J_n}\notin\mathcal{U}$ for all $n\in\mathbb{N}$. Hence, $(x_{J_n})_{n\in\mathbb{N}}$ cannot converge to α. ***o.ε.δ.***

Prop. 48 implies that we can legitimately identify a net $(x_J)_{J\in\mathcal{P}_0(\mathbb{N})}\subseteq\mathbb{C}$, with the family of sequences

$$\Big\{\ (x_{J_n})_{n\in\mathbb{N}}\ \Big|\ (J_n)_{n\in\mathbb{N}}\in\mathfrak{F}\ \Big\}.$$

To say that the net converges to α is to say that all the sequences in this family converge to the same $\alpha \in \mathbb{C}$. It is helpful again to think of the points $x_J \in \mathbb{C}$ as vectors in $\mathbb{R}^2$ (which is equivalent for topological purposes). Then, by Prop. 48

- the net of *magnitude*-products $(\prod_{j\in J}|x_j|)_{J\in\mathcal{P}_0(\mathbb{N})} \subseteq \mathbb{C}$ converges *if and only if* every sequence of finite partial products $(\prod_{j\in J_n}|x_j|)_{n\in\mathbb{N}}$, $(J_n)_{n\in\mathbb{N}} \in \mathfrak{F}$ eventually gets arbitrarily close to the same sphere of the plane.
- On the other hand, the net of *argument*-sums $(\sum_{j\in J}\theta_j)_{J\in\mathcal{P}_0(\mathbb{N})} \subseteq \mathbb{C}$ fails to converge *exactly when* there is a partial sum sequence $(\sum_{j\in J_n}\theta_j)_{n\in\mathbb{N}}$, $(J_n)_{n\in\mathbb{N}} \in \mathfrak{F}$ that does not converge. That is, when there is some sequence $(\prod_{j\in J_n}x_j)_{n\in\mathbb{N}}$ for which the directions of its vectors $\prod_{j\in J_n}x_j$ do not converge. In particular, if $\prod_{j\in I}|x_j|$ converges, then $(\prod_{j\in J_n}x_j)_{n\in\mathbb{N}}$ must wander around more than one cluster point.[a] One can imagine that such a sequence traces either a spiral or a possibly irregular polygon (with the cluster points as vertices) as it approaches the sphere in magnitude.

One could desire that a reasonable generalized net limit equates with a Banach limit for each sequence in the family indexed by $\mathfrak{F}$. But if so, in view of Prop. 47, the generalized net limit would presumably yield the weighted barycenter of the joint set of cluster points of all sequences in the $\mathfrak{F}$ family. Hence, if they end up being distributed symmetrically around the origin, any such generalized limit would expectably yield 0.

---

[a] If $\prod_{j\in I}|x_j|$ converges, $(\prod_{j\in J_n}x_j)_{n\in\mathbb{N}}$ is contained in a compact set and thus it has at least one cluster point. Hence, if even still $(\prod_{j\in J_n}x_j)_{n\in\mathbb{N}}$ does not converge it must be because there are several cluster points.

A.1.c The Infinite Tensor Product's Construction

At this step, one must choose which of the alternative definitions of the finite tensor product is the most convenient one to be generalized. The one using an abstract Cantor completion could yield a mathematically elegant notion of ITP —in the sense that it would presuppose minimal mathematical structure. However, von Neumann opted to generalize Def. 23, which is based on multilinear forms. The reason could be that in a Cantor-style definition, the vectors of the resulting space would be *equivalence classes of Cauchy sequences of abstract linear combinations of abstract symbols*. On the one hand, the manipulation of such structures in an actual calculation after the theory is built could be quite cumbersome. On the other hand, although there are many theorems and results that we can apply to multi-linear forms, there might not be such a dense literature on the abstract Cantor structure. In practice, perhaps the most critical hindrance von Neumann could have found if he ever tried to use a Cantor-style definition is in proving positive-definiteness of the inner product. It is a step that comes almost "for free" in the multilinear form definition, while in the abstract setting it is slightly more involved.

We will now describe a reorganized and only slightly augmented version of von Neumann's original construction, making manifest how little modification Def. 23 requires to make the generalization work. In what follows, I denotes an arbitrary index set and for each $j \in I$, $\mathcal{H}_j$ is a Hilbert space.

Definition 34. A *$\mathscr{C}$-sequence* is a tuple $(f_j)_{j\in I} \in \prod\limits_{j\in I}\mathcal{H}_j$ for which $\prod\limits_{j\in I}\|f_j\|$ exists. We will denote the set of $\mathscr{C}$-sequences by $\mathscr{C}$. ♦

Corollary 21. $(f_j)_{j\in I}, (g_j)_{j\in I} \in \mathscr{C} \quad \Longrightarrow \quad {}^q\prod_{j\in I}\langle f_j, g_j\rangle$ exists. ♦

Proof: See Lemma 2.5.2 in (von Neumann, 1939). ☐

We will denote by $(f_j)_{j\in I\setminus k} \times g_k$ the tuple in $\prod\limits_{j\in I} \mathcal{H}_j$ that is equal to $(f_j)_{j\in I}$ except that in the entry $k \in I$ it is g_k. Its associated elementary tensor product will be denoted by $\bigotimes_{j\in I\setminus k} f_j \otimes g_k$.

Definition 35. **(i)** $\widetilde{\mathcal{L}}_I$ denotes the complex vector space of *conjugate multilinear forms* over the $\mathscr{C}$-sequences,[11] namely,

$$\widetilde{\mathcal{L}}_I := \left\{ \Phi\colon\ \mathscr{C} \longrightarrow \mathbb{C} \ \middle|\ \begin{array}{c} \Phi\Big((f_j)_{j\in I\setminus k} \times (\lambda f_k + \mu g_k)\Big) = \overline{\lambda}\Phi\Big((f_j)_{j\in I}\Big) + \overline{\mu}\Phi\Big((f_j)_{j\in I\setminus k} \times g_k\Big) \\ \forall (f_j)_j \in \mathscr{C},\ \forall k\in I,\ g_k \in \mathcal{H}_k,\ \mu,\lambda\in\mathbb{C} \end{array} \right\},$$

where the vector space structure is defined as: given $\Phi, \Psi \in \widetilde{\mathcal{L}}$, $(f_j)_j \in \mathscr{C}$ and $\lambda, \mu \in \mathbb{C}$,

$$(\lambda\Phi + \mu\Psi)((f_j)_{j\in I}) := \lambda\,(\Phi((f_j)_{j\in I})) + \mu\,(\Psi((f_j)_{j\in I})). \tag{A.10}$$

(ii) For any $(f_j)_{j\in I} \in \mathscr{C}$, we denote by $\bigotimes_{j\in I} f_j$ the form given by

$$\bigotimes_{j\in I} f_j\Big((g_j)_{j\in I}\Big) :={}^q \prod_{j\in I}\langle g_j, f_j\rangle \qquad \text{with } (g_j)_j \in \mathscr{C}. \tag{A.11}$$

We call it an *elementary tensor product* and it is an element of $\widetilde{\mathcal{L}}$.[12]

(iii) V denotes the linear span of the elementary tensor products:

$$V := span\left\{ \bigotimes_{j\in I} f_j \ \middle|\ (f_j)_{j\in I} \in \mathscr{C} \right\} := \left\{ \sum_{k=1}^{m} c_k \bigotimes_{j\in I} f_j^k \ \middle|\ m\in\mathbb{N},\ c_k\in\mathbb{C},\ (f_j^k)_{j\in I}\in\mathscr{C} \right\} \tag{A.12}$$

(iv) Since each $\Phi, \Psi \in V$ admit expansions $\Phi = \sum_{k=1}^m c_k \bigotimes_{j\in I} f_j^k$, $\Psi = \sum_{l=1}^r d_l \bigotimes_{j\in I} g_j^l$ for some $c_k, d_l \in \mathbb{C}$ and $(f_j^k)_{j\in I}, (g_j^l)_{j\in I} \in \mathscr{C}$, one can define the form $\langle\cdot,\ \cdot\rangle\colon V\times V \to \mathbb{C}$

$$\Big\langle \Psi,\ \Phi\Big\rangle := \sum_{l=1}^{r}\sum_{k=1}^{m} \overline{d_l}\, c_k \bigotimes_{j\in I} f_j^k((g_j^l)_{j\in I}) = \sum_{l=1}^{r}\sum_{k=1}^{m} \overline{d_l}\, c_k\, {}^q\prod_{j\in I}\langle g_j^l, f_j^k\rangle. \tag{A.13}$$

Claim: It is well-defined (independent of the chosen expansion). ♦

Proof: See Lemma 3.2.1 in (von Neumann, 1939). ☐

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[11] There are two main reasons why it is enough (and perhaps even necessary) to take the arguments of the maps in $\widetilde{\mathcal{L}}$ to be the $\mathscr{C}$-sequences (as opposed to *any* sequence $(g_j)_{j\in I} \in \prod\limits_{j\in I}\mathcal{H}_j$):

(i) The only forms that we need to define explicitly have the shape $(g_j)_{j\in I} \mapsto {}^q\prod_{j\in I}\langle g_j, f_j\rangle$, represent the action of $\otimes_{j\in I} f_j$ with $(f_j)_j \in \mathscr{C}$ and are used to define the inner product of elementary tensor products $\otimes_{j\in I} f_j, \otimes_{j\in I} g_j$ for which $(f_j)_{j\in I}, (g_j)_{j\in I}$ must be $\mathscr{C}$-sequences. Since any other multilinear form in the construction will be a linear combination or point-wise limit of such $\otimes_j f_j$ (in the spirit of Def. 23), their inner products will be definable in terms of inner products of elementary tensors. With all, there is no need to define the action of the multilinear forms outside $\mathscr{C}$ at any point of the construction.

(ii) It is an integral part of the proof of *positive definiteness* of the inner product that a vector Φ with $\Phi((g_j)_j) = 0$ for all $(g_j)_j \in \mathscr{C}$ is itself the zero form, $\Phi = 0$, which would not follow if Φ was a form taking possibly any sequence $(g_j)_j \in \prod_{j\in I}\mathcal{H}_j$.

[12] Obvious from conjugate linearity of the first argument in each inner product.

Corollary 22. **(i)** For all Φ, $\otimes_{j\in I} f_j \in V$, $\left\langle \bigotimes_{j\in I} f_j,\ \Phi \right\rangle = \Phi\Big((f_j)_{j\in I}\Big)$.

(ii) $\langle\cdot,\ \cdot\rangle$ is *conjugate symmetric* and *sesquilinear* (conjugate linear in the *first* argument and linear in the second one).

♦

We are now at a crucial point: if we prove that $\langle\cdot,\ \cdot\rangle$ is positive definite, it will be an inner product and the rest will follow exactly as in Def. 23. One could be tempted to use the proof we sketched in footnote [4] within Def. 23 for the generalized situation. So seemed to be von Neumann, since in order to use it, one needs to elaborate first the following equivalence relation (we will point out afterwards where this was fundamental to prove definiteness). First, he gets rid of some of the elementary tensor products that equal the zero vector:

Definition 36. A $\mathscr{C}_0$*-sequence* is a tuple $(f_j)_{j\in I} \in \prod_{j\in I} \mathcal{H}_j$ for which $\sum_{j\in I} |\|f_j\| - 1|$ exists. We denote the set of $\mathscr{C}_0$-sequences by $\mathscr{C}_0$.

♦

Proposition 50. Given $f_j \in \mathcal{H}_j$ for $j \in I$,

$$\left(\sum_{j\in I} |\|f_j\| - 1| \text{ exists} \right) \iff \left(\sum_{j\in I} |\|f_j\|^2 - 1| \text{ exists} \right)$$

♦

Proof: See Lemma 3.3.2 in von Neumann (1939). □

Lemma 36. **(i)** All $\mathscr{C}_0$-sequences are $\mathscr{C}$-sequences, i.e., $\mathscr{C}_0 \subseteq \mathscr{C}$.

(ii) $(f_j)_{j\in I} \in \mathscr{C} \setminus \mathscr{C}_0 \implies \bigotimes_{j\in I} f_j = 0$.

(*So, one can dismiss* $\mathscr{C} \setminus \mathscr{C}_0$ *from the construction, centering only on* $\mathscr{C}_0$*-sequences".*)

More generally, any $(f_j)_{j\in I} \in \mathscr{C}$ satisfies:

$$\bigotimes_{j\in I} f_j = 0 \iff \begin{cases} \bullet \text{ either:} & (f_j)_{j\in I} \notin \mathscr{C}_0 \\ \bullet \text{ or:} & \Big((f_j)_{j\in I} \in \mathscr{C}_0\Big) \ \& \ \left(\begin{array}{c} f_j = 0 \ \text{ for at least one} \\ \text{and at most finitely many } j \in I \end{array} \right). \end{cases}$$

(Consequently, any $(f_j)_{j\in I} \in \mathscr{C}_0$ satisfies: $\bigotimes_{j\in I} f_j \neq 0 \iff f_j \neq 0\ \forall j \in I$.)

(iii) In particular, any $(f_j)_{j\in I} \in \mathscr{C}$ satisfies:

$$\left. \begin{cases} \bullet \text{ if:} & f_j = 0 \text{ infinitely often} \\ \bullet \text{ or if:} & f_j \neq 0\ \forall j \in I \text{ and } \prod\limits_{j\in I} \|f_j\| = 0. \end{cases} \right\} \implies (f_j)_{j\in I} \in \mathscr{C} \setminus \mathscr{C}_0.$$

(Hence, only considering $\mathscr{C}_0$-sequences spares the possibility that $\otimes_{j\in I} f_j$ satisfies any of those two inconvenient situations.)

♦

Proof: **(i)** See Lemma 3.3.1 in (von Neumann, 1939).

(ii) We prove first that in both "either" and "or" cases, $(\Leftarrow)$ follows.

- Negating Lemma 3.3.1 in (von Neumann, 1939) we get: $(f_j)_{j\in I} \in \mathscr{C} \setminus \mathscr{C}_0 \implies \bigotimes_{j\in I} f_j = 0$.
- If $(f_j)_{j\in I} \in \mathscr{C}_0$, it is also in $\mathscr{C}$, so there is a corresponding multilinear form $\bigotimes_j f_j \in \widetilde{\mathcal{L}}$ acting as $\bigotimes_j f_j((h_j)_{j\in I}) =^q \prod_j \langle h_j, f_j\rangle$ on $(h_j)_{j\in I} \in \mathscr{C}$. But if at least one of $f_j = 0$, then $\bigotimes_j f_j = 0$.

Now we prove that if $(f_j)_j \in \mathscr{C}$ with $\bigotimes_j f_j = 0$, the only possibilities are the "either" and "or". Indeed, any $\mathscr{C}$-sequence is either in $\mathscr{C}_0$ or it is not (proving the "either" case). If it is a $\mathscr{C}_0$-sequence, then we claim that $\bigotimes_j f_j = 0$ only holds if $f_j = 0$ for at least one and at most finitely many j. To see why:

- Assume $f_j \neq 0 \ \forall j \in I$. Then (by Prop. 50), $\mathscr{C}_0$-sequence means that $\sum_{j\in I} |\|f_j\|^2 - 1|$ converges, so, by Cor. 19, we get that $\prod_{j\in I} \|f\|_j^2 = \langle \bigotimes_{j\in I} f_j, \bigotimes_{j\in I} f_j \rangle$ converges to a *non-zero* number. But because by hypothesis, $\bigotimes_{j\in I} f_j = 0$, in particular, $\langle \otimes_{j\in I} f_j, \otimes_{j\in I} f_j \rangle = (\otimes_{j\in I} f_j)((f_j)_{j\in I}) = 0$. Absurd! This proves that $f_j = 0$ for at least one $j \in I$.
- Now, assume $f_j = 0$ for infinitely many $j \in I$. Then, $\sum_{j\in I} |\|f\|_j - 1|$ cannot converge because it contains a sum of infinitely many ones. This contradicts that $(f_j)_j$ is a $\mathscr{C}_0$-sequence. Hence, $f_j = 0$ at most for finitely many $j \in I$.

(iii) We just showed that if $f_j = 0$ infinitely often, $(f_j)_{j\in I} \notin \mathscr{C}_0$. On the other hand, if $(f_j)_{j\in I} \in \mathscr{C}$ satisfies that $f_j \neq 0 \ \forall j \in I$ and $\prod_{j\in I} \|f_j\| = 0$, it cannot be that $(f_j)_{j\in I} \in \mathscr{C}_0$, because if so, $\sum_{j\in I} |\|f_j\| - 1|$ would exist, implying by Prop. 44, that $\prod_{j\in I} \|f_j\| \neq 0$. Hence, $(f_j)_{j\in I} \in \mathscr{C} \setminus \mathscr{C}_0$. ***o.ε.δ.***

One could wonder why we are excluding only those zero tensor products outside $\mathscr{C}_0$ (after all, there are still more than one in $\mathscr{C}_0$). The reason is that they were avoiding us from finding the following equivalence relation, which groups elementary tensor products with "asymptotically equal tails" —$\langle f_j, g_j \rangle \simeq 1$ for most $j \in I$— and allows to prove positive definiteness.

Theorem 23. **(i)** Given $(f_j)_{j\in I}, (g_j)_{j\in I} \in \mathscr{C}_0$, the relation

$$(f_j)_{j\in I} \approx (g_j)_{j\in I} \quad :\Longleftrightarrow \quad \left(\sum_{j\in I} |\langle f_j, g_j \rangle - 1| \text{ exists} \right), \tag{A.14}$$

is an *equivalence relation.*

Notation: The set of all $\approx$-equivalence classes will be denoted throughout the whole present document by Γ and the equivalence class of $(f_j)_{j\in I}$ by $\mathfrak{C}[(f_j)_{j\in I}]$. We will abuse notation saying that $\bigotimes_{j\in I} f_j$ is a $\mathscr{C}_0$-sequence or belongs to some $\approx$-class if $(f_j)_{j\in I}$ does so.

(ii) Given $(f_j)_{j\in I}, (g_j)_{j\in I} \in \mathscr{C}_0$,

$$\Big\langle \bigotimes_{j\in I} f_j, \bigotimes_{j\in I} g_j \Big\rangle = 0 \Longleftrightarrow \begin{cases} \bullet \ \textit{either:} \quad (f_j)_{j\in I} \not\approx (g_j)_{j\in I} \\ \bullet \ \textit{or:} \qquad (f_j)_{j\in I} \approx (g_j)_{j\in I} \ \ \textit{and} \ \ (\ \langle f_j, g_j \rangle = 0 \text{ for some } j \in I\ ) \end{cases}$$

(iii) Given $(f_j)_{j\in I}, (g_j)_{j\in I} \in \mathscr{C}_0$,

$$\begin{cases} \bullet \ (f_j)_{j\in I} \approx (g_j)_{j\in I} \Longrightarrow & \displaystyle\prod_{j\in I} \langle f_j, g_j \rangle \text{ exists. It is } \neq 0 \text{ unless } \exists j_0 \in I : \langle f_j, g_j \rangle = 0. \\ \bullet \ (f_j)_{j\in I} \not\approx (g_j)_{j\in I} \Longrightarrow & \textit{either} \ \displaystyle\prod_{j\in I} \langle f_j, g_j \rangle = 0 \ \textit{or} \text{ it does not exist.}^{[13]} \end{cases}$$

(In particular, if $\prod_{j\in I} \langle f_j, g_j \rangle$ does not exist, then $(f_j)_{j\in I} \not\approx (g_j)_{j\in I}$.) ♦

---

[13] If it does not exist, then necessarily $(f_j)_{j\in I} \overset{q}{\not\approx} (g_j)_{j\in I}$ (see Prop. 58).

Proof: For items (i) and (ii) see respectively, Lemma 3.3.3 and Theorem I in (von Neumann, 1939). For **item (iii):** By Cor. 19, using the definition of $\approx$,

$$\left(\prod_{j\in I}\langle f_j, g_j\rangle \quad \text{exists} \ \ \& \ \ \text{is} \ \neq 0 \right) \iff \left(\langle f_j, g_j\rangle \neq 0 \ \forall j \in I \ \ \& \ \ (f_j)_{j\in I} \approx (g_j)_{j\in I} \right).$$

If $\langle f_j, g_j\rangle = 0$ for some j, $\prod_{j\in I}\langle f_j, g_j\rangle = 0$, so the last statement proves the first sub-item of (iii). Likewise, its negation,

$$\left(\prod_{j\in I}\langle f_j, g_j\rangle \quad \text{does not exist} \ \ \textit{or} \ \ \text{is} \ = 0 \right) \iff \left(\exists j_0 \in I : \langle f_{j_0}, g_{j_0}\rangle = 0 \quad \textit{or} \quad (f_j)_{j\in I} \not\approx (g_j)_{j\in I} \right)$$

proves the second sub-item of (iii). ***o.ε.δ.***

After this characterization, we can now prove positive definiteness of $\langle \cdot, \cdot \rangle$.

Proposition 51. **(i)** $\forall \Phi \in V$, $\langle \Phi, \Phi\rangle \geq 0$ *(semi-definiteness).*

(ii) $\forall \Phi, \Psi \in V$, $|\langle \Psi, \Phi\rangle| \leq \sqrt{\langle \Psi, \Psi\rangle \, \langle \Phi, \Phi\rangle}$ *(Cauchy-Schwarz inequality).* ♦

Proof: (We follow the proof in (von Neumann, 1939)) **(i)** First, note that as multilinear forms, for all $z \in \mathbb{C}$, $z \bigotimes_{j\in I} f_j = \bigotimes_{j\in I\setminus k} f_j \otimes (z f_k)$. Hence, by definition of V, we can expand an arbitrary $\Phi \in V$ as $\Phi = \sum_{k=1}^{a} \bigotimes_{j\in I} f_j^k$ (i.e., we can always absorb the coefficient in some factor). If for some k, $(f_j^k)_{j\in I}$ is not a $\mathscr{C}_0$-sequence, then, by Lemma 36 $\bigotimes_{j\in I} f_j^k = 0$. Thus, we can assume that all $(f_j^k)_{j\in I}$ in the expansion of Φ are $\mathscr{C}_0$-sequences. But then, we can group them according to which $\approx$-equivalence class they belong to. Let us say there are p ($\leq a$) different classes and denote by Φ_s the sum of the $\bigotimes_{j\in I} f_j^k$ belonging to the same s-th class. If so, $\Phi = \sum_{s=1}^{p} \Phi_s$. By Theorem 23 $\langle \Phi_s, \Phi_k\rangle = 0$ for all $s \neq k$, so $\langle \Phi, \Phi\rangle = \sum_{s=1}^{p}\langle \Phi_s, \Phi_s\rangle$. Hence, it is enough to prove that $\langle \Phi_s, \Phi_s\rangle \geq 0$ for every s. That is, we just need to prove that $\sum_{k=1}^{m} \bigotimes_{j\in J} f_j^k$, composed by $(f_j^k)_{j\in J}$ in the same $\approx$-equivalence class, has non-negative norm:

$$0 \overset{\text{(need to show)}}{\leq} \langle \Phi_s, \Phi_s\rangle = \sum_{j,k=1}^{m} {}^{q}\prod_{l\in I}\langle f_l^j, f_l^k\rangle \overset{\text{(Thm. 23.(iii))}}{=} \sum_{j,k=1}^{m} \prod_{l\in I}\langle f_l^j, f_l^k\rangle. \tag{A.15}$$

Now, because each product in the sum is convergent (by Cor. 21 and Thm. 23.(iii)), if we prove that for any $\{i_1, ..., i_n\} \subseteq I$, $n \in \mathbb{N}$, the sum of partial products $\sum_{j,k=1}^{m} \prod_{l=1}^{n}\langle f_{i_l}^j, f_{i_l}^k\rangle$ is non-negative, the sum of arbitrary products (A.15) will need to be non-negative. From here the proof continues exactly as for finitely many factors (footnote [4] of Def. 23).

The Gram matrix $(\langle f_l^j, f_l^k\rangle)_{j,k=1}^{m} \in \mathbb{C}^{m\times m}$ is positive semi-definite for each $l \in \{i_1, ..., i_n\}$ since

$$\sum_{j,k=1}^{m} x^j \langle f_l^j, f_l^k\rangle x^k = \left\| \sum_{j=1}^{m} x^j \, f_l^j \right\|^2 \geq 0, \quad \forall (x_j)_j \in \mathbb{C}^m. \tag{A.16}$$

Hence, it can be diagonalized in an orthonormal basis $(u^{lr})_{r=1}^{m} \subset \mathbb{C}^m$ with non-negative eigenvalues $\lambda_r \geq 0$:

$$(\langle f_l^j, f_l^k\rangle)_{j,k} = \sum_{r=1}^{m} \lambda_r \, (u_j^{lr} \, \overline{u_k^{lr}})_{j,k}. \tag{A.17}$$

Consequently,

$$\sum_{j,k=1}^{m}\prod_{l=1}^{n}\langle f^j_{i_l}, f^k_{i_l}\rangle = \sum_{j,k=1}^{m}\prod_{l=1}^{n}\left(\sum_{r=1}^{m}\lambda_r\, u_j^{i_l r}\,\overline{u_k^{i_l r}}\right) = \sum_{j,k=1}^{m}\sum_{r_1,\dots,r_n=1}^{m}\lambda_{r_1}\cdots\lambda_{r_n} u_j^{i_1 r_1}\overline{u_k^{i_1 r_1}}\cdots u_j^{i_n r_n}\overline{u_k^{i_n r_n}} = \tag{A.18}$$

$$= \sum_{r_1,\dots,r_n=1}^{m}\lambda_{r_1}\cdots\lambda_{r_n}\left(\sum_{j=1}^{m}u_j^{i_1 r_1}\cdots u_j^{i_n r_n}\right)\left(\sum_{k=1}^{m}\overline{u_k^{i_1 r_1}}\cdots\overline{u_k^{i_n r_n}}\right) = \sum_{r_1,\dots,r_n=1}^{m}\lambda_{r_1}\cdots\lambda_{r_n}\left|\sum_{j=1}^{m}u_j^{i_1 r_1}\cdots u_j^{i_n r_n}\right|^2,$$

which is trivially non-negative.

Note on why one needs to introduce the $\approx$-equivalence relation before this proof: As we said, for general $\Phi \in V$, $\Phi = \sum_{j=1}^{a}\bigotimes_{l\in I} f_l^j$ with $\langle\Phi,\Phi\rangle = \sum_{j,k=1}^{a}{}^{q}\prod_{l\in I}\langle f_l^j, f_l^k\rangle$. Unless we got rid of the possibility that ${}^{q}\prod_{l\in I}\langle f_l^j, f_l^k\rangle$ is quasi-convergent but not convergent, we would be unable to make the argument after equation (A.15) that allowed the use of the finite dimensional proof. For quasi-convergent but non-convergent ${}^{q}\prod_{l\in I}\langle f_l^j, f_l^k\rangle$, the product is defined to be zero, so, $\sum_{j,k=1}^{m}{}^{q}\prod_{l\in I}\langle f_l^j, f_l^k\rangle = \sum_{j,k=1}^{m}\theta_{j,k}\prod_{l\in I}\langle f_l^j, f_l^k\rangle$, where $\theta_{j,k}$ is 0 if the j,k product is not convergent and 1 otherwise. But then, the third equality of (A.18) —which was the key thing that proved ≥ 0— would not be possible.

(ii) The proof is exactly the same as for finitely many factors and it only uses that $\langle\cdot,\cdot\rangle$ is sesquilinear together with item (i). See for instance Lemma 3.4.2 in (von Neumann, 1939).
o.ε.δ.

Corollary 23. $\forall\Phi \in V\setminus\{\vec{0}\}$, $\langle\Phi,\Phi\rangle > 0$. Hence, $\langle\cdot,\,\cdot\rangle$ is an *inner product* in V.

In particular, $v \in V \mapsto \|v\| := \sqrt{\langle v, v\rangle}$ defines a *norm* and hence a *topology* in V. ♦

Proof: (Following (von Neumann, 1939).) Let $\Phi \in V$, such that $\langle\Phi,\Phi\rangle = 0$. By Prop. 51.(ii), $|\langle\Psi,\Phi\rangle| \leq \cancelto{0}{\sqrt{\langle\Phi,\Phi\rangle}}\sqrt{\langle\Psi,\Psi\rangle} = 0$ for all $\Psi \in V$. Hence, $\langle\Psi,\Phi\rangle = 0$ for all $\Psi \in V$. In particular, $\langle\bigotimes_{j\in I} f_j,\ \Phi\rangle = 0$ for all elementary tensor products $\otimes_{j\in I} f_j$. By Corollary 22 this is to say that $\Phi((f_j)_{j\in I}) = 0$ for all $(f_j)_{j\in I} \in \mathscr{C}$. Therefore, as a form in $\widetilde{\mathcal{L}}$, $\Phi = 0$. That is, the only $\Phi \in V$ with $\langle\Phi,\Phi\rangle = 0$ is $\Phi = 0$.
o.ε.δ.

Now we can proceed exactly as in Def. 23 in order to "complete" the tensor product space.[14]

Lemma 37. Let $(\phi_n)_{n\in\mathbb{N}} \subseteq V$ be a $\|\cdot\|$-Cauchy sequence (i.e., $\lim_{N\to\infty}\sup_{m,n\geq N}\|\phi_m - \phi_n\| = 0$).

(i) For all $(f_j)_{j\in I} \in \mathscr{C}$, the limit $\lim_{n\to\infty}\phi_n\Big((f_j)_{j\in I}\Big)$ exists.

(ii) The form $(f_j)_{j\in I} \in \mathscr{C} \longmapsto \lim_{n\to\infty}\phi_n\Big((f_j)_{j\in I}\Big)$ is an element of $\widetilde{\mathcal{L}}$ (i.e., a conjugate multilinear form). We call it the *point-wise limit* of $(\phi_n)_{n\in\mathbb{N}}$. ♦

Proof: See Lemma 3.5.2 in von Neumann (1939). □

Definition 37. We define $\bigotimes_{j\in I}\mathcal{H}_j$ to be the subspace of $\widetilde{\mathcal{L}}$ constituted by point-wise limits of $||\cdot||-$Cauchy sequences of elements in V (i.e., of linear combinations of elementary tensor products). Explicitly, $\Psi \in \bigotimes_{j\in I}\mathcal{H}_j \subseteq \widetilde{\mathcal{L}}$ *if and only if* $\exists(\phi_n)_{n\in\mathbb{N}} \subset V$ with

$$(i)\ \Phi((f_j)_{j\in I}) = \lim_{n\to\infty}\phi_n((f_j)_{j\in I})\ \ \forall (f_j)_{j\in I} \in \mathscr{C} \qquad (ii)\ \lim_{N\to\infty}\sup_{m,n\geq N}\|\phi_m - \phi_n\| = 0. \tag{A.19}$$
♦

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[14] What we do is something between taking a *closure* —which we really do not because in principle, there is no topology in the ambient space $\widetilde{\mathcal{L}}$— and taking an *abstract completion* —it is not abstract, because the limit points already exist as members of $\widetilde{\mathcal{L}}$.

Lemma 38. $\|\cdot\|$-Cauchy sequences $(\phi_n)_{n\in\mathbb{N}}, (\psi_n)_{n\in\mathbb{N}} \subset V$ satisfy that

$$\lim_{n\to\infty} \|\phi_n - \psi_n\| = 0 \quad\Longleftrightarrow\quad \left(\text{ point-wise } \lim_{n\to\infty} \phi_n = \lim_{n\to\infty} \psi_n. \right)$$

(Informally, *Cauchy sequences with convergent tails represent the same limit vector.*) ♦

Proof: See Lemma 3.5.2 in von Neumann (1939). □

Proposition 52. **(i)** For any pair $\Phi, \Psi \in \bigotimes_{j\in I} \mathcal{H}_j$, and any pair of $\|\cdot\|$-Cauchy sequences $(\phi_n)_{n\in I}$, $(\psi_n)_{n\in I}$ whose point-wise limits are respectively Φ and Ψ,[15] $\lim_{n\to\infty} \langle \phi_n, \psi_n\rangle$ exists and is independent of the chosen Cauchy sequences.

(ii) The form $\langle\cdot,\cdot\rangle : \bigotimes_{j\in I} \mathcal{H}_j \times \bigotimes_{j\in I} \mathcal{H}_j \to \mathbb{C}$ defined by

$$\Big\langle \Phi,\, \Psi \Big\rangle := \lim_{n\to\infty} \langle \phi_n, \psi_n\rangle,$$

(using the notation in (i)) is well-defined and is an *inner product.* For $\Phi, \Psi \in V$ it coincides with $\langle\cdot,\cdot\rangle$ as in Def. 36 (iv). As such, $\Phi \in \bigotimes_{j\in I} \mathcal{H}_j \mapsto \|\Phi\| := \sqrt{\langle \Phi, \Phi\rangle}$ is a norm extending that of Cor. 23.

(iii) Given $\Phi \in \bigotimes_{j\in I} \mathcal{H}_j$ and a sequence $(\phi_n)_{n\in\mathbb{N}} \subset V$,

$$\left(\begin{array}{c} (\phi_n)_{n\in\mathbb{N}} \text{ is a } \|\cdot\|\text{-Cauchy sequence} \\ \text{and } \phi_n \xrightarrow[n\to\infty]{\text{point-wise}} \Phi \end{array} \right) \Longleftrightarrow \left(\begin{array}{c} \lim_{n\to\infty} \|\Phi - \phi_j\| = 0, \ i.e., \\ \phi_n \xrightarrow[n\to\infty]{\|\cdot\|} \Phi \end{array} \right).$$

(iv) V is a *dense* vector subspace of $\bigotimes_{j\in I} \mathcal{H}_j$ (in the $\|\cdot\|$ topology).

(v) $(\bigotimes_{j\in I} \mathcal{H}_j,\ \|\cdot\|)$ is *Cauchy complete.* ♦

Proof: See Lemmas 3.5.3, 3.5.4, 3.5.7, 3.5.8 and 3.5.9 in von Neumann (1939). □

Definition 38. We call the Hilbert space $(\bigotimes_{j\in I} \mathcal{H}_j,\ \langle\cdot,\cdot\rangle)$, the *proper infinite tensor product.* Whenever we simply say "infinite tensor product" (ITP), we mean the *proper* ITP. ♦

Naturally, we recover many of the properties of finite tensor products, such as:

Proposition 53. For each $j_0 \in I$, and each $(f_j)_{j\in I} \in \mathscr{C}$, the mapping

$$g_{j_0} \in \mathcal{H}_{j_0} \longmapsto \bigotimes_{j\in I\setminus j_0} f_j \otimes g_{j_0} \in \bigotimes_{j\in I} \mathcal{H}_j \tag{A.20}$$

is *linear* and *continuous.* ♦

Proof: See Lemma 4.1.3 in (von Neumann, 1939). □

A.1.d The "Well-Generalized" Theorem

Now, one could wonder whether alternative ways to generalize the tensor product would yield a similar construction. Remarkably, although he employed conjugate multilinear forms to construct his ITP, von Neumann also proved the following uniqueness result:

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[15] There are always such sequences by Def. 37.

Theorem 24. A Hilbert space $(\mathscr{H}, \langle\cdot,\cdot\rangle_{\mathscr{H}})$ satisfies the following two conditions:

(i) for each $(f_j)_{j\in I}\mathscr{C}$, one can designate an element $\tilde{\otimes}_{j\in I} f_j \in \mathscr{H}$ (not necessarily injectively), such that for all $(f_j)_{j\in I}, (g_j)_{j\in I} \in \mathscr{C}$:

$$\left\langle \tilde{\otimes}_{j\in I} f_j,\ \tilde{\otimes}_{j\in I} g_j \right\rangle_{\mathscr{H}} = ^q \prod_{j\in I} \langle f_j,\ g_j\rangle,$$

(ii) $\tilde{V} := span\left\{\ \tilde{\otimes}_{j\in I} f_j \in \mathscr{H} \mid (f_j)_{j\in I} \in \mathscr{C}\right\}$ is dense in $\mathscr{H}$,

if and only if there *exists a linear isomorphism* $\mathscr{U} : \bigotimes_{j\in I} \mathcal{H}_j \longrightarrow \mathscr{H}$ with $\mathscr{U}(\otimes_{j\in I} f_j) = \tilde{\otimes}_{j\in I} f_j$. If it exists, U is unitary and *unique.*

The statement still holds if we replace $\mathscr{C}$ by $\mathscr{C}_0$. ♦

Proof: See Theorem IV in von Neumann (1939). □

That is, whichever generalization of the tensor product one could construct, as long as the two points in the theorem hold —both of which being obvious desiderata that any generalization should satisfy—, for all mathematical purposes[16] the new construction would still be von Neumann's.

Recovering the Finite Case

As one of the most important parts in checking well-generalization, we must study if the construction falls back to the usual tensor product under the appropriate circumstances. To do this, Theorem 24 is utmost important.

Proposition 54.

- If I has only one element, say $I = \{0\}$, then $\bigotimes_{j\in I} \mathcal{H}_j = \mathcal{H}_0$.
- If I has finitely many elements, say $I = \{1, ..., n\}$, then $\bigotimes_{j\in I} \mathcal{H}_j = \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n$. ♦

Proof:

- For $I = \{0\}$, using the notation of Theorem 24, we declare $\tilde{\otimes}_{j\in I} f_j \equiv \tilde{\otimes}_0 f_0 := f_0 \in \mathcal{H}_0$ and $\mathscr{H} := \mathcal{H}_0$. Then, all the conditions are trivially satisfied for the existence of a unique unitary map $U : \bigotimes_{j\in I} \mathcal{H}_j = \bigotimes_0 \mathcal{H}_0 \to \mathcal{H}_0$ such that $U(\otimes_0 f_0) = \tilde{\otimes}_0 f_0 = f_0$. Now, all vectors of $\otimes_0 \mathcal{H}_0$ are elementary tensor products $\tilde{\otimes}_0 f_0 = f_0$ (also their linear combinations), so, $\otimes_0 \mathcal{H}_0$ equals the set of elementary tensor products $\otimes_0 f_0$. Hence, canonically, $\bigotimes_0 \mathcal{H}_0 = \mathcal{H}_0$.
- Whichever definition of $\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n$ we employ, the theorem will give a unique unitary identification relating the elementary tensor products of that definition and von Neumann's generalization. ***o.ε.δ.***

The ITP of Subspaces seen as a Subspace of the full ITP

A priori, it could happen that given W_j is a vector subspace of the Hilbert space $\mathcal{H}_j$ for each $j \in I$, the tensor product $\bigotimes_{j\in I} W_j$ is not comparable to the tensor product of the full Hilbert spaces $\bigotimes_{j\in I} \mathcal{H}_j$. However,

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[16] By which we mean that a proposition holds in one of the constructions *if and only if* after the identification, it holds in the other one.

Corollary 24. Given a closed vector subspace $\mathcal{W}_j \subseteq \mathcal{H}_j$ for each $j \in I$, such that $\mathcal{W}_j \neq \{\vec{0}\}$, there is a canonical way to see $\bigotimes_{j\in I} \mathcal{W}_j$ as a subspace of $\bigotimes_{j\in I} \mathcal{H}_j$. ♦

Proof: (We follow (von Neumann, 1939).) Denote by $\tilde{\otimes}_{j\in I}\mathcal{W}_j$ the closure of the span of elementary tensor products $\bigotimes_{j\in I} f_j \in \bigotimes_{j\in I} \mathcal{H}_j$ for which $f_j \in \mathcal{W}_j\ \forall j$. This is a closed vector subspace of $\bigotimes_{j\in I} \mathcal{H}_j$ and hence inherits a Hilbert space structure. To apply Theorem 24 on $\mathscr{H} := \tilde{\otimes}_{j\in I}\mathcal{W}_j$, we now designate $\tilde{\otimes}_j f_j := \otimes_j f_j \in \bigotimes_j \mathcal{H}_j$ for each $(f_j)_j \in \mathscr{C}$. Then, the two conditions are trivially satisfied and the theorem ensures that there is a unique unitary isomorphism $U : \bigotimes_{j\in I} \mathcal{W}_j \longrightarrow \tilde{\otimes}_j\mathcal{W}_j \subseteq \bigotimes_{j\in I} \mathcal{H}_j$ that identifies the basic tensor products of one and the other space in the obvious way. ***o.ε.δ.***

On the "Commutativity" of $\otimes$

In the following sense, there is no intrinsic dependence on the order in which we place the Hilbert spaces in an ITP.

Lemma 39. Given:

- I_1, I_2 arbitrary index sets with the same cardinality,
- $(\mathcal{H}_j)_{j\in I_1}$, $(\mathcal{K}_j)_{j\in I_2}$ families of Hilbert spaces such that there exists a bijection

$$\sigma : I_1 \longrightarrow I_2 \quad \text{with} \quad \mathcal{H}_j = \mathcal{K}_{\sigma(j)} \quad \text{(that we call a \textit{rearrangement})},$$

then, there exists a canonical identification between $\bigotimes_{j\in I_1} \mathcal{H}_j$ and $\bigotimes_{j\in I_2} \mathcal{K}_j$ associated to that rearrangement. Namely, the one identifying

$$\bigotimes_{j\in I_1} f_j \in \bigotimes_{j\in I_1} \mathcal{H}_j \quad \longleftrightarrow \quad \bigotimes_{j\in I_2} g_j \in \bigotimes_{j\in I_2} \mathcal{K}_j \qquad \text{when } f_j = g_{\sigma(j)}.$$

♦

Proof: The notion of infinite product we used to define the inner products of elementary tensor products was order independent. Following that, the construction had no dependence on the particular order of the factors. In fact, I does not even need to possess an order. With that, the lemma is essentially trivial. ***o.ε.δ.***

For instance, given a countable family of Hilbert spaces $(\mathcal{H}_j)_{j\in\mathbb{N}}$, there is a canonical identification between $\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots$ and $\mathcal{H}_{\sigma(1)} \otimes \mathcal{H}_{\sigma(2)} \otimes \cdots$ for any bijection $\sigma : \mathbb{N} \to \mathbb{N}$.

Importantly, note that this does not mean that the construction is "bosonic" when we put all $\mathcal{H}_j$ equal to a same space $\mathcal{H}$. One still needs to keep track of which factor in an elementary tensor product should be paired with which from another elementary tensor product. That is, consider $f_1 \otimes f_2 \otimes f_3 \cdots \in \mathcal{H} \otimes \mathcal{H} \otimes \cdots$. Then, note that

$$\Big\langle f_1 \otimes f_2 \otimes f_3 \cdots\,,\ \underbrace{f_2 \otimes f_3 \otimes f_1} \otimes f_4 \otimes f_5 \otimes \cdots \Big\rangle = \langle f_1, f_2\rangle\langle f_2, f_3\rangle\langle f_3, f_1\rangle \prod_{j\in\mathbb{N}\setminus\{1,2,3\}} \|f_j\|^2 \tag{A.21}$$

which in general, is not even a real number! (So certainly, $f_1 \otimes f_2 \otimes f_3 \otimes \cdots \neq f_2 \otimes f_3 \otimes f_1 \otimes \cdots$.)

A.2 Dissecting the Infinite Tensor Product

In order to manifest the vastness of the ITP, we enunciate the following fact. In a finite tensor product $\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n$, given an ONB $\{f_j^k\}_{k\in K_j} \subseteq \mathcal{H}_j$ per each factor space, one obtains an ONB of the tensor product $\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n$ by considering all their possible elementary tensor products $f_1^{k_1} \otimes \cdots \otimes f_n^{k_n}$. This is no longer true for the ITP:

Proposition 55. Given an arbitrary index set I and a family of Hilbert spaces $(\mathcal{H}_j)_{j\in I}$, consider a choice of ONB $\{f_j^k\}_{k\in K_j} \subseteq \mathcal{H}_j$ for each $\mathcal{H}_j$, where K_j is an index set (of cardinality equal to the Hilbert dimension of $\mathcal{H}_j$). Any choice of "one vector per each basis", namely, any $(f_j^{k_j})_{j\in I}$, $k_j \in K_j$, is a $\mathscr{C}$-sequence because all $f_j^{k_j}$ are norm 1 vectors. Hence, their corresponding elementary tensor products,

$$B := \left\{ \bigotimes_{j\in I} f_j^{k_j} \;\middle|\; (k_j)_{j\in I} \in \prod_{j\in I} K_j \right\} \tag{A.22}$$

are well-defined and have norm 1. In particular, they are trivially orthogonal to each other, so B is an *orthonormal family.*

Now, if I is infinite and if *infinitely* many of the spaces $\mathcal{H}_j$ have Hilbert dimension greater than 1 (i.e., $|K_j| \geq 2$), then:

(i) B is an *uncountable* set. **(ii)** B is *not an ONB* of $\bigotimes_{j\in I} \mathcal{H}_j$. ♦

Proof:

(i) By assumption, there exists a subset of countably many indices $R = (r_1, r_2, \ldots) \subseteq I$ such that for each $r \in R$, $|K_r| \geq 2$. Thus, we can assume without loss of generality that for all $r \in R$ the indices 0 and 1 are in K_r. The set of possible sequences made of 0 and 1, denoted by $\prod_{r\in R} \{0,1\}$, is then a subset of the set of possible choices of an element in K_r per each $r \in R$. This has a smaller or equal cardinality than the set of possible choices of an element in K_j per each $j \in I$ (without restricting only to R), namely, than $\prod_{j\in I} K_j$. But $\prod_{j\in I} K_j$ indexes B, so they have the same cardinality.[a] Finally, the number of possible 0,1 sequences is uncountable by Cantor's diagonal argument. Therefore, B must be uncountable as well.

(ii) Consider $g_r := \frac{1}{\sqrt{2}}(f_r^0 + f_r^1) \in \mathcal{H}_r$ for each $r \in R$ and for $j \in I \setminus R$ let g_j be a fixed vector of each ONB. Then, $\|g_j\| = 1\ \forall j \in I$ and as such, $\bigotimes_{j\in I} g_j$ has norm 1. But this vector is orthogonal to all the elements in B as we proceed to prove. For any choice of $(k_j)_{j\in I} \in \prod_{j\in I} K_j$

$$\sum_{j\in I} |1 - \langle f_j^{k_j}, g_j\rangle| \geq \sum_{r\in R} |1 - \langle f_j^{k_j}, g_j\rangle| = \sum_{r\in R} \left\{ \begin{array}{ll} 1 - \dfrac{1}{\sqrt{2}} & \text{if } k_j \in \{0,1\} \\ 1 & \text{if } k_j \notin \{0,1\} \end{array} \right\} \geq \sum_{r\in R} 0.2 = \infty. \tag{A.23}$$

Hence, by definition $(g_j)_{j\in I} \not\approx (f_j^{k_j})_{j\in I}$ and by Theorem 23, $\langle \bigotimes_{j\in I} f_j^{k_j}, \bigotimes_{j\in I} g_j\rangle = 0$. Therefore, the span of B cannot "expand" $\bigotimes_{j\in I} g_j$ in any way and a fortiori, B cannot be an ONB of $\bigotimes_{j\in I} \mathcal{H}_j$. ***o.ε.δ.***

---

[a] Note that $\otimes_{j\in I} f_j^k$ for different choices of $(k_j)_{j\in I} \in \prod_{j\in I} K_j$ are non-zero and orthogonal to each other. Thus, they are all different to each other and there is a bijection between B and $\prod_{j\in I} K_j$.

This exposes the main challenge (or perhaps miracle) of the ITP:

Corollary 25. Given an infinite index set I and a family of Hilbert spaces $(\mathcal{H}_j)_{j\in I}$, if *infinitely* many of the spaces $\mathcal{H}_j$ have *dimension greater than 1,* then $\bigotimes_{j\in I}\mathcal{H}_j$ is a *non-separable* Hilbert space (i.e., any ONB has uncountable cardinality). ♦

Proof: Because an ONB in a Hilbert space is an orthonormal family of vectors with maximal cardinality, by Prop. 55, any ONB of $\bigotimes_{j\in I}\mathcal{H}_j$ must have uncountable cardinality. **$o.\varepsilon.\delta.$**

That is, even the simplest non-trivial ITP, $\mathbb{C}^2\otimes\mathbb{C}^2\otimes\mathbb{C}^2\otimes\cdots$ is a non-separable Hilbert space. This explains the importance of finding well-behaved "dissections" of the space. Ideally, we would like to make most of the work on a generic (preferably separable) subspace of the ITP and then obtain the results for their assembly with minimal "pain". In the tongue of an ancient tactician: *"divide and conquer".*

By "dissection" we mean "direct sum decomposition into orthogonal subspaces". The obvious candidate for such a decomposition, in light of Theorem 23, are the span of $\approx$-classes.

Definition 39. For each $\approx$-equivalence class $\mathfrak{C}\in\Gamma$ (see Theorem 23), we denote the closure of the span of all its associated elementary tensor products by

$$\otimes^{\mathfrak{C}}_{j\in I}\mathcal{H}_j := \overline{\operatorname{span}\left\{ \bigotimes_{j\in I} f_j \in \bigotimes_{j\in I}\mathcal{H}_j \;\middle|\; (f_j)_{j\in I}\in\mathfrak{C} \right\}}. \tag{A.24}$$

We call it the $\mathfrak{C}$-th *improper infinite tensor product* or $\mathfrak{C}$-th *layer of the infinite tensor product.* ♦

Theorem 25. Let $(\mathcal{H}_j)_{j\in I}$ be a family of Hilbert spaces.

(i) If $\mathfrak{C},\mathfrak{N}\in\Gamma$ are $\mathfrak{C}\neq\mathfrak{N}$, then $\otimes^{\mathfrak{C}}_{j\in I}\mathcal{H}_j \perp \otimes^{\mathfrak{N}}_{j\in I}\mathcal{H}_j$.

(ii)

$$\bigotimes_{j\in I}\mathcal{H}_j = \overline{\operatorname{span}\left\{ \bigcup_{\mathfrak{C}\in\Gamma}\left(\otimes^{\mathfrak{C}}_{j\in I}\mathcal{H}_j\right)\right\}}. \tag{A.25}$$

Hence, by Proposition 15, $\bigotimes_{j\in I}\mathcal{H}_j = \bigoplus_{\mathfrak{C}\in\Gamma}\left(\otimes^{\mathfrak{C}}_{j\in I}\mathcal{H}_j\right)$. This is why we called $\otimes^{\mathfrak{C}}_{j\in I}\mathcal{H}_j$ a *layer* of the ITP.

(iii) For every $\mathfrak{C}\in\Gamma$, $\exists(f^{\mathfrak{C}}_j)_{j\in I}\in\mathfrak{C}$ such that $\left\|f^{\mathfrak{C}}_j\right\|=1$. We call them (and their associated tensor products) *generators of the $\mathfrak{C}$-th layer* because for each and every one of them,

$$\otimes^{\mathfrak{C}}_{j\in I}\mathcal{H}_j = \overline{\operatorname{span}\left\{ \bigotimes_{j\in I} f_j \in \bigotimes_{j\in I}\mathcal{H}_j \;\middle|\; f_j\neq f^{\mathfrak{C}}_j \quad \text{only for finitely many } j\in I \right\}}. \tag{A.26}$$

(iv) For each $\mathfrak{C}\in\Gamma$ and each generator $\bigotimes_{j\in I} f^{\mathfrak{C}}_j \in \otimes^{\mathfrak{C}}_{j\in I}\mathcal{H}_j$, given an ONB $(\phi^k_j)_{k\in K_j}\subseteq\mathcal{H}_j$ of each $\mathcal{H}_j$ (where K_j is its index set), such that $0\in K_j$ and $\phi^0_j := f^{\mathfrak{C}}_j$ for all $j\in I$, then,

$$\left\{ \bigotimes_{j\in I}\phi^{\beta(j)}_j \;\middle|\; \beta\in\prod_{j\in I}K_j \quad \text{s.th.} \quad \beta(j)\neq 0 \quad \text{only for finitely many } j\in I \right\} \tag{A.27}$$

is an ONB of $\otimes^{\mathfrak{C}}_{j\in I}\mathcal{H}_j$.

(v) If $\mathcal{H}_j$ are all *separable* and I is *countable*, then $\otimes^{\mathfrak{C}}_{j\in I}\mathcal{H}_j$ is a *separable* Hilbert space $\forall\mathfrak{C}\in\Gamma$. ♦

Proof: In (von Neumann, 1939), see Lemma 4.1.1 for (i) and (ii), Lemmas 3.3.7 and 4.1.2 for (iii) and Lemma 4.1.4 and its subsequent discussion for (iv) and (v). ☐

With all, even if the full $\bigotimes_{j\in I}\mathcal{H}_j$ is non-separable, in many cases of interest (say, in the main text's case), each layer is a separable Hilbert space —such that certain things are simpler to be proven focusing first on the layers.

A.2.1 A Note on the Associativity of $\otimes$

As an example usage of this dissection technique, von Neumann studied the following associativity properties of the ITP. Given an index set I and a partition for it $(I_\alpha)_{\alpha\in P}$, one could either build $\otimes_{j\in I}\mathcal{H}_j$ directly or first build $\otimes_{j\in I_\alpha}\mathcal{H}_j$ for each $\alpha\in P$ and only at the end do $\otimes_{\alpha\in P}\Big(\otimes_{j\in I_\alpha}\mathcal{H}_j\Big)$. The question would then be: when are the results canonically identifiable?

Theorem 26. Given an arbitrary index set I and a partition $(I_\alpha)_{\alpha\in P}$ of I (i.e., a family of disjoint subsets of I such that $\cup_{\alpha\in P}I_\alpha = I$), consider a family of Hilbert spaces $(\mathcal{H}_j)_{j\in I}$.

- **(i)** $(f_j)_{j\in I}$ is a $\mathscr{C}$-sequence $\Longrightarrow$ for each $\alpha\in P$, $(f_j)_{j\in I_\alpha}$ and $(\bigotimes_{j\in I_\alpha}f_j)_{\alpha\in P}$ are $\mathscr{C}$-sequences.
- **(ii)** For any representative $(f_j)_{j\in I}$ of a fixed $\approx$-class $\mathfrak{C}$ in $\bigotimes_{j\in I}\mathcal{H}_j$, $(f_j)_{j\in I_\alpha}$ (resp. $(\bigotimes_{j\in I_\alpha}f_j)_{\alpha\in P}$) belongs to the same class $\mathfrak{C}^\alpha$ of $\bigotimes_{j\in I_\alpha}\mathcal{H}_j$ (resp. $\mathfrak{C}^0$ of $\otimes_{\alpha\in P}\Big(\otimes_{j\in I_\alpha}\mathcal{H}_j\Big)$). That is, each $\mathfrak{C}$ in $\bigotimes_{j\in I}\mathcal{H}_j$ determines unique $\approx$-equivalence classes $\mathfrak{C}^\alpha$ and $\mathfrak{C}^0$ in $\bigotimes_{j\in I_\alpha}\mathcal{H}_j$ and $\otimes_{\alpha\in P}\Big(\otimes_{j\in I_\alpha}\mathcal{H}_j\Big)$, respectively.
- **(iii)** For each $\mathfrak{C}\in\Gamma$, there *exists a unique* isomorphism (which is in particular *unitary*)

$$\begin{array}{rrcl} U: & \otimes^{\mathfrak{C}}_{j\in I}\mathcal{H}_j & \longrightarrow & \otimes^{\mathfrak{C}^0}_{\alpha\in P}\Big(\otimes^{\mathfrak{C}^\alpha}_{j\in I_\alpha}\mathcal{H}_j\Big) \\ \text{such that} & \bigotimes_{j\in I}f_j & \longmapsto & \bigotimes_{\alpha\in P}\Big(\bigotimes_{j\in I_\alpha}f_j\Big). \end{array} \tag{A.28}$$

- **(iv)** If there are *finitely many* "boxes" in the partition, i.e., $|P|<+\infty$, then the isomorphism of (iii) can be extended to a unique isomorphism $\mathscr{U}:\bigotimes_{j\in I}\mathcal{H}_j\longrightarrow\bigotimes_{\alpha\in P}\Big(\bigotimes_{j\in I_\alpha}\mathcal{H}_j\Big)$.
- **(v)** If $|P|=+\infty$, in general, the "associative law" given by item (iii) *cannot* be extended to an isomorphism $\mathscr{U}:\bigotimes_{j\in I}\mathcal{H}_j\longrightarrow\bigotimes_{\alpha\in P}\Big(\bigotimes_{j\in I_\alpha}\mathcal{H}_j\Big)$. ♦

Proof: See Theorems VI and VII in (von Neumann, 1939) for items (i),(ii), (iii) and (iv).

Due to its simplicity (and hence, its drama) we now show a counterexample by von Neumann (1939) that exemplifies (v). Let $I=\mathbb{N}$. We partition it in couples: $I_1=\{1,2\}$, $I_2=\{3,4\},\ldots$ For each $j\in\mathbb{N}$ choose some $f_j\in\mathcal{H}_j$ such that $\|f_j\|=1$ and define $g_j:=-f_j$. Then, $(f_j)_{j\in I},(g_j)_{j\in I}\in\mathscr{C}_0$ and

$$\sum_{j\in\mathbb{N}}|\langle f_j,g_j\rangle-1|=\sum_{j\in I}2 \quad\Longrightarrow\quad (f_j)_{j\in\mathbb{N}}\not\approx(-f_j)_{j\in\mathbb{N}}. \tag{A.29}$$

Hence, by Thm. 23, $\bigotimes_{j\in\mathbb{N}}f_j\perp\bigotimes_{j\in\mathbb{N}}(-f_j)$, i.e., $\otimes_{j\in\mathbb{N}}f_j\perp\otimes_{j\in\mathbb{N}}g_j$.

However, for each $\alpha\in\mathbb{N}$, $\bigotimes_{j\in I_\alpha}f_j=f_{2\alpha-1}\otimes f_{2\alpha}$ and

$$\bigotimes_{j\in I_\alpha}g_j=g_{2\alpha-1}\otimes g_{2\alpha}=(-f_{2\alpha-1})\otimes(-f_{2\alpha})=f_{2\alpha-1}\otimes f_{2\alpha}=\bigotimes_{j\in I_\alpha}f_j.$$

So, $\otimes_{j\in I_\alpha} f_j = \otimes_{j\in I_\alpha} g_j$ and thus,

$$\bigotimes_{\alpha\in\mathbb{N}}\Big(\bigotimes_{\alpha\in I_\alpha} g_\alpha\Big) = \bigotimes_{\alpha\in\mathbb{N}}\Big(\bigotimes_{\alpha\in I_\alpha} f_\alpha\Big).$$

But then, as a function of the association we choose to make, we can cause the resulting elementary tensor products to be either orthogonal or be the same element!

o.ε.δ.

Luckily, this is no issue in the context of QM because there (say, in the standard model of particle physics) one only considers finitely many different species of particles. As such, by virtue of Theorem 26, it is the same to

- study the ITP space of each species separately and then their composite system,
- or to consider them altogether in the same ITP from the very beginning.

A.2.2 A Coarser Grained Dissection: the "Quasi" Equivalence Relation

For any $\bigotimes_{j\in I} f_j \neq 0$, even when $\|f_j\| = 1$, the vector $\bigotimes_{j\in I}(-f_j)$ is orthogonal to it (as soon as) I is infinite, because:

$$\Big\langle \bigotimes_{j\in I} f_j,\ \bigotimes_{j\in I}(-f_j)\Big\rangle =^q \prod_{j\in I}\langle f_j,\ -f_j\rangle =^q \prod_{j\in I}(-1),$$

which is quasi-convergent but not convergent —so yields 0. The same happens whenever we take $\bigotimes_{j\in I}(e^{i\theta_j} f_j)$ such that $\prod_{j\in I} e^{i\theta_j}$ does not converge. So, even if factor-wise the two vectors seem to be proportional to each other, overall they end up being even orthogonal. In general:

Proposition 56. Let $(z_j)_{j\in I} \subseteq \mathbb{C}\setminus\{0\}$ with $\prod_{j\in I}|z_j|$ existent and let $\bigotimes_{j\in I} f_j \neq 0$.

(i) If $\prod_{j\in I}|z_j| \neq 0$ then $(f_j)_{j\in I},\ (z_j f_j)_{j\in I} \in \mathscr{C}_0$ and

$$\begin{cases} \bullet\ \textit{either :} & \displaystyle\bigotimes_{j\in I}\Big(z_j f_j\Big) = \Big(\prod_{j\in I} z_j\Big)\cdot\bigotimes_{j\in I} f_j \iff \Big(\prod_{j\in I} e^{i\theta_j}\quad \text{exists}\Big) \\ \bullet\ \textit{or :} & \displaystyle\bigotimes_{j\in I}\Big(z_j f_j\Big) \perp \bigotimes_{j\in I} f_j \iff \Big(z_j f_j\Big)_{j\in I} \not\approx \Big(f_j\Big)_{j\in I} \end{cases}.$$

(ii) If $\prod_{j\in I}|z_j| = 0$, then $(z_j f_j)_{j\in I} \in \mathscr{C}$ and $\bigotimes_{j\in I}\Big(z_j f_j\Big) = 0$. ♦

Proof: • **Item (i):** By Prop. 44, $\sum_{j\in I}\big||z_j|-1\big|$ exists, and by Lemma 36.(ii) $(f_j)_{j\in I} \in \mathscr{C}_0$ because $\bigotimes_{j\in I} f_j \neq 0$. Hence, by Lemma 3.3.6.(ii) in (von Neumann, 1939) $(z_j f_j)_{j\in I} \in \mathscr{C}_0$. Now, either $\prod_{j\in I} z_j$ exists or it does not. By Lemma 3.3.6.(iii) in (von Neumann, 1939), given our hypotheses, $\bigotimes_{j\in I}\Big(z_j f_j\Big) = \Big(\prod_{j\in I} z_j\Big)\cdot\bigotimes_{j\in I} f_j$ holds *if and only if* $\prod_{j\in I} z_j$ exists. On the other hand, by Lemma 3.3.6.(iv) in (von Neumann, 1939), in our situation, $(f_j)_{j\in I} \approx (z_j f_j)_{j\in I}$ *if and only if* $\sum_{j\in I}|z_j - 1|$ exists. By Cor. 19 the latter holds *if and only if* $\prod_{j\in I} z_j$ exists (and is $\neq 0$ —which in our case would be guaranteed by $\prod_{j\in I}|z_j| \neq 0$). Hence, in our setting, $\prod_{j\in I} z_j$ does not exist *if and only if* $(z_j f_j)_j \not\approx (f_j)_j$. This proves that

$$\begin{cases} \bullet\ \textit{either :} & \displaystyle\bigotimes_{j\in I}\Big(z_j f_j\Big) = \Big(\prod_{j\in I} z_j\Big)\cdot\bigotimes_{j\in I} f_j \iff \Big(\prod_{j\in I} z_j\quad \text{exists}\Big) \\ \bullet\ \textit{or :} & \displaystyle\Big(z_j f_j\Big) \not\approx \Big(f_j\Big)_{j\in I} \overset{(\text{Thm. }23)}{\Longrightarrow} \bigotimes_{j\in I}\Big(z_j f_j\Big) \perp \bigotimes_{j\in I} f_j \end{cases}. \qquad (A.30)$$

Now, by Prop. 45, $\sum_{j\in I} |arg(z_j)|$ being convergent is necessary and sufficient for $\prod_{j\in I} z_j$ to converge because by hypothesis, $\prod_{j\in I} |z_j| \in (0, +\infty)$. But, by Example 3, for that, the convergence of $\prod_{j\in I} e^{i\theta_j}$ is necessary and sufficient. This proves the "either" of the claim.

Finally, given $\otimes_{j\in I}(z_j f_j) \perp \otimes_{j\in I} f_j$, assume that $\prod_{j\in I} z_j$ exists. Then, by (A.30), $\otimes_{j\in I}(z_j f_j) = (\prod_{j\in I} z_j) \cdot \otimes_{j\in I} f_j$, which implies that $(\prod_{j\in I} z_j) \cdot \otimes_{j\in I} f_j = 0$ (since, the only vector orthogonal to itself is the 0 vector). But then, given that by hypothesis, $\prod_{j\in I} |z_j| \neq 0$, $|\prod_{j\in I} z_j| \neq 0$ and thus, $\prod_{j\in I} z_j \neq 0$. This, together with the hypothesis that $\otimes_{j\in I} f_j \neq 0$ would imply that $(\prod_{j\in I} z_j) \cdot \otimes_{j\in I} f_j \neq 0$. Absurd! Therefore, it must be that $\otimes_{j\in I}(z_j f_j) \perp \otimes_{j\in I} f_j$ implies the non-existence of $\prod_{j\in I} z_j$, which by (A.30) implies that $(z_j f_j)_{j\in I} \not\approx (f_j)_{j\in I}$. This ends the proof of the "or" case.

- **Item (ii):** By Lemma 3.3.6.(i) in (von Neumann, 1939), since $(f_j) \in \mathscr{C}$, then $(z_j f_j)_{j\in I} \in \mathscr{C}$. By Prop. 45, $\prod_{j\in I} |z_j| = 0$ implies that $\prod_{j\in I} z_j = 0$. Hence, $(z_j)_{j\in I}$ has a convergent product and as such, by Lemma 3.3.6.(iii) in (von Neumann, 1939), $\bigotimes_{j\in I} \Big(z_j f_j\Big) = \Big(\prod_{j\in I} z_j\Big) \cdot \bigotimes_{j\in I} f_j = 0$. ***o.ε.δ.***

Now, if we employ $\bigotimes_{j\in I} \mathcal{H}_j$ for QM, one could desire that the unit vectors $\otimes_{j\in I} \psi_j$ and $\otimes_{j\in I}(e^{i\theta_j}\psi_j)$ represent the same physical state irrespective of $(\theta_j)_{j\in\mathbb{N}}$. Prop. 56 shows that such vectors are orthogonal whenever $\prod_{j\in I} e^{i\theta_j}$ does not converge. And yet, it turns out that the ITP accepts the desired identification, as we proceed to see now. Furthermore, the key equivalence relation for the identification, $\overset{q}{\approx}$, will turn out to explain when quasi-convergence is needed and when not. Moreover, $\overset{q}{\approx}$ will also emerge naturally from within the structure: the operator algebra generated by lifts from the factor spaces will be remarkably simplified by the sectorization associated to $\overset{q}{\approx}$.

Proposition 57. Given $(f_j)_{j\in I}, (g_j)_{j\in I} \in \mathscr{C}_0$, the relation

$$(f_j)_{j\in I} \overset{q}{\approx} (g_j)_{j\in I} \quad :\Longleftrightarrow \quad \left(\quad \sum_{j\in I} \Big| \big|\langle f_j,\ g_j\rangle\big| - 1 \Big| \quad \text{exists} \quad \right) \tag{A.31}$$

is an *equivalence relation*. We call it *quasi-equivalence* or *q-equivalence*,[17] and we denote its family of equivalence classes by Γ^q. Alternatively,

$$(f_j)_{j\in I} \overset{q}{\approx} (g_j)_{j\in I} \qquad \Longleftrightarrow \qquad \left(\quad \exists (\theta_j)_{j\in I} \subseteq [-\pi, \pi) :\ (f_j)_{j\in I} \approx (e^{i\theta_j}\, g_j)_{j\in I} \quad \right). \tag{A.32}$$

(Informally, "$(f_j)_{j\in I} \overset{q}{\approx} (g_j)_{j\in I}$ *when up to a factor-wise global phase they are in the same layer".*) ♦

Proof: See Lemmas 6.1.1, 6.1.2 and 6.1.3 in (von Neumann, 1939) ☐

It turns out that the splitting into $\overset{q}{\approx}$-classes is a coarser dissection than that of the $\approx$-equivalence relation. Namely, each $\overset{q}{\approx}$-class is exactly partitionable into entire $\approx$-classes:

Lemma 40. Given $(f_j)_{j\in I}, (g_j)_{j\in I} \in \mathscr{C}_0$,

$$(f_j)_{j\in I} \approx (g_j)_{j\in I} \qquad \Longrightarrow \qquad (f_j)_{j\in I} \overset{q}{\approx} (g_j)_{j\in I}.$$

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[17] Note that von Neumann calls it *almost-equivalence*. We decided to change it to emphasize its relationship with the *quasi*-convergence and the potential application within *quantum* mechanics.

- Hence, for each $\mathfrak{C}\in\Gamma$, there exists a unique $\mathfrak{C}^q\in\Gamma^q$ such that $\mathfrak{C}\subseteq\mathfrak{C}^q$. In particular, for all $\mathfrak{C}^q\in\Gamma^q$,
$$\mathfrak{C}^q = \bigsqcup_{\mathfrak{C}\in\Gamma:\ \mathfrak{C}\subseteq\mathfrak{C}^q}\mathfrak{C}. \tag{A.33}$$
- Then, defining the $\mathfrak{C}^q$*-th quasi-layer of the ITP* to be
$$\otimes^{\mathfrak{C}^q}_{j\in I}\mathcal{H}_j := \overline{\operatorname{span}\Big\{\bigotimes_{j\in I} f_j \;\Big|\; (f_j)_{j\in I}\in\mathfrak{C}^q\Big\}}, \tag{A.34}$$

one gets that $\otimes^{\mathfrak{C}^q}_{j\in I}\mathcal{H}_j \perp \otimes^{\mathfrak{N}^q}_{j\in I}\mathcal{H}_j$ for all $\mathfrak{C}^q,\mathfrak{N}^q\in\Gamma^q$ with $\mathfrak{C}^q\neq\mathfrak{N}^q$. As such,
$$\bigotimes_{j\in I}\mathcal{H}_j = \overline{\operatorname{span}\Big\{\bigcup_{\mathfrak{C}^q\in\Gamma^q}\otimes^{\mathfrak{C}^q}_{j\in I}\mathcal{H}_j\Big\}} \quad\text{and}\quad \otimes^{\mathfrak{C}^q}_{j\in I}\mathcal{H}_j = \overline{\operatorname{span}\Big\{\bigcup_{\mathfrak{C}\in\Gamma\ :\ \mathfrak{C}\subseteq\mathfrak{C}^q}\otimes^{\mathfrak{C}}_{j\in I}\mathcal{H}_j\Big\}}$$
for each $\mathfrak{C}^q\in\Gamma^q$. Therefore, by Proposition 15,
$$\bigotimes_{j\in I}\mathcal{H}_j = \bigoplus_{\mathfrak{C}^q\in\Gamma^q}\Big(\otimes^{\mathfrak{C}^q}_{j\in I}\mathcal{H}_j\Big) = \bigoplus_{\mathfrak{C}^q\in\Gamma^q}\Bigg(\bigoplus_{\mathfrak{C}\in\Gamma:\ \mathfrak{C}\subseteq\mathfrak{C}^q}\Big(\otimes^{\mathfrak{C}}_{j\in I}\mathcal{H}_j\Big)\Bigg). \tag{A.35}$$

♦

Proof: See Definitions 6.1.2, 6.1.3 and the discussion in between in (von Neumann, 1939).□

Remarkably, it turns out that the concept of *quasi-convergence* is only required to explain the layers within the same quasi-layers.

Proposition 58. Let I be infinite and let $\otimes_{j\in I}f_j,\ \otimes_{j\in I}g_j\in\otimes_{j\in I}\mathcal{H}_j$. Then,

(i)
$$(f_j)_{j\in I}\overset{q}{\not\approx}(g_j)_{j\in I} \quad\Longrightarrow\quad \Big(\prod_{j\in I}\langle f_j,g_j\rangle \text{ converges to } 0.\Big)$$

(ii)
$$(f_j)_{j\in I}\overset{q}{\approx}(g_j)_{j\in I}\Longrightarrow\begin{cases}\textbf{(a)}\text{ if }\ (f_j)_{j\in I}\approx(g_j)_{j\in I} & \Longrightarrow\Big(\prod_{j\in I}\langle f_j,g_j\rangle\text{ converges.}\Big)\\[2ex] \textbf{(b)}\text{ if }\ (f_j)_{j\in I}\not\approx(g_j)_{j\in I} & \Longrightarrow\left(\begin{array}{c}\prod_{j\in I}|\langle f_j,g_j\rangle|\text{ converges \&}\\ \prod_{j\in I}e^{i\arg(\langle f_j,g_j\rangle)}\ \textit{does not converge}\end{array}\right)\\[2ex] & \Longrightarrow\left(\begin{array}{c}\text{unless }\langle f_j,g_j\rangle=0\text{ for some }j\in I,\\ \prod_{j\in I}\langle f_j,g_j\rangle\ \textit{does not converge.}\end{array}\right)\end{cases}$$

♦

Proof: **Item (i):** If $f_j=0$ or $g_j=0$ for some $j\in I$, then, $\langle f_j,g_j\rangle=0$ and trivially, $\prod_{j\in I}\langle f_j,g_j\rangle=0$. Hence, assume $g_j,f_j\neq0\ \forall j\in I$. By definition, $(f_j)_{j\in I}\overset{q}{\not\approx}(g_j)_{j\in I}$ implies that $\sum_{j\in I}||\langle f_j,g_j\rangle|-1|$ does not converge. But then, by Prop. 44, either $\prod_{j\in I}|\langle f_j,g_j\rangle|$ does not exist or $\prod_{j\in I}|\langle f_j,g_j\rangle|=0$. The first option is not possible due to Lemma 3.1.1 in (von Neumann, 1939), so it must be that $\prod_{j\in I}|\langle f_j,g_j\rangle|=0$. But then, by Prop. 45, $\prod_{j\in I}\langle f_j,g_j\rangle=0$.

Item (ii).(a): If $(f_j)_{j\in I}\approx(g_j)_{j\in I}$, by Thm. 23.(iii), $\prod_{j\in I}\langle f_j,g_j\rangle$ converges.

Item (ii).(b): It cannot be that $\langle f_j,g_j\rangle=0$ for infinitely many $j\in I$: else, $\sum_{j\in I}||\langle f_j,g_j\rangle|-1|$ would not exist, violating the hypothesis $(f_j)_{j\in I}\overset{q}{\approx}(g_j)_{j\in I}$. As such, the set $K:=\{k\in I\mid\langle f_k,g_k\rangle\neq0\}$ differs from I in finitely many elements (so, in particular, K is infinite). But then, the divergence of $\sum_{j\in I}|\langle f_j,g_j\rangle-1|$ (due to $(f_j)_{j\in I}\not\approx(g_j)_{j\in I}$) implies that of $\sum_{j\in K}|\langle f_j,g_j\rangle-1|$, and thus, $(f_j)_{j\in K}\not\approx(g_j)_{j\in K}$. Similarly, $(f_j)_{j\in I}\overset{q}{\approx}(g_j)_{j\in I}$ gives $(f_j)_{j\in K}\overset{q}{\approx}(g_j)_{j\in K}$. Now, $(f_j)_{j\in K}\not\approx(g_j)_{j\in K}$ implies, by Thm. 23.(iii), that either $\prod_{j\in K}\langle f_j,g_j\rangle=0$

or that the product does not exist. Let us prove that the first case is impossible: since $(f_j)_{j\in K} \overset{q}{\approx} (g_j)_{j\in K}$, $\sum_{j\in K} ||\langle f_j, g_j\rangle| - 1|$ converges, but then, by definition of K and Prop. 44, $\prod_{j\in K} |\langle f_j, g_j\rangle|$ converges and it is non-zero. Therefore, it must be that $\prod_{j\in K}\langle f_j, g_j\rangle$ does not converge. But by Prop. 45, this can only be if $\sum_{j\in K} |arg(\langle f_j, g_j\rangle)|$ does not converge, implying by Example 3 that $\prod_{j\in K} e^{i\arg(\langle f_j,g_j\rangle)}$ does not converge.

Because I just differs in finitely many elements from K, it must be that $\prod_{j\in I} |\langle f_j, g_j\rangle|$ also converges and that $\prod_{j\in I} e^{i\arg(\langle f_j,g_j\rangle)}$ does not converge either. Moreover, we proved that if $I = K$, then, $\prod_{j\in I}\langle f_j, g_j\rangle$ does not converge, so, unless $K \neq I$ (i.e., unless $\exists j \in I : \langle f_j, g_j\rangle = 0$), $\prod_{j\in I}\langle f_j, g_j\rangle$ does not converge. ***o.ε.δ.***

Now, since in QM one often identifies the vectors that differ by a "global phase", one might want to go a step further and identify the elements $(f_j)_{j\in I}, (g_j)_{j\in I}$ within the same $\mathfrak{C}^q \in \Gamma^q$, for which there exist $(\theta_j)_{j\in I} \subseteq [-\pi, \pi)$ such that $g_j = e^{i\theta_j} f_j$ for all $j \in I$ (namely, as in $\overset{q}{\approx}$ but with an actual equality). What follows is a proper realization of this idea.

Proposition 59. Given $\theta := (\theta_j)_{j\in I} \subseteq [-\pi, \pi)$ there *exists a unique* linear operator $U_\theta : \bigotimes_{j\in I} \mathcal{H}_j \to \bigotimes_{j\in I} \mathcal{H}_j$ such that

$$U_\theta\Big(\bigotimes_{j\in I} f_j\Big) = \bigotimes_{j\in I} \Big(e^{i\theta} f_j\Big), \qquad \text{for all } (f_j)_{j\in I} \in \mathscr{C}_0. \tag{A.36}$$

Such a U_θ is in particular a *unitary* operator. Moreover, given that $P_{\mathfrak{C}}$ and $P_{\mathfrak{C}^q}$ denote the orthogonal projectors onto the $\mathfrak{C}$-th layer and $\mathfrak{C}^q$-th quasi-layer (i.e., onto $\otimes^{\mathfrak{C}}_{j\in I}\mathcal{H}_j$ and $\otimes^{\mathfrak{C}^q}_{j\in I}\mathcal{H}_j$ respectively), U_θ satisfies the following properties:

- **(i)** $P_{\mathfrak{C}^q} U_\theta = U_\theta P_{\mathfrak{C}^q}$ for all $\mathfrak{C}^q \in \Gamma^q$, namely, each quasi-layer reduces U_θ and in particular,

$$U_\theta\Big(\otimes^{\mathfrak{C}^q}_{j\in I}\mathcal{H}_j\Big) = \otimes^{\mathfrak{C}^q}_{j\in I}\mathcal{H}_j \qquad \forall \mathfrak{C}^q \in \Gamma^q. \tag{A.37}$$

- **(ii)** For each $\mathfrak{C} \in \Gamma$ it holds that

$$\Bigg(\quad P_{\mathfrak{C}} U_\theta = U_\theta P_{\mathfrak{C}} \quad \text{s.th.,} \quad U_\theta\Big(\otimes^{\mathfrak{C}}_{j\in I}\mathcal{H}_j\Big) = \otimes^{\mathfrak{C}}_{j\in I}\mathcal{H}_j \quad \Bigg) \iff \Bigg(\quad \prod_{j\in I} e^{i\theta_j} \quad \text{exists} \quad \Bigg),$$

  and in such a case, $U_\theta = (\prod_{j\in I} e^{i\theta_j}) \cdot Id$.

- **(iii)** Moreover, if the condition in (ii) does not hold, then U_θ maps $\otimes^{\mathfrak{C}}_{j\in I}\mathcal{H}_j$ bijectively into a different layer $\otimes^{\mathfrak{N}}_{j\in I}\mathcal{H}_j$. In particular, $\mathfrak{N} \in \Gamma$ ($\mathfrak{N} \neq \mathfrak{C}$), is the class where $(e^{i\theta} f_j)_{j\in I}$ belongs to for one (and hence any) $(f_j)_{j\in I} \in \mathfrak{C}$. As such, $\mathfrak{N}$ is fully determined by $\mathfrak{C}$ and θ. ♦

Proof: In (von Neumann, 1939), see Lemma 6.2.1 for existence and uniqueness, Lemma 6.2.2 for (i) and Lemma 6.2.3 (and its proof) for (ii) and (iii). □

Lemma 41. Given $\theta := (\theta_j)_{j\in I}$, $\omega := (\omega_j)_{j\in I} \subseteq [-\pi, \pi)$, denoting $-\theta := (-\theta_j)_{j\in I}$ and $\theta + \omega := (\theta_j + \omega_j)_{j\in I}$,

$$\text{(i)} \quad U_{-\theta} = (U_\theta)^{-1} \;, \qquad\qquad \text{(ii)} \quad U_\theta \circ U_\omega = U_{\theta+\omega} \;.$$

♦

Proof: Item (i): By Prop. 59, for all $(f_j)_{j\in I} \in \mathscr{C}$, $U_\theta \circ U_{-\theta}(\otimes_{j\in I} f_j) = U_\theta\Big(\otimes_{j\in I} (e^{-i\theta_j} f_j)\Big) = \otimes_{j\in I} f_j$. Hence, restricted to the elementary tensor products and their span, $U_\theta \circ U_{-\theta} = Id$. But by construction, the span of elementary tensors is dense in $\bigotimes_{j\in I} \mathcal{H}_j$. Hence, $U_\theta \circ U_{-\theta}$ and Id must agree everywhere (by Lemma 32).

- Item (ii): for all $(f_j)_{j\in I} \in \mathscr{C}$, $U_\theta \circ U_\omega(\otimes_{j\in I} f_j) = U_\theta(\otimes_{j\in I} (e^{i\omega_j} f_j)) = \otimes_{j\in I}(e^{i(\omega_j+\theta_j)} f_j) = U_{\theta+\omega}(\otimes_{j\in I} f_j)$. The rest follows by Lemma 32 as in (i). ***o.ε.δ.***

If $\theta = (0)_{j\in I}$, $U_\theta = Id$, so this suggests that $\mathscr{U} := \{ \, U_\theta \mid \theta \in \prod_{j\in I} \mathbb{S}^1 \, \}$ is an I-parameter abelian group of unitaries and a unitary representation of the group $\prod_{j\in I} \mathbb{S}^1$ (under addition). Note that it can happen that for $\theta \neq \omega$, $U_\theta = U_\omega$, so, one would need to take especial care. Be that as it may, one can do the following.

Definition 40. For each $\Psi \in \bigotimes_{j\in I} \mathcal{H}_j$ we define its *quasi-global phase sphere* as

$$S[\Psi] := \Big\{ \, U_\theta \Psi \, \Big| \, \theta = (\theta_j)_{j\in I} \subseteq [-\pi, \pi) \, \Big\}, \tag{A.38}$$

and we define the *quasi-ray along* Ψ as

$$R[\Psi] := \overline{\operatorname{span}\Big\{ \, U_\theta \Psi \, \Big| \, \theta = (\theta_j)_{j\in I} \subseteq [-\pi, \pi) \, \Big\}} = \overline{\operatorname{span} S[\Psi]}. \tag{A.39}$$

♦

With that, we can culminate our desire to identify vectors that differ "by a global phase" (in the generalized sense mentioned above) by noting that:

Proposition 60. Given $\Psi, \Phi \in \bigotimes_{j\in I} \mathcal{H}_j$, the relation

$$\Phi \overset{gl.ph.}{=} \Psi \qquad :\Longleftrightarrow \quad \Big(\quad \Phi \in S[\Psi] \quad \Big) \tag{A.40}$$

is an *equivalence relation.* We call it *equality up to a quasi-global phase.* ♦

Proof: • It is *reflexive* because setting $\theta = (0)_{j\in I}$, $U_\theta = Id$ and thus, $\forall \Psi \in \bigotimes_{j\in I} \mathcal{H}_j$, $U_\theta \Psi = \Psi$. It is *symmetric* because given $\Psi \in S[\Phi]$, there exists θ such that $U_\theta \Phi = \Psi$, but then, by Lemma 41, $\Phi = U_{-\theta} \Psi$ and hence $\Phi \in S[\Psi]$. It is *transitive* because given $\Phi, \Psi, \Xi \in \bigotimes_{j\in I} \mathcal{H}_j$ such that $\Phi \in S[\Psi]$ and $\Psi \in S[\Xi]$, there exist θ, ω such that $U_\theta \Phi = \Psi$ and $U_\omega \Psi = \Xi$. Hence, $U_\omega \circ U_\theta \, \Phi = \Xi$. But then, by Lemma 41, $U_\eta \Phi = \Xi$ for $\eta = (\theta_j + \omega_j)_{j\in I}$ and thus, $\Xi \in S[\Phi]$. ***o.ε.δ.***

There are many things that demand more research here. Firstly, given $\mathscr{U} := \{ \, U_\theta \mid \theta \in \prod_{j\in I} \mathbb{S}^1 \, \}$, what we defined as $R[\Psi]$ is the smallest closed $\mathscr{U}$-invariant subspace of $\bigotimes_{j\in I} \mathcal{H}_j$ that contains Ψ. But then, there are questions one would like to answer, such as whether $\Phi \in R[\Psi]$ implies $R[\Phi] = R[\Psi]$ —which could allow to define a projective Hilbert space where each ray$\setminus\{0\}$ of $\bigotimes_{j\in I} \mathcal{H}_j$ is a different class. Another interesting question would be, noting that $\mathscr{U}$ leaves each $\mathfrak{C}^q \in \Gamma^q$ layer invariant but relates all $\mathfrak{C} \subseteq \mathfrak{C}^q$, $\mathfrak{C} \in \Gamma$, whether one can reasonably "thin out" each $\otimes^{\mathfrak{C}^q}_{j\in I} \mathcal{H}_j$ into a single "representative" $\otimes^{\mathfrak{C}}_{j\in I} \mathcal{H}_j$ with $\mathfrak{C} \subseteq \mathfrak{C}^q$. If so, we would erase quasi-convergence from the structure. In the direction of these questions, von Neumann already gave us some results:

Proposition 61. Given an arbitrary I and a family of Hilbert spaces $(\mathcal{H}_j)_{j\in I}$, denote by $P_{\mathfrak{C}}$ and $P_{\mathfrak{C}^q}$ respectively, the orthogonal projectors onto the $\mathfrak{C}$-th layer and $\mathfrak{C}^q$-th quasi-layer ($\otimes^{\mathfrak{C}}_{j\in I} \mathcal{H}_j$ and $\otimes^{\mathfrak{C}^q}_{j\in I} \mathcal{H}_j$). Let U_θ denote the unitary of Prop. 59 for a given $\theta \in \prod_{j\in I}[-\pi, \pi)$ and for an arbitrary $\Psi \in \bigotimes_{j\in I} \mathcal{H}_j$, denote by E_Ψ the orthogonal projector onto $R[\Psi]$.

(i) If $\Psi \in \otimes^{\mathfrak{C}}_{j\in I}\mathcal{H}_j$, then E_Ψ commutes with $P_{\mathfrak{C}}$ and their product is the orthogonal projector to $\operatorname{span}(\Psi)$.

(ii) If $\Psi \in \otimes^{\mathfrak{C}^q}_{j\in I}\mathcal{H}_j$, then $E[\Psi] \leq P_{\mathfrak{C}^q}$.

(iii) Every E_Ψ commutes with U_θ. ♦

Proof: See Lemma 6.3.2 in (von Neumann, 1939). □

We will study the questions prompted above in future work —although it might be the case that they have already been studied in the literature. After all, one can find an expression like the following one in a recent textbook that mentions that mentions the ITP:

> *"Each weak* [for us $\overset{q}{\approx}$-] *equivalence class can be further decomposed into mutually orthogonal strong* [namely, $\approx$-] *equivalence classes. Since the latter differ only by phase factors within a given weak equivalence class, they contain the same physical information."* (Thirring, 1979)

A.3 The Quasi-Local Bounded Operators

It is a natural question whether one can take an operator from one of the factor Hilbert spaces and "putting identities" in the other factors, lift it to "its version" acting on the ITP. In the main text, we gave some hints to answer this for unbounded operators (and we will provide several more in Appendix B), but for bounded operators von Neumann already gave an exhaustive characterization in 1939. Such a characterization reveals some sort of redundancy in the structure of the ITP that could be attributed to the notion of *quasi convergence.* (Note that as usual, we will denote by $\mathcal{L}(\mathcal{H})$ the space of bounded operators on the Hilbert space $\mathcal{H}$.)

Proposition 62. For each fixed $j_0 \in I$ and $A_{j_0} \in \mathcal{L}(\mathcal{H}_j)$, there *exists a unique* $\widehat{A_{j_0}} \in \mathcal{L}(\bigotimes_{j\in I}\mathcal{H}_j)$ such that

$$\widehat{A_{j_0}}\Big(\bigotimes_{j\in I} f_j\Big) = \bigotimes_{j\in I\setminus\{j_0\}} f_j \otimes \Big(A_{j_0} f_{j_0}\Big) \qquad \forall (f_j)_{j\in I} \in \mathscr{C}. \tag{A.41}$$

We call it the *lift of* A_{j_0} *to* $\bigotimes_{j\in I}\mathcal{H}_j$ and $\left\|\widehat{A_{j_0}}\right\|_{op} = \|A_{j_0}\|_{op}$. In the two appendices, when applied on elements of $\mathcal{L}(\mathcal{H}_{j_0})$ the "widehat" $\widehat{\cdot}$ will denote the mapping $\widehat{\cdot} : A_{j_0} \in \mathcal{L}(\mathcal{H}_{j_0}) \longmapsto \widehat{A_{j_0}} \in \mathcal{L}(\bigotimes_{j\in I}\mathcal{H}_j)$. In particular, we will denote the set of lifts of bounded operators from the factor $\mathcal{H}_{j_0}$ as

$$\widehat{\mathcal{L}(\mathcal{H}_{j_0})} := \left\{ \widehat{A_{j_0}} \;\middle|\; A_{j_0} \in \mathcal{L}(\mathcal{H}_j) \right\} \subseteq \mathcal{L}(\bigotimes_{j\in I}\mathcal{H}_j). \tag{A.42}$$

♦

Proof: See Lemma 5.1.1 in (von Neumann, 1939). The claim about the operator norm is not explicitly given, but we can complete his proof as follows. von Neumann shows that for $C > 0$,

$$\Big(\; \|A_{j_0} f\| \leq C\|\psi\| \quad \forall f \in \mathcal{H}_{j_0} \; \Big) \Longrightarrow \Big(\; \left\|\widehat{A_{j_0}}\Psi\right\| \leq C\|\Psi\| \quad \forall \Psi \in D \; \Big)$$

where $D := \operatorname{span}\{\phi \otimes f \mid \phi \in K,\ f \in \mathcal{H}_{j_0}\}$ and K is an arbitrary ONB of $\bigotimes_{j\in I\setminus\{j_0\}}\mathcal{H}_j$. This proves that $\left\|\widehat{A_{j_0}}\restriction_D\right\|_{op} \leq \|A_{j_0}\|_{op}$. Now, by definition of operator norm, there exists a unit norm sequence $(f^n_{j_0})_{n\in\mathbb{N}} \subset \mathcal{H}_{j_0}$ realizing the bound $\|A_{j_0}\|_{op}$, i.e., such that $\lim_{n\to\infty}\left\|A_{j_0} f^n_{j_0}\right\| =$

$\|A_{j0}\|_{op}$. But then, for any $\phi \in K$, $\lim_{n\to\infty} \left\|\widehat{A_{j0}}(\phi \otimes f^n_{j0})\right\| = \lim_{n\to\infty} \left\|A_{j0}(f^n_{j0})\right\| = \|A_{j0}\|_{op}$. Hence, $\left\|\widehat{A_{j0}} \restriction_D\right\|_{op} = \|A_{j0}\|_{op}$. Since von Neumann proves that D is dense in $\bigotimes_{j\in I} \mathcal{H}_j$, by Thm. 8, the unique bounded extension of $\widehat{A_{j_0}} \restriction_D$, namely, $\widehat{A_{j_0}}$, must have operator norm $\|A_{j0}\|_{op}$. **ο.ε.δ.**

Proposition 63. The map $\widehat{\cdot} : \mathcal{L}(\mathcal{H}_{j_0}) \to \widehat{\mathcal{L}(\mathcal{H}_{j_0})}$ is a bijection that is isomorphic with respect to the $*$-algebra operations

$$\lambda\, A_{j_0} \text{ (scalar multipl.)}, \quad A_{j_0} + B_{j_0} \text{ (addit.)}, \quad A^*_{j_0} \text{ (adjoint)}, \quad A_{j_0} \circ B_{j_0} \text{ (composit.)},$$

for all $A_{j_0}, B_{j_0} \in \mathcal{L}(\mathcal{H}_{j_0})$ and $\lambda \in \mathbb{C}$. Moreover, it preserves the operator norms, so, it is a *C^*-algebra isomorphism.* In particular, it carries the zero and identity operators of the factor space to the zero and identity operators of the tensor product. ♦

Proof: See Lemma 5.1.1 in (von Neumann, 1939). For the isometry part see Proposition 62. ☐

Definition 41. A *W^*-algebra-*, a *von Neumann algebra-* or (in von Neumann's own terminology) a *ring- of operators $\mathcal{B}$* on a Hilbert space $\mathcal{H}$, is a vector subspace of $\mathcal{L}(\mathcal{H})$ that is closed under compositions (considered to be the multiplication of the algebra), adjoints (the * in the name) and is a closed set in the weak operator topology (hence the W). It is equivalently closed in the strong topology.[18] It is called *unital* if it contains a multiplicative identity (the identity operator). ♦

Proposition 64. For each fixed $j_0 \in I$,

(i) $\widehat{\mathcal{L}(\mathcal{H}_{j_0})}$ is a unital W^*-algebra.

(ii) Given a subset $R_{j_0} \subseteq \mathcal{L}(\mathcal{H}_{j_0})$, $\Big(R_{j_0} \text{ is a } W^*\text{-algebra}\Big) \iff \Big(\widehat{R_{j_0}} \text{ is a } W^*\text{-algebra.}\Big)$ ♦

Proof: In (von Neumann, 1939), see Lemmas 5.2.1 and 5.2.3 in for items (i) and (ii), resp. ☐

Hereafter, every W^*-algebra will be assumed to be unital.

In a sense, the lifts of bounded operators from the factors, describe the operators that are "directly related" to each factor —if, say, each factor Hilbert space represents a certain region of space, they would be the "local operators" acting in those regions. Hence, an interesting algebra of operators is that generated by taking compositions, linear combinations and limits of such local operators. We call them the "quasi-local operators" because they no longer need to act on finitely many factor spaces alone.

Definition 42. We denote by $\mathcal{L}^q(\bigotimes_{j\in I} \mathcal{H}_j)$ the smallest W^*-algebra containing all $\widehat{\mathcal{L}(\mathcal{H}_j)}$, $j \in I$, and call it the algebra of *quasi-local operators.* Equivalently, using Thm. 32,

$$\mathcal{L}^q\Big(\bigotimes_{j\in I} \mathcal{H}_j\Big) = \left(\bigcup_{j\in I} \widehat{\mathcal{L}(\mathcal{H}_j)}\right)''. \tag{A.43}$$

♦

One could expect that these operators make up all the bounded operators in $\bigotimes_{j\in I} \mathcal{H}_j$. But, for general I this is quite far from true, as the following shows.

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[18] See for instance §15.1 in (Arai, 2018).

Proposition 65. Given an arbitrary family of Hilbert spaces $(\mathcal{H}_j)_{j\in I}$,

$$\left(\quad \mathcal{L}^q\Big(\bigotimes_{j\in I}\mathcal{H}_j\Big) = \mathcal{L}\Big(\bigotimes_{j\in I}\mathcal{H}_j\Big) \quad \right) \iff \left(\quad |I| \quad \text{is finite} \quad \right).$$

Else, $\mathcal{L}^q\Big(\bigotimes_{j\in I}\mathcal{H}_j\Big) \subsetneqq \mathcal{L}\Big(\bigotimes_{j\in I}\mathcal{H}_j\Big)$. ♦

To prove this, we first need to understand the deep connection between $\approx, \overset{q}{\approx}$ and the quasi-local operators. von Neumann developed the following sequence of technical results that culminate in two big theorems exposing the connection.

Proposition 66. Following the notation in Proposition 61,

- **(i)** Each $A \in \mathcal{L}^q(\otimes_{j\in I}\mathcal{H}_j)$ commutes with every $P_{\mathfrak{C}}, P_{\mathfrak{C}^q}$ and U_θ.

  (Informally, *no quasi-local operator can take a vector out of its layer.*)

- **(ii)** Given $(f_j^o)_{j\in I} \in \mathfrak{C}$ with $\left\|f_j^o\right\| = 1$ (a generator of $\mathfrak{C}$) and an arbitrary $\Psi \in \otimes^{\mathfrak{C}}_{j\in I}\mathcal{H}_j$, there exists some $A \in \mathcal{L}^q(\otimes_{j\in I}\mathcal{H}_j)$ such that $A(\bigotimes_{j\in I} f_j^o) = \Psi$.

  (*Any vector in a layer can be reached from a generating vector by applying quasi-local operators.*)

- **(iii)** For each $\Psi \in \otimes^{\mathfrak{C}}_{j\in I}\mathcal{H}_j$, $E_\Psi \in \mathcal{L}^q(\otimes_{j\in I}\mathcal{H}_j)$. If we merely say $\Psi \in \bigotimes_{j\in I}\mathcal{H}_j$ (so, if we admit linear combinations of different layers), the claim is false.

  (*Elements that superpose different layers may have rays that get out of the scope of quasi-local operators.*)

- **(iv)** If an orthogonal projector Q commutes with $P_{\mathfrak{C}}$ $\forall \mathfrak{C} \in \Gamma$ and with U_θ $\forall\theta$, then $Q \in \mathcal{L}^q(\otimes_{j\in I}\mathcal{H}_j)$. ♦

Proof: In (von Neumann, 1939), see respectively Lemmas 6.3.1, 6.3.4, 6.3.5, 6.3.6 for items (i), (ii), (iii), (iv). □

Theorem 27. Given an arbitrary family of Hilbert spaces $(\mathcal{H}_j)_{j\in I}$ and an $A \in \mathcal{L}(\otimes_{j\in I}\mathcal{H}_j)$,

$$A \in \mathcal{L}^q\Big(\bigotimes_{j\in I}\mathcal{H}_j\Big) \iff \left(\quad A \ \text{ commutes with } \ P_{\mathfrak{C}} \ \text{ and } \ U_\theta \quad \forall \mathfrak{C} \in \Gamma, \ \text{ and } \ \forall \theta \in \prod_{j\in I}[-\pi,\pi) \quad \right).$$ ♦

Proof: See Theorem IX in (von Neumann, 1939).□

Hence, if we find a bounded operator that does not preserve the layers, we know immediately that it is *not* a quasi-local operator. As such, we already know plenty of examples of non-quasi-local bounded operators: all the U_θ for which $\prod_{j\in I} e^{i\theta_j}$ does not converge are of this kind.

Proof **of Prop. 65:** If $|I|$ is finite, that $\mathcal{L}^q(\otimes_{j\in I}\mathcal{H}_j) = \mathcal{L}(\bigotimes_{j\in I}\mathcal{H}_j)$ is given in page 135 of (Murray and Neumann, 1936).

Now, if $|I|$ is infinite, consider an arbitrary $\bigotimes_{j\in I} f_j \in \bigotimes_{j\in I}\mathcal{H}_j$ with $\|f_j\| = 1$. Then, the vector $\bigotimes_{j\in I}(-f_j)$ is in a different $\approx$-layer, $\bigotimes_{j\in I} f_j \not\approx \bigotimes_{j\in I}(-f_j)$, since $\sum_{j\in I} |\langle f_j, \, -f_j\rangle - 1| = \sum_{j\in I} 2$ (so it has no limit). But, the unitary $U_{(\pi)_{j\in I}}$ acts as: $U_{(\pi)_{j\in I}}(\bigotimes_{j\in I} f_j) = \bigotimes_{j\in I}(e^{i\pi} f_j) = \bigotimes_{j\in I}(-f_j)$. Hence, it cannot commute with the orthogonal projector to the layer of $\bigotimes_{j\in I} f_j$.

This implies by Theorem 27 that $U_{(\pi)_{j\in I}} \notin \mathcal{L}^q(\otimes_{j\in I}\mathcal{H}_j)$, despite it is certainly a bounded operator. ***o.ε.δ.***

So, the "extra" operators —which one could call "non-quasi-local"— are precisely those that cause "superpositions" and "transitions" of elements between different layers of the ITP —precisely the operators that Chapter 7 suggests to look at in order to model rigorous interacting QFTs.

Lastly, an impressive characterization of the quasi-local operators follows from Thm. 27:

Theorem 28. Let $(\mathcal{H}_j)_{j\in I}$ be an arbitrary family of Hilbert spaces $(\mathcal{H}_j)_{j\in I}$.

(i) Each $A \in \mathcal{L}^q(\otimes_{j\in I}\mathcal{H}_j)$ is *reduced* by every quasi-layer $\otimes_{j\in I}^{\mathfrak{C}^q}\mathcal{H}_j$, $\mathfrak{C}^q \in \Gamma^q$, such that

$$A = \bigoplus_{\mathfrak{C}^q\in\Gamma^q} A|_{\otimes_{j\in I}^{\mathfrak{C}^q}\mathcal{H}_j}. \tag{A.44}$$

(ii) Within each quasi-layer $\otimes_{j\in I}^{\mathfrak{C}^q}\mathcal{H}_j$, $\mathfrak{C}^q \in \Gamma^q$, each $A \in \mathcal{L}^q(\otimes_{j\in I}\mathcal{H}_j)$ is further *reduced* by the layers $\otimes_{j\in I}^{\mathfrak{C}}\mathcal{H}_j$, $\mathfrak{C} \in \Gamma$ contained in it, $\mathfrak{C} \subseteq \mathfrak{C}^q$. Hence, all,

$$A = \bigoplus_{\mathfrak{C}\in\Gamma} A|_{\otimes_{j\in I}^{\mathfrak{C}}\mathcal{H}_j}. \tag{A.45}$$

Most importantly: the action of A in *just one* $\approx$-layer $\otimes_{j\in I}^{\mathfrak{C}}\mathcal{H}_j$, $\mathfrak{C} \in \Gamma$, fully determines its action in any other layer $\mathfrak{N} \in \Gamma$ of the same $\overset{q}{\approx}$-class. That is, for each $\mathfrak{C}^q \in \Gamma^q$ and any $\mathfrak{C}, \mathfrak{N} \in \Gamma$ with $\mathfrak{C}, \mathfrak{N} \subseteq \mathfrak{C}^q$, $A|_{\mathfrak{N}}$ is determined by $A|_{\mathfrak{C}}$ (and vice-versa).

Stated more carefully: for each $\mathfrak{C}^q \in \Gamma^q$, select one $\mathfrak{C} \in \Gamma$ inside $\mathfrak{C}^q$ and denote it $\mathfrak{C}_{sel}(\mathfrak{C}^q)$. Then,

(iii) Consider for each $\mathfrak{C}^q \in \Gamma^q$ a bounded operator $A^{\mathfrak{C}^q} \in \mathcal{L}\big(\otimes_{j\in I}^{\mathfrak{C}_{sel}(\mathfrak{C}^q)}\mathcal{H}_j\big)$ acting on the selected $\mathfrak{C}_{sel}(\mathfrak{C}^q)$ layer. Then,

$$\left(\begin{array}{c} \exists A \in \mathcal{L}^q\Big(\bigotimes_{j\in I}\mathcal{H}_j\Big) : \\ A\Big|_{\otimes_{j\in I}^{\mathfrak{C}_{sel}(\mathfrak{C}^q)}\mathcal{H}_j} = A^{\mathfrak{C}^q} \quad \forall \mathfrak{C}^q \in \Gamma^q \end{array}\right) \iff \sup_{\mathfrak{C}^q\in\Gamma^q}\left\{\left\|A^{\mathfrak{C}_q}\right\|_{op}\right\} < +\infty,$$

and in such a case, A is *unique*.

(iv) For each $A \in \mathcal{L}^q\big(\bigotimes_{j\in I}\mathcal{H}_j\big)$,

$$\|A\|_{op} = \sup_{\mathfrak{C}\in\Gamma}\left\{\left\|A|_{\otimes_{j\in I}^{\mathfrak{C}}\mathcal{H}_j}\right\|_{op}\right\} = \sup_{\mathfrak{C}^q\in\Gamma^q}\left\{\left\|A|_{\otimes_{j\in I}^{\mathfrak{C}_{sel}(\mathfrak{C}^q)}\mathcal{H}_j}\right\|_{op}\right\}. \tag{A.46}$$

♦

Proof: See Theorem X in (von Neumann, 1939) and the remarks that follow it. □

This proves that those $\mathfrak{C} \in \Gamma$ contained in a common quasi-class $\mathfrak{C}^q \in \Gamma^q$ are somewhat redundant. Recall that these are precisely the $\mathfrak{C}$ layers that differ from each other by a "quasi-global phase" (see Prop. 60), namely, the layers $\mathfrak{C}$ related by those U_θ with non-convergent $\prod_{j\in I} e^{i\theta_j}$. That is, as proven in Prop. 58, the ultimate cause for the vectors in a same $\overset{q}{\approx}$-layer to be in different $\approx$-layers, and thus, for them to be orthogonal, was that their relative global phase $\prod_{j\in I} e^{i\theta_j}$ is not convergent. But then, in a sense, it is the definition of quasi-convergence what forces all such “redundant” $\mathfrak{C}$ inside the same $\mathfrak{C}^q$ to be different subspaces. As the promised clarification of the “ad-hoc-ness” of quasi-convergence, it is at this point where von Neumann says:

> “ *Now (II)* [for us, Theorem 28’s item (iii) and its preceding paragraph], *as indeed the entire difference between the subdivisions of* $\bigotimes_{j\in I}\mathcal{H}_j$ *into* $\otimes^{\mathfrak{C}}_{j\in I}\mathcal{H}_j$*’s resp.* $\otimes^{\mathfrak{C}^q}_{j\in I}\mathcal{H}_j$ *is ultimately due to our way of handling the non-convergent but quasi-convergent case in 2.5. (The* U_θ *which map an* $\mathfrak{C} \subset \mathfrak{C}^q$, $\mathfrak{C} \in \Gamma$, *on other* $\mathfrak{N} \subset \mathfrak{C}^q$, $\mathfrak{N} \in \Gamma$, *have non-convergent but quasi-convergent* $\prod_{j\in I} e^{i\theta_j}$ *cf. Lemma 6.2.3.). A more complicated procedure in dealing with such infinite products, using generalised-Banach-limits, would have permitted us to avoid this. Compared with our present method, however, it would have been highly artificial and arbitrary, and would have implied serious difficulties in the formulation of an associative law.”*
>
> (von Neumann, 1939)

A.4 How many Layers and Quasi-Layers are there?

Finally, after finding this apparent “redundancy” in the structure above, one might be tempted to think that this could resolve the non-separability of $\bigotimes_{j\in I}\mathcal{H}_j$, because say, if we choose to employ only one $\mathfrak{C}$-layer per quasi-layer-$\mathfrak{C}^q$, then the number of “genuine layers” could get reduced enough for that. Unfortunately, for any truly *infinite* tensor product, even when the factor spaces are finite dimensional, the number of $\mathfrak{C}^q \in \Gamma^q$ classes is not only infinite, but uncountably infinite! See (iii).b in what follows:

Proposition 67. Given a family of Hilbert spaces $(\mathcal{H}_j)_{j\in I}$,

- **(i)** If I is *finite*, say, if $I = \{j_1, ..., j_n\}$, then $|\Gamma| = |\Gamma^q| = 1$. Denote the unique $\approx$- and $\overset{q}{\approx}$-classes $\mathfrak{C}_0, \mathfrak{C}_0^q$. Then $\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n = \otimes_{j\in I}^{\mathfrak{C}_0^q}\mathcal{H}_j = \otimes_{j\in I}^{\mathfrak{C}_0^q}\mathcal{H}_j$. In particular, $U_\theta = (\prod_{j\in I} e^{i\theta_j})\cdot Id$, $\forall\theta \in \prod_{j\in I}(-\pi, \pi]$, and $P_{\mathfrak{C}_0} = Id = P_{\mathfrak{C}_0^q}$.
- **(ii)** If $|I|$ is *infinite*, then, for all $\mathfrak{C}^q \in \Gamma^q$,

$$\left|\ \left\{\mathfrak{C} \in \Gamma \ :\ \mathfrak{C} \subseteq \mathfrak{C}^q\ \right\}\ \right| = 2^{|I|}. \tag{A.47}$$

  (*So, even for* $\mathbb{C}\otimes\mathbb{C}\otimes\cdots$, *although dim*$(\otimes_{j\in I}^{\mathfrak{C}}\mathbb{C}) = 1$, *there are uncountably many* $\mathfrak{C}$*-layers.*)
- **(iii)** If $|I|$ is *infinite:*
  - **(a)** If *finitely many* $\mathcal{H}_j$ have dim > 1, then $|\Gamma^q| = 1$.

    (*So, we save* $\mathbb{C}\otimes\mathbb{C}\otimes\cdots$, *because up to the “redundancy”, there is only one layer.*)
  - **(b)** If *infinitely many* $\mathcal{H}_j$ have dim > 1, then $|\Gamma^q| \geq |\mathbb{R}|$.

    (*So, not even the “redundancy” saves* $\mathbb{C}^2\otimes\mathbb{C}^2\otimes\cdots$ *from uncountably many layers*).

  (*Therefore, any genuine ITP is indeed non-separable! —no matter if we take one or all the* $\mathfrak{C}\in\Gamma$ *within each* $\mathfrak{C}^q \in \Gamma^q$.) ♦

Proof: See Lemma 6.4.1 in (von Neumann, 1939) for the full proof.

- Here, we provide a more explicit proof of (iii).(b), which is the most relevant statement.

Let $I = \mathbb{N}$ for simplicity. For each $j \in \mathbb{N}$ choose unit vectors $\psi_j, \varphi_j \in \mathcal{H}_j$ such that $\psi_j \perp \varphi_j$. Then, define for each subset $B \subseteq \mathbb{N}$

$$g_j^B := \begin{cases} \psi_j & \text{if } j \in B \\ \varphi_j & \text{if } j \notin B \end{cases}. \tag{A.48}$$

Immediately, we get that $\bigotimes_{j\in\mathbb{N}} g_j^B \perp \bigotimes_{j\in I} g_j^{\widetilde{B}}$ if $B \neq \widetilde{B}$ (they will disagree at least in one factor).

Moreover, given $B \ \Delta\ \widetilde{B} := (B\cup\widetilde{B}) \setminus (B\cap\widetilde{B})$ (the elements where they do not coincide), if $|B \ \Delta\ \widetilde{B}| = +\infty$, then $\langle g_j^B,\ g_j^{\widetilde{B}}\rangle = 0$ will occur infinitely often, so in particular

$$\sum_{j\in\mathbb{N}} \big|\, |\langle g_j^B,\ g_j^{\widetilde{B}}\rangle| - 1 \,\big| = \sum_{j\in\mathbb{N}} 1 \quad\Longrightarrow\quad (g_j^B)_{j\in\mathbb{N}} \overset{q}{\not\approx} (g_j^{\widetilde{B}})_{j\in\mathbb{N}},$$

so they will belong to two different $\mathfrak{C}^q \in \Gamma^q$.

- **Claim :** There exists a family $\{\,(g_j^{B_x})_{j\in I}\,\}_{x\in\mathbb{R}}$ such that pairwise $(g_j^{B_x}\,)_{j\in I} \overset{q}{\not\approx} (g_j^{B_{\widetilde{x}}}\,)_{j\in I}$ $(x, \widetilde{x} \in \mathbb{R},\, x \neq \widetilde{x})$. (Thjis would prove that $|\Gamma^q| \geq |\mathbb{R}|$.)

Check :[a] For each $x \in \mathbb{R}$ take a strictly increasing Cauchy sequence of rational numbers approximating x and call it $(r_k(x))_{k\in\mathbb{N}}$.

Take an arbitrary bijection $h : \mathbb{Q} \longrightarrow \mathbb{N}$ and for each $x \in \mathbb{R}$ define

$$B_x := \Big\{\; h\Big(r_k(x)\Big) \;\mid\; k \in \mathbb{N} \;\Big\}. \tag{A.49}$$

If $x, y \in \mathbb{R}$ are $x \neq y$, $|B_x \cap B_y| < +\infty$ because $(r_k(x))_{k\in\mathbb{N}}$, $(r_k(y))_{k\in\mathbb{N}}$ must be different after some finite $k_0 \in \mathbb{N}$ —otherwise they would limit to the same number. At the same time, $|B_x| = +\infty$ because $(r_k(x))_{k\in\mathbb{N}}$ was chosen strictly increasing —so, has no repetition. Hence, $|B_x \;\Delta\; B_y| = +\infty$. But then, $\Big\{ \bigotimes_{j\in N} g_j^{B_x} \Big\}_{x\in\mathbb{R}}$ satisfies what we wanted.

o.ε.δ.

---

[a] The trick is to find a so-called *"almost-disjoint"* family of subsets of ℕ that is uncountable. The credit of the one we use here goes to Brian M. Scott in https://math.stackexchange.com/questions/278837/uncountably-many-sets-of-natural-numbers-with-finite-intersections.

B

Appendix: The Obvious CCR Representation of $\otimes_{j\in\mathbb{N}}L^2(\mathbb{R},dx)$

In this appendix we provide a rigorous account of some of the ideas mentioned in Chapter 7 about representations of the *canonical commutation relations* (CCR) of $\otimes_{j\in\mathbb{N}}L^2(\mathbb{R},dx)$. As we mentioned there, many interesting results on this topic were discovered in the 1960's and they can be found scattered in different publications of the time (in disparate notations, levels of rigor and proof-detail). As such, most (if not all) of the results that follow are a mere re-discovery of that knowledge, this time gathered a common place and with full proofs.

B.1 Basic Definitions

Let us make the language used in the following sections precise using the next definitions (mostly due to Arai (2018)). First of all, let us clarify the notation employed for operators.

Definition 43. Given a Hilbert space $\mathcal{H}$, we will denote by $\mathfrak{L}(\mathcal{H})$ the set of *densely defined linear operators* on $\mathcal{H}$. In particular, for an arbitrary $A\in\mathfrak{L}(\mathcal{H})$, $D(A)$ will denote its *domain*. For $A,B\in\mathfrak{L}(\mathcal{H})$:

- $A=B$ means that $D(A)=D(B)$ and $A\psi=B\psi\ \forall\psi\in\mathcal{H}$.
- Unless otherwise stated, by $A+B$ we mean the operator $(A+B,\ D(A)\cap D(B))$.

Note that $\mathcal{L}(\mathcal{H})$ still denotes the space of *bounded* operators on $\mathcal{H}$.

◆

Definition 44. Let $\mathcal{H}$ be a Hilbert space, $A\in\mathfrak{L}(\mathcal{H})$ and let $M\subseteq\mathcal{H}$ be a subspace,

(i) *if $M\subseteq D(A)$,* we denote by $A\upharpoonright_M$ the *restriction of A to its sub-domain M*, i.e., the operator of domain M and action $A\upharpoonright_M\psi:=A\psi$ for $\psi\in M$. It is still an operator on $\mathcal{H}$ (note that the outcome can be anywhere in $\mathcal{H}$), namely, if M is dense in $\mathcal{H}$, $A\upharpoonright_M\in\mathfrak{L}(\mathcal{H})$.

(ii) *if M is closed* with orthogonal projector P^M, we say that A is *reduced by M* when $P^MA\subseteq AP^M$, i.e., when $\forall\psi\in D(A)$, *(i)* $P^M\psi\in D(A)$ and *(ii)* $AP^M\psi=P^MA\psi$. We denote by $A|_M$ the operator of domain $M\cap D(A)$ and action $A|_M\psi=A\psi$ for $\psi\in M\cap D(A)$. It is an operator on M —as a Hilbert sub-space— (note that its outcomes are always in M). We call it the *reduced part* of A and by Thm. 1.47.(i) (Arai, 2018) it is still densely defined (now on M), namely, $A|_M \in \mathfrak{L}(M)$. Note that by §1.9 & Prop. 1.51 in (Arai, 2018), A is automatically also reduced by $M^{\perp}$ and $A = A|_M \oplus A|_{M^{\perp}}$ relative to $\mathcal{H} = M \oplus M^{\perp}$.

For $B \in \mathcal{L}(\mathcal{H})$ one trivially has that

(ii)' *if M is closed* with orthogonal projector P^M, then, B is *reduced by M if and only if* $P^M B = B P^M$. The reduced part $B|_M$ is well-defined for every $\psi \in M$ (such that $B|_M \psi = B\psi$) and it is still bounded, i.e., $B|_M \in \mathcal{L}(M)$ —with $\|B|_M\|_{op} \leq \|B\|_{op}$ (see Thm. 1.47.(vi) (Arai, 2018)).

For bounded operators $B \in \mathcal{L}(\mathcal{H})$ let us extend the definition of $B|_M$ to include cases where B is *not* reduced by the closed subspace $M \subseteq \mathcal{H}$. In general, we define for $\psi \in M$, $B|_M \psi := P^M B \psi$.

Lastly, given a set $\mathcal{B} \subseteq \mathfrak{L}(\mathcal{H})$, whenever it makes sense, we will denote $\mathcal{B}|_M := \{ A|_M \ : \ A \in \mathcal{B} \}$. ♦

Definition 45. Let $(\mathcal{H}_j)_{j\in I}$ be a family of Hilbert spaces. If for some $\mathfrak{C} \in \Gamma$ the layer $\otimes^{\mathfrak{C}}_{j\in I} \mathcal{H}_j$ reduces an operator $A \in \mathfrak{L}(\otimes_{j\in I}\mathcal{H})$, we will denote $A|_{\mathfrak{C}} := A|_{\otimes^{\mathfrak{C}}_{j\in I}\mathcal{H}_j}$. Likewise, if $\otimes^{\mathfrak{C}}_{j\in I}\mathcal{H}_j \subseteq D(A)$, we will denote $A \restriction_{\mathfrak{C}} := A \restriction_{\otimes_{j\in I}\mathcal{H}_j}$. ♦

Following Arai (2018), we now provide concrete definitions for the three structures that are called "CCR representation" in the literature —we will use the qualifying labels "Heisenberg", "Weyl" and "creation-annihilation" to distinguish them.

Definition 46. Let $\mathcal{K}, \mathscr{H}$ be respectively a *real* and a *complex* Hilbert space, with respective dense subspaces $\mathcal{W}$ and $\mathscr{D}$. Given unbounded-operator-valued maps $\widehat{\phi}, \widehat{\pi} : \mathcal{W} \to \mathfrak{L}(\mathcal{H})$, the triplet $(\ \mathscr{H},\ \mathscr{D},\ \{\widehat{\phi}(f),\ \widehat{\pi}(f)\}_{f\in\mathcal{W}}\ )$ is called a *representation of the Heisenberg CCR (with test-function space $\mathcal{W}$)* when all the following are satisfied:

(i) $\forall f \in \mathcal{W}$, $\widehat{\phi}(f), \widehat{\pi}(f)$ are *symmetric* with

$$\mathscr{D} \subseteq D\Big(\widehat{\phi}(f)\Big) \bigcap D\Big(\widehat{\pi}(f)\Big), \quad \widehat{\phi}(f)\mathscr{D} \subseteq \mathscr{D} \quad \text{and} \quad \widehat{\pi}(f)\mathscr{D} \subseteq \mathscr{D}$$

—i.e., $\mathscr{D}$ is a *common sub-domain* and is *preserved* by all the operators.

(ii) $\forall f, g \in \mathcal{W}$ and $\forall \alpha, \beta \in \mathbb{R}$, restricted to $\mathscr{D}$,

$$\widehat{\phi}(\alpha f + \beta g) = \alpha\ \widehat{\phi}(f) + \beta\ \widehat{\phi}(g) \quad \text{and} \quad \widehat{\pi}(\alpha f + \beta g) = \alpha\ \widehat{\pi}(f) + \beta\ \widehat{\pi}(g) \tag{B.1}$$

—i.e., the operator-maps are *linear on the "test functions"*.

(iii) $\forall f, g \in \mathcal{W}$, they satisfy the *Heisenberg CCR* on $\mathscr{D}$, i.e., restricted to $\mathscr{D}$,

$$\Big[\widehat{\phi}(f),\ \widehat{\pi}(g)\Big] = i\langle f,\ g\rangle, \qquad \Big[\widehat{\phi}(f),\ \widehat{\phi}(g)\Big] = 0 = \Big[\widehat{\pi}(f), \widehat{\pi}(g)\Big]. \tag{B.2}$$

Two representations $(\ \mathscr{H},\ \mathscr{D},\ \{\widehat{\phi}(f),\ \widehat{\pi}(f)\}_{f\in\mathcal{W}}\ )$ and $(\ \mathscr{H}',\ \mathscr{D}',\ \{\widehat{\phi}'(f),\ \widehat{\pi}'(f)\}_{f\in\mathcal{W}}\ )$ are said to be *equivalent* when there exists a unitary $U : \mathscr{H} \to \mathscr{H}'$ such that

$$U\ \overline{\widehat{\phi}(f)}\ U^{-1} = \overline{\widehat{\phi}'(f)} \quad \text{and} \quad U\ \overline{\widehat{\pi}(f)}\ U^{-1} = \overline{\widehat{\pi}'(f)} \qquad \forall f \in \mathcal{W}.$$

♦

Definition 47. Let $\mathcal{K}, \mathscr{H}$ be respectively a *real* and a *complex* Hilbert space, with $\mathcal{W}$ a dense vector subspace of $\mathcal{K}$. Given unbounded-operator-valued maps $\widehat{\phi}, \widehat{\pi} : \mathcal{W} \to \mathfrak{L}(\mathscr{H})$, the tuple $(\ \mathscr{H},\ \{\widehat{\phi}(f),\ \widehat{\pi}(f)\}_{f\in\mathcal{W}}\ )$ is called a *representation of the Weyl CCR (with test-function space $\mathcal{W}$)* when all the following are satisfied:

(i) For every $f \in \mathcal{W}$, $\widehat{\pi}(f)$ and $\widehat{\phi}(f)$ are *self-adjoint* operators.

(ii) $\forall f, g \in \mathcal{W}$ and $\forall t \in \mathbb{R}$,

$$\widehat{\phi}(tf) = t\ \widehat{\phi}(f) \quad \text{and} \quad \widehat{\pi}(tf) = t\ \widehat{\pi}(f) \tag{B.3}$$

$$\widehat{\phi}(f) + \widehat{\phi}(g) \subseteq \widehat{\phi}(f+g) \quad \text{and} \quad \widehat{\pi}(f) + \widehat{\pi}(g) \subseteq \widehat{\pi}(f+g). \tag{B.4}$$

—i.e., "up to unboundedness" the operator-maps are *linear on the "test functions"*.

(iii) $\forall f, g \in \mathcal{W}$, they satisfy the *Weyl relations*, i.e.,

$$e^{i\widehat{\phi}(f)} e^{i\widehat{\pi}(g)} = e^{-i\langle f,g\rangle} e^{i\widehat{\pi}(g)} e^{i\widehat{\phi}(f)}, \tag{B.5}$$

$$e^{i\widehat{\phi}(f)} e^{i\widehat{\phi}(g)} = e^{i\widehat{\phi}(g)} e^{i\widehat{\phi}(f)} \quad \text{and} \quad e^{i\widehat{\pi}(f)} e^{i\widehat{\pi}(g)} = e^{i\widehat{\pi}(g)} e^{i\widehat{\pi}(f)} \tag{B.6}$$

Two representations $(\ \mathscr{H},\ \{\widehat{\phi}(f),\ \widehat{\pi}(f)\}_{f\in\mathcal{W}}\ )$ and $(\ \mathscr{H}',\ \{\widehat{\phi}'(f),\ \widehat{\pi}'(f)\}_{f\in\mathcal{W}}\ )$ are said to be *equivalent* when there exists a unitary $U : \mathscr{H} \to \mathscr{H}'$ such that

$$U\ \widehat{\phi}(f)\ U^{-1} = \widehat{\phi}'(f) \quad \text{and} \quad U\ \widehat{\pi}(f)\ U^{-1} = \widehat{\pi}'(f) \qquad \forall f \in \mathcal{W}.$$

♦

Definition 48. Let $\mathcal{K}, \mathscr{H}$ be *complex* Hilbert spaces, with respective dense subspaces $\mathcal{V}$ and $\mathscr{D}$. Given an unbounded-operator-valued map $\widehat{C} : \mathcal{V} \to \mathfrak{L}(\mathscr{H})$, the triplet $(\ \mathscr{H},\ \mathscr{D},\ \{\widehat{C}(f)\}_{f\in\mathcal{V}}\ )$ is called a *representation of the creation-annihilation CCR (with test-function space $\mathcal{V}$)* when all the following are satisfied:

(i) $\forall f \in \mathcal{V}$, $\widehat{C}(f)$ is a *closed* operator with

$$\mathscr{D} \subseteq D\Big(\widehat{C}(f)\Big) \bigcap D\Big(\widehat{C}(f)^*\Big), \quad \widehat{C}(f)\mathscr{D} \subseteq \mathscr{D} \quad \text{and} \quad \widehat{C}(f)^*\mathscr{D} \subseteq \mathscr{D}$$

—i.e., $\mathscr{D}$ is a *common sub-domain* and is *preserved* by all the operators and their adjoints.

(ii) $\forall f, g \in \mathcal{V}$ and $\forall \alpha, \beta \in \mathbb{C}$, restricted to $\mathscr{D}$,

$$\widehat{C}(\alpha f + \beta g) = \overline{\alpha}\ \widehat{C}(f) + \overline{\beta}\ \widehat{C}(g) \tag{B.7}$$

—i.e., the operator-map is *conjugate linear on the "test functions"*.

(iii) $\forall f, g \in \mathcal{V}$, they satisfy the *creation-annihilation CCR* on $\mathscr{D}$, i.e., restricted to $\mathscr{D}$,

$$\Big[\widehat{C}(f),\ \widehat{C}(g)^*\Big] = \langle f,\, g\rangle, \qquad \Big[\widehat{C}(f),\, \widehat{C}(g)\Big] = 0 = \Big[\widehat{C}(f)^*,\ \widehat{C}(g)^*\Big]. \tag{B.8}$$

We call $\widehat{C}(f)$ and $\widehat{C}(f)^*$, respectively, the *annihilation and creation operators* (of state $f \in \mathcal{V}$). Two representations $(\ \mathscr{H},\ \mathscr{D},\ \{\widehat{C}(f)\}_{f\in\mathcal{V}}\ )$ and $(\ \mathscr{H}',\ \mathscr{D}',\ \{\widehat{C}'(f)\}_{f\in\mathcal{W}}\ )$ are said to be *equivalent* when there exists a unitary $U : \mathscr{H} \to \mathscr{H}'$ such that $U\ \widehat{C}(f)\ U^{-1} = \widehat{C}'(f)$. ♦

Lemma 42. Property (ii) in Definition 48 implies that, $\forall f, g \in \mathcal{V}$ and $\forall \alpha, \beta \in \mathbb{C}$,

$$\widehat{C}(\alpha f + \beta g)^* = \alpha\ \widehat{C}(f)^* + \beta\ \widehat{C}(g)^*,$$

when restricted to $\mathscr{D}$, i.e., the creation operators are *linear on the test functions.* ♦

Proof: See Remark 5.23 in (Arai, 2018). □

Definition 49. One calls the real space $\mathcal{W}$ of a Heisenberg or a Weyl CCR representation, as well as the complex space $\mathcal{V}$ of a creation-annihilation CCR representation, a *test-function space*. For any such representation one says that they have $\dim(\mathcal{K})$ many *degrees of freedom* and the space $\mathscr{H}$ is called the *representation space*.

A set $\mathcal{B} \subseteq \mathcal{L}(\mathcal{H})$ is said to be *reducible* when there exists a closed subspace $M \subseteq \mathcal{H}$ different from $\{\vec{0}\}$ and $\mathscr{H}$ that reduces every operator in $\mathcal{B}$. We say that a CCR representation is reducible if the following sets are so: for the Heisenberg CCR, $\{\widehat{\phi}(f), \widehat{\pi}(f)\}_{f\in\mathcal{W}}$, for the Weyl CCR, $\{e^{i\widehat{\phi}(f)},\ e^{i\widehat{\pi}(f)}\}_{f\in\mathcal{W}}$ and for the creation-annihilation CCR, $\{\widehat{C}(f), \widehat{C}^*(f)\}_{f\in\mathcal{V}}$. A set $\mathcal{B} \subseteq \mathcal{L}(\mathcal{H})$ or a representation featuring $\mathcal{B}$ is said to be *irreducible* when it is *not* reducible. ♦

Next, we review the notion of "conjugation" and "complexification" of a Hilbert space.

Definition 50. Let $\mathcal{H}$ be a complex Hilbert space. A mapping $C : \mathcal{H} \to \mathcal{H}$ that is

(i) *conjugate-linear:* $C(\alpha f + \beta g) = \overline{\alpha} C(f) + \overline{\beta} C(g), \quad \forall f, g \in \mathcal{H}\ \forall \alpha, \beta \in \mathbb{C}$,

(ii) *involutive:* $C^2 = Id$,

(iii) *norm-preserving:* $\|Cf\| = \|f\|$ for all $f \in \mathcal{H}$,

is called a *conjugation (on $\mathcal{H}$).* ♦

Proposition 68. Let $\mathcal{H}$ be a complex Hilbert space and C a conjugation on $\mathcal{H}$. Then,

(i) for all $f, g \in \mathcal{H}$, $\langle Cf, Cg\rangle = \langle g, f\rangle$.

(ii) The set $(\mathcal{H})^C := \{\psi \in \mathcal{H} \mid C\psi = \psi\}$ is a *real Hilbert space* when equipped with the inner product induced from $\mathcal{H}$. We call it the *real subspace of $\mathcal{H}$ with respect to C*.

(iii) For each $f \in \mathcal{H}$ there exist unique $f_{Re}, f_{Im} \in (\mathcal{H})^C$ such that $f = f_{Re} + i f_{Im}$, namely

$$f_{Re} := \frac{1}{2}\Big(f + Cf\Big) \quad \text{and} \quad f_{Im} := \frac{1}{2i}\Big(f - Cf\Big).$$

One calls them, respectively, the *real and imaginary parts of f with respect to C*, and one writes $\operatorname{Re} f := f_{Re}$ and $\operatorname{Im} f := f_{Im}$.

(iv) Given a subspace $M \subseteq \mathcal{H}$, we define $(M)^C := M \cap (\mathcal{H})^C$. If M is dense in $\mathcal{H}$ and $CM \subseteq M$, then $(M)^C$ is dense in $(\mathcal{H})^C$. ♦

Proof: See §5.13 in (Arai, 2018). □

Definition 51. Let $\mathcal{K}$ be a *real* vector-space. The *complexification* of $\mathcal{K}$, denoted $\mathcal{K}_{\mathbb{C}}$, is the set $\mathcal{K} \times \mathcal{K}$ equipped with the complex vector-space structure: $\forall (f, g), (f', g') \in \mathcal{K} \times \mathcal{K}$ and $\forall z = a + ib \in \mathbb{C}$,

$$(f, g) + (f', g') := (f + f',\ g + g') \quad \text{and} \quad z \cdot (f, g) := (af - bg,\ ag + bf)$$

Note that since we can uniquely decompose any $\psi := (f, g) \in \mathcal{K}_{\mathbb{C}}$ as $\psi = (f, 0) + i(g, 0)$, if we identify $(f, 0) \cong f$, then $\psi = f + ig$. The vector f (resp. g) is called the *real* (resp. *imaginary*) *part of ψ* and we write $f = \operatorname{Re}\psi$ (resp. $g = \operatorname{Im}\psi$).

- Given a Cauchy complete inner product $\langle\cdot,\cdot\rangle_{\mathcal{K}}$ for $\mathcal{K}$ (so that $\mathcal{K}$ is a real Hilbert space), a natural Cauchy complete inner product for $\mathcal{K}_{\mathbb{C}}$ is given by

$$\Big\langle f+ig,\ f'+ig'\Big\rangle_{\mathcal{K}_{\mathbb{C}}} := \langle f,f'\rangle_{\mathcal{K}} + \langle g,g'\rangle_{\mathcal{K}} - i\langle g,f'\rangle_{\mathcal{K}} + i\langle f,g'\rangle_{\mathcal{K}}$$

with $f,f',g,g' \in \mathcal{K}$ arbitrary (as though we applied sesquilinearity). We call the complex Hilbert space $(\mathcal{K}_{\mathbb{C}},\langle\cdot,\cdot\rangle)$ the *complexification of (the real Hilbert space)* $(\mathcal{K},\langle\cdot,\cdot\rangle_{\mathcal{K}})$. ♦

Proposition 69. • Let $\mathcal{K}$ be a *real* Hilbert space. The mapping $C_{\mathcal{K}}:\mathcal{K}_{\mathbb{C}}\to\mathcal{K}_{\mathbb{C}}$ such that

$$C_{\mathcal{K}}(f+ig) := f-ig, \quad \forall f,g\in\mathcal{K} \tag{B.9}$$

is a conjugation and is called the *complex conjugation of* $\mathcal{K}_{\mathbb{C}}$. In particular, $\mathcal{K} = (\mathcal{K}_{\mathbb{C}})^{C_{\mathcal{K}}}$ —the *"real subspace of* $\mathcal{K}_{\mathbb{C}}$ *with respect to the complex conjugation* C *is* $\mathcal{K}$*"*.

- Let $\mathcal{H}$ be a *complex* Hilbert space and C a conjugation on $\mathcal{H}$. Then, there exists a unique unitary identification $((\mathcal{H})^{C})_{\mathbb{C}} \cong \mathcal{H}$ such that $(f_{Re},f_{Im}) \in ((\mathcal{H})^{C})_{\mathbb{C}}$ is identified with $f = f_{Re} + if_{Im} \in \mathcal{H}$. ♦

Proof: See §5.13 in (Arai, 2018). □

Finally, let us fix some notation that will be used everywhere in this chapter.

Definition 52. For fixed $n\in\mathbb{N}$, denote the space of *complex valued polynomials (of finite degree) over* $\mathbb{R}^n$ by $B_{poly}(\mathbb{R}^n)$. ♦

Lemma 43. For $n\in\mathbb{N}_0$ let $h_n(x) := (-1)^n e^{x^2}\frac{d^n}{dx^n}e^{-x^2}$ be called the *n-th Hermite polynomial* (note that it is an n-th order polynomial with real coefficients) and for $\omega\in(0,+\infty)$ define

$$\phi_n(x) := c_n h_n(\sqrt{\omega}x)e^{-\omega x^2/2} \tag{B.10}$$

with $c_n := \frac{\omega^{1/4}}{\pi^{1/4}\sqrt{2^n n!}}$ a normalization constant. Then, $\{\phi_n\}_{n\in\mathbb{N}_0}$ is an ONB of $L^2(\mathbb{R},dx)$ that we will call the *"harmonic oscillator's" ONB.*[1] In particular, defining $\hat{a}_{(\omega)} := \frac{1}{\sqrt{2}}\Big(\sqrt{\omega}x + \frac{1}{\sqrt{\omega}}\frac{d}{dx}\Big)$ and $\hat{a}^{\dagger}_{(\omega)} := \frac{1}{\sqrt{2}}\Big(\sqrt{\omega}x - \frac{1}{\sqrt{\omega}}\frac{d}{dx}\Big)$, one gets that for $n\in\mathbb{N}$,

$$\text{(i)}\ \ \hat{a}_{(\omega)}\phi_0 = 0, \qquad \text{(ii)}\ \ \hat{a}_{(\omega)}\phi_n = \sqrt{n}\ \phi_{n-1}, \qquad \text{(iii)}\ \ \phi_n = \frac{1}{\sqrt{n!}}(\hat{a}^{\dagger}_{(\omega)})^n\phi_0.$$ ♦

Proof: See §7.3.1 in (Porta, 2019). □

Definition 53. Let $(\phi_n)_{n\in\mathbb{N}_0} \subseteq L^2(\mathbb{R},dx)$ be the ONB of Lemma 43 with $\omega = 1$ (the so-called *Hermite functions*). We will denote $B_{Her} := \operatorname{span}(\phi_n)_{n\in\mathbb{N}_0}$. It has the following properties:

(i) By the ONB property of $(\phi_n)_{n\in\mathbb{N}_0}$, B_{Her} is dense in $L^2(\mathbb{R},dx)$.

(ii) Using that one can write any $p\in B_{poly}(\mathbb{R})$ as a finite linear combination of Hermite polynomials $(h_n)_{n\in\mathbb{N}_0}$,[2] note that

---

[1] They are all eigenvectors of the operator $\Big(-\frac{d^2}{dx^2}+\omega^2x^2,\ \mathcal{S}(\mathbb{R})\Big)$.

[2] We know that $h_n(x) = \sum_{j=0}^{n} b_{n,j}\,x^j$ for for some $b_{n,j}\in\mathbb{C}$ such that $b_{n,n}\neq 0$. But then, we can prove that for any $n\in\mathbb{N}_0$, the monomial $x^n\in\operatorname{span}(h_j)_{j\in\mathbb{N}_0}$ as follows. $1 \equiv \frac{1}{b_{0,0}}h_0(x)$, $x \equiv \frac{1}{b_{1,1}}(h_1(x) - b_{1,0}\cdot 1)$, $x^2 \equiv \frac{1}{b_{2,2}}(h_2(x) - b_{2,0}1 - b_{2,1}x)$ and in general, $x^n \equiv \frac{1}{b_{n,n}}(h_n(x) - \sum_{j=0}^{n-1}b_{n,j}x^j)$. Thus, $\operatorname{span}(1,x,x^2,\ldots) \equiv B_{poly}(\mathbb{R}) \subseteq \operatorname{span}(h_n)_{n\in\mathbb{N}_0}$.

$$B_{Her} = \left\{ x \in \mathbb{R} \mapsto p(x)e^{-x^2/2} \;\middle|\; p \in B_{poly}(\mathbb{R}) \right\} = \text{“}B_{poly}(\mathbb{R})e^{-x^2/2}\text{”}.$$

(In words, *the elements of* B_{Her} *are “polynomials times* $e^{-x^2/2}$*”*.)

(iii) Every $\eta \in B_{Her} \setminus \{0\}$ is a function with finitely many zeros[3] and thus such that $\eta(x) \neq 0$ for almost every $x \in \mathbb{R}$. ♦

Definition 54. Denoting the polynomial $x \in \mathbb{R} \mapsto x^\alpha$ for each $\alpha \in \mathbb{N}$ as x^α and denoting for $f : \mathbb{R} \to \mathbb{C}$, $\|f\|_\infty := \sup_{x \in \mathbb{R}} |f(x)|$, we call the set

$$\mathcal{S}(\mathbb{R}) := \left\{ f \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{C}) \;\middle|\; \left\| x^\alpha \partial^\beta f \right\|_\infty < +\infty \quad \forall \alpha, \beta \in \mathbb{N}_0 \right\} \tag{B.11}$$

the *Schwartz function space.* ♦

Proposition 70. **(i)** $\mathcal{S}(\mathbb{R})$ is a complex vector space.

(ii) if $f \in \mathcal{S}(\mathbb{R})$ then $x^\alpha \partial^\beta f \in \mathcal{S}(\mathbb{R})$ $\forall \alpha, \beta \in \mathbb{N}_0$.

(iii) $B_{Her} \subseteq \mathcal{S}(\mathbb{R}) \subseteq L^2(\mathbb{R}, dx)$. ♦

Proof: In (Teschl, 2014), see §7.1 for (i), (ii) and $\mathcal{S}(\mathbb{R}) \subseteq L^2(\mathbb{R}, dx)$. For $B_{Her} \subseteq \mathcal{S}(\mathbb{R})$ use that $e^{-(\cdot)^2/2} \in \mathcal{S}(\mathbb{R})$, so, by (ii), $\forall q \in B_{poly}(\mathbb{R})$, $qe^{-x^2/2} \in \mathcal{S}(\mathbb{R})$. □

B.2 The Lifted Position and Momentum Operators Compose the Obvious CCR Representation of $\otimes_{j \in \mathbb{N}} L^2(\mathbb{R}, dx)$

In this section we prove that the lifted position and momentum operators of $L^2(\mathbb{R}, dx)$ constitute a (reducible) representation of both, the Heisenberg and Weyl CCR and that they naturally give a representation of the creation-annihilation CCR, which turns out to be the lift of the creation and annihilation operators seen in §1.3. In order to prove all this, we will derive several results that are interesting on their own for the analysis of ITPs.

B.2.1 The Heisenberg CCR Representation

A CCR representation in the ITP will consist of a family of operators, each acting non-trivially in different factors of the tensor product. Since we need them to have a common dense domain, we shall start from the following results.

Lemma 44. Let $\mathcal{H}$ be a Hilbert space, $\mathcal{D} \subseteq \mathcal{H}$ a dense subspace and let $\psi \in \mathcal{H}$ be such that $\|\psi\| = 1$. Then, $\forall \varepsilon > 0$, there exists $\varphi \in \mathcal{D}$ with

$$\textbf{(i)}\ \|\varphi\| = 1 \quad \text{and} \quad \textbf{(ii)} |\langle \psi, \varphi \rangle - 1| \leq \varepsilon.$$

♦

Proof: By density $\exists \eta_\varepsilon \subset \mathcal{D}$ such that $\|\psi - \eta_\varepsilon\| \leq \min(\varepsilon/2, 1/2)$. In particular, by the reverse triangle inequality, $|\|\psi\| - \|\eta_\varepsilon\|| < 1$, so $\eta_\varepsilon \neq 0$. Define $\varphi := \eta_\varepsilon / \|\eta_\varepsilon\|$. Trivially, $\|\varphi\| = 1$.

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[3] Each $\eta \in B_{Her}$ is a polynomial $q \in B_{poly}(\mathbb{R})$ times $e^{-(\cdot)^2/2}$. Since the latter has no zeros and a polynomial has no more zeros than its degree, η has at most finitely many zeros. A finite set has Lebesgue measure zero, so, η is almost everywhere non-zero.

Moreover,

$$|\langle\psi,\varphi\rangle-1| \leq \Big|\langle\psi,\frac{\eta_\varepsilon}{\|\eta_\varepsilon\|}\rangle-\langle\psi,\eta_\varepsilon\rangle\Big|+\Big|\langle\psi,\eta_\varepsilon\rangle-1\Big| \overset{(1=\langle\psi,\psi\rangle)}{=} \Big|\langle\psi,\ \eta_\varepsilon(\frac{1}{\|\eta_\varepsilon\|}-1)\rangle\Big|+\Big|\langle\psi,\eta_\varepsilon-\psi\rangle\Big| \overset{\text{(Cau.Sch))}}{\leq}$$

$$\leq \cancelto{1}{\|\psi\|}\|\eta_\varepsilon\|\Big|\frac{1}{\|\eta_\varepsilon\|}-1\Big| + \cancelto{1}{\|\psi\|}\|\eta_\varepsilon-\psi\| = \Big|\|\eta_\varepsilon\| - \underbrace{1}_{\|\psi\|}\Big| + \|\eta_\varepsilon-\psi\| \overset{\text{(rev.trgl)}}{\leq} 2\|\eta_\varepsilon-\psi\| \leq \varepsilon.$$

o.ε.δ.

Proposition 71. Let $(\mathcal{H}_j)_{j\in\mathbb{N}}$ be a sequence of Hilbert spaces and $\mathcal{D}_j \subseteq \mathcal{H}_j$ a dense subspace for each $\mathcal{H}_j$. Then, for each $\mathfrak{C}\in\Gamma$ there *exists* a vector $\bigotimes_{j\in\mathbb{N}}\rho_j^{\mathfrak{C}} \in \bigotimes_{j\in\mathbb{N}}\mathcal{H}_j$ generating $\bigotimes_{j\in\mathbb{N}}^{\mathfrak{C}}\mathcal{H}_j$ such that $\rho_j^{\mathfrak{C}}\in\mathcal{D}_j$ $\forall j\in\mathbb{N}$. Moreover,

$$\otimes_{j\in\mathbb{N}}^{\mathscr{F}}\mathcal{D}_j := \operatorname{span}\left\{ \bigotimes_{j\in\mathbb{N}}\psi_j \in \bigotimes_{j\in\mathbb{N}}\mathcal{H}_j \;\middle|\; \psi_j\in\mathcal{D}_j \quad \forall j\in\mathbb{N}\right\} \tag{B.12}$$

is a *dense* vector subspace of $\bigotimes_{j\in\mathbb{N}}\mathcal{H}_j$. ♦

Proof: By Prop. 9, for any fixed $\mathfrak{C}\in\Gamma$, $\exists(\psi_j)_{j\in\mathbb{N}}\in\mathfrak{C}$ with $\|\psi_j\|=1$ $\forall j\in\mathbb{N}$. Then, by Lemma 44, for each $j\in\mathbb{N}$, there exists a $\rho_j\in\mathcal{D}_j$ such that $\|\rho_j\|_{L^2}=1$ and $|\langle\psi_j,\rho_j\rangle-1|\leq 1/j^2$. Hence,

$$\sum_{j\in\mathbb{N}}|\langle\psi_j,\rho_j\rangle-1| \leq \sum_{j\in\mathbb{N}}\frac{1}{j^2} = \frac{\pi^2}{6} \quad (\textit{exists}), \tag{B.13}$$

which, by definition, implies that $(\psi_j)_{j\in\mathbb{N}}\approx(\rho_j)_{j\in\mathbb{N}}$. With all, $\otimes_{j\in\mathbb{N}}\rho_j$ is a generator of the same layer $\mathfrak{C}$ as $\otimes_{j\in\mathbb{N}}\psi_j$ but such that $\rho_j\in\mathcal{D}_j$ —proving the first claim.

- Now, given such a generator $\bigotimes_{j\in\mathbb{N}}\rho_j^{\mathfrak{C}}$ for each $\mathfrak{C}\in\Gamma$, by Thm. 25.(iii), for each $\mathfrak{C}\in\Gamma$, the set

$$S^{\mathfrak{C}} := \operatorname{span}\left\{\ \psi_1\otimes\cdots\otimes\psi_N\otimes\rho_{N+1}^{\mathfrak{C}}\otimes\cdots \;\middle|\; \psi_j\in\mathcal{H}_j,\ N\in\mathbb{N}\ \right\} \tag{B.14}$$

is a dense vector subspace of $\bigotimes_{j\in\mathbb{N}}^{\mathfrak{C}}\mathcal{H}_j$. We just miss transferring this fact to a subset of $S^{\mathfrak{C}}$ where also the first N vectors are from $\mathcal{D}_j$.

- Let there be an arbitrary $\psi_1\otimes\cdots\otimes\psi_N$ with $\psi_j\in\mathcal{H}_j$. For any fixed $\delta_j>0$, by density of $\mathcal{D}_j\subseteq\mathcal{H}_j$, there is an $\eta_j\in\mathcal{D}_j$ such that $\|\psi_j-\eta_j\|\leq\min(\delta_j,\ 1/2)$. In particular,

$$\Big\|(\psi_1\otimes\cdots\otimes\psi_N\otimes\rho_{N+1}^{\mathfrak{C}}\otimes\cdots)-(\eta_1\otimes\cdots\otimes\eta_N\otimes\rho_{N+1}^{\mathfrak{C}}\otimes\cdots)\Big\| \overset{(\star)}{=} \tag{B.15}$$

$$= \Big\|(\psi_1\otimes\cdots\otimes\psi_N-\eta_1\otimes\cdots\otimes\eta_N)\otimes(\rho_{N+1}^{\mathfrak{C}}\otimes\cdots)\Big\| =$$

$$= \|\psi_1\otimes\cdots\otimes\psi_N-\eta_1\otimes\cdots\otimes\eta_N\|\ \cancelto{1}{\Big\|(\rho_{N+1}^{\mathfrak{C}}\otimes\cdots)\Big\|} = \|\psi_1\otimes\cdots\otimes\psi_N-\eta_1\otimes\cdots\otimes\eta_N\| \leq$$

$$\overset{\left(\substack{\pm\psi_1\otimes\eta_2\otimes\cdots\eta_N \\ \text{\& triang. ineq}}\right)}{\leq} \|\psi_1\otimes\cdots\otimes\psi_N-\psi_1\otimes\eta_2\otimes\cdots\otimes\eta_N\| + \underbrace{\|\psi_1\otimes\eta_2\otimes\cdots\otimes\eta_N-\eta_1\otimes\cdots\otimes\eta_N\|}_{\|(\psi_1-\eta_1)\otimes\eta_2\otimes\cdots\|=\|\psi_1-\eta_1\|\prod_{j=2}^N\|\eta_j\|} \leq$$

$$\overset{\left(\substack{\pm\psi_1\otimes\psi_2\otimes\eta_3\otimes\cdots\eta_N \\ \text{\& trigl. ineq}}\right)}{\leq} \|\psi_1\otimes\cdots\otimes\psi_N-\psi_1\otimes\psi_2\otimes\eta_3\otimes\cdots\otimes\eta_N\| +$$

$$+\|\psi_1-\eta_1\|\prod_{j>1}^{N}\|\eta_j\| + \underbrace{\|\psi_1\otimes\psi_2-\psi_1\otimes\eta_2\|}_{\|\psi_1\|\|\psi_2-\eta_2\|}\prod_{j>2}^{N}\|\eta_j\| \quad \leq \overset{(\star\star)}{\cdots} \leq$$

$$\leq \sum_{k=1}^{N} \Big[\Big(\prod_{j=1}^{k-1} \|\psi_j\| \Big) \|\psi_k - \eta_k\| \Big(\prod_{j=k+1}^{N} \|\eta_j\| \Big) \Big] \overset{\text{(by def)}}{\leq} \sum_{k=1}^{N} \Big(\delta_k \prod_{j=1}^{k-1} \|\psi_j\| \prod_{j=k+1}^{N} \|\eta_j\| \Big).$$

In $(\star)$ we used that von Neumann's tensor product $\otimes$ is linear (in the sense of Prop. 53) together with the associativity property given by Thm 26. In $(\star\star)$ we added and subtracted terms like $\psi_1 \otimes \cdots \otimes \psi_k \otimes \eta_{k+1} \otimes \cdots \otimes \eta_N$ finitely many times (from $k = 3$ to $k = N$) and then applied the triangle inequality to split the resulting terms.

Now, let us fix the δ_j to make the last computation smaller than any fixed $\varepsilon > 0$. Start from $k = N$ with

$$\delta_N := \frac{\varepsilon}{N \prod_{j=1}^{N-1} \|\psi_j\|}.$$

Then, fix consecutively, for descending k from $k = N - 1$ until $k = 1$,

$$\delta_k := \frac{\varepsilon}{N \prod_{j=1}^{k-1} \|\psi_j\| \prod_{j=k+1}^{N} \|\eta_j\|}$$

—note that δ_k only depends on the η_j with $j > k$, so the iteration is well-defined. Retaking (B.15) these choices leave,

$$\Big\| \psi_1 \otimes \cdots \otimes \psi_N \otimes \rho^{\mathfrak{C}}_{N+1} \otimes \cdots - \eta_1 \otimes \cdots \otimes \eta_N \otimes \rho^{\mathfrak{C}}_{N+1} \otimes \cdots \Big\| \leq \varepsilon.$$

- Now, let $\Psi \in S^{\mathfrak{C}}$ be an arbitrary element. Then, $\Psi = \sum_{\ell=1}^{M} \psi_1^\ell \otimes \cdots \otimes \psi_{N_\ell} \otimes \rho^{\mathfrak{C}}_{N_\ell+1} \otimes \cdots$ for some $\psi_j \in \mathcal{H}_j,\ N_\ell, M \in \mathbb{N}$ —the linear combination coefficients were absorbed in some factor. By the above, for any $\varepsilon > 0$ and each $\ell \in \{1, ..., M\}$ there exists an $\eta_1^\ell \otimes \cdots \otimes \eta_N^\ell \otimes \rho^\ell_{N+1} \otimes \cdots$ such that $\Big\| \psi_1^\ell \otimes \cdots \otimes \psi_N^\ell \otimes \rho^\ell_{N+1} \otimes \cdots - \eta_1^\ell \otimes \cdots \otimes \eta_N^\ell \otimes \rho^\ell_{N+1} \otimes \cdots \Big\| \leq \varepsilon/M$. As such, by the triangle inequality, $\Big\| \Psi - \sum_{\ell=1}^{M} \eta_1^\ell \otimes \cdots \otimes \eta_N^\ell \otimes \rho^\ell_{N+1} \otimes \cdots \Big\| \leq \varepsilon$. Hence, there exists an element of

$$S^{\mathfrak{C}}_{\mathcal{D}} := \operatorname{span}\Big\{ \eta_1 \otimes \cdots \otimes \eta_N \otimes \rho^{\mathfrak{C}}_{N+1} \otimes \cdots \ \Big|\ \eta_j \in \mathcal{D}_j,\ N \in \mathbb{N} \Big\}$$

arbitrarily close to any $\Psi \in S^{\mathfrak{C}}$. That is, $S^{\mathfrak{C}} \subseteq \overline{S^{\mathfrak{C}}_{\mathcal{D}}}$. But, we proved that $S^{\mathfrak{C}}$ is dense in $\otimes^{\mathfrak{C}}_{j\in\mathbb{N}} \mathcal{H}_j$, so, $\otimes^{\mathfrak{C}}_{j\in\mathbb{N}} \mathcal{H}_j = \overline{S^{\mathfrak{C}}} \subseteq \overline{S^{\mathfrak{C}}_{\mathcal{D}}}$, implying that $\overline{S^{\mathfrak{C}}_{\mathcal{D}}} = \otimes^{\mathfrak{C}}_{j\in\mathbb{N}} \mathcal{H}_j$ and thus, that $S^{\mathfrak{C}}_{\mathcal{D}}$ is dense in $\otimes^{\mathfrak{C}}_{j\in\mathbb{N}} \mathcal{H}_j$. By Lemma 16, this fact, together with $\bigotimes_{j\in\mathbb{N}} \mathcal{H}_j = \oplus_{\mathfrak{C}\in\Gamma} (\otimes^{\mathfrak{C}}_{j\in\mathbb{N}} \mathcal{H}_j)$, imply that $\operatorname{span}(\cup_{\mathfrak{C}\in\Gamma} S^{\mathfrak{C}}_{\mathcal{D}})$ is dense in $\bigotimes_{j\in\mathbb{N}} \mathcal{H}_j$. But trivially, $\operatorname{span}(\cup_{\mathfrak{C}\in\Gamma} S^{\mathfrak{C}}_{\mathcal{D}}) \subseteq \otimes^{\mathscr{F}}_{j\in\mathbb{N}} \mathcal{D}_j$, so, the latter must be dense in $\otimes_{j\in\mathbb{N}} \mathcal{H}_j$ as well. ***o.ε.δ.***

On the other hand, we will need slightly more than a trivial lift of an operator to the ITP (say, as in Lemma 24). In particular, we need to make sense of polynomials of such lifts.

Definition 55. Given a Hilbert space $\mathcal{H}$ and $A \in \mathfrak{L}(\mathcal{H})$, in order to define A's *n-th power* ($n \in \mathbb{N}$) one declares recursively, $D(A^n) := \{\psi \in D(A^{n-1}) \mid A^{n-1}\psi \in D(A)\}$ (which trivially satisfies $D(A^n) \subseteq D(A^\ell)\ \forall \ell \leq n$), together with $A^n\psi := \underbrace{(A \circ \cdots \circ A)}_{n}\ \psi$ for $\psi \in D(A^n)$. ♦

Lemma 45. If $(A, D(A))$ is essentially self-adjoint, given a dense subspace $D \subseteq D(A^n)$ such that $(A^n) \restriction_D$ is essentially self-adjoint, for all $\ell \leq n$, $(A^\ell) \restriction_D$ is also *essentially self-adjoint.* ♦

Proof: Trivially, $D \subseteq D(A^n) \subseteq D((\overline{A})^n)$ and $A^n \restriction_D = (\overline{A})^n \restriction_D$. Thus, $(\overline{A})^n$ is an extension of $A^n \restriction_D$, and by Thm. 1.16 in (Arai, 2018) it is self-adjoint. But, since $A^n \restriction_D$ is essentially self-adjoint, $\overline{A^n \restriction_D}$ is also a self-adjoint extension of $A^n \restriction_D$ and an essentially self-adjoint operator has a unique self-adjoint extension (Prop. 1.21 in (Arai, 2018)), so, it must be that $\overline{A^n \restriction_D} = (\overline{A})^n$. Therefore, D is a core of $(\overline{A})^n$. By Prop. 1.43 in (Arai, 2018), this implies that D is also a core of $(\overline{A})^{\ell}$ $\forall \ell \leq n$. Since by Thm. 1.16 in (Arai, 2018) all such $(\overline{A})^{\ell}$ are self-adjoint, by Prop. 1.32 in (Arai, 2018), $(\overline{A})^{\ell} \restriction_D$ is essentially self-adjoint. Finally, since by the above definition $D \subseteq D(A^n) \subseteq D(A^{\ell})$ for all $\ell \leq n$, $(\overline{A})^{\ell} \restriction_D = A^{\ell} \restriction_D$, proving the lemma. ***o.ε.δ.***

Proposition 72. Let $Q \in B_{poly}(\mathbb{R}^N)$, such that

$$Q(x_1, ..., x_N) = \sum_{j_1,...,j_N=1}^{M_1,...,M_N} c_{j_1,...,j_N} x_1^{j_1} \cdots x_N^{j_N}$$

for some $M_{\ell} \in \mathbb{N}$, $c_{j_1,....,j_N} \in \mathbb{C}$. For each $k \in \{1, ..., N\}$, let $\mathcal{H}_k$ be a Hilbert space and $D_k \subseteq \mathcal{H}_k$ a dense subspace of $\mathcal{H}_k$. Also, let $A_k \in \mathfrak{L}(\mathcal{H}_k)$ be a self-adjoint operator such that $D_k \subseteq D((A_k)^{M_k})$ and $(A_k)^{M_k} \restriction_{D_k}$ is essentially self-adjoint. Then, given

$$\otimes^{\mathscr{F}}_{j\in\{1,...,N\}} D_j := \operatorname{span}\Big\{\psi_1 \otimes \cdots \otimes \psi_N \;\Big|\; \psi_j \in D_j \; \forall j\Big\},$$

the operator $\Big(Q(A_1, ..., A_N),\ \otimes^{\mathscr{F}}_{j\in\{1,...,N\}} D_j\Big)$ defined for $\psi_1 \otimes \cdots \otimes \psi_N \in \otimes^{\mathscr{F}}_{j\in\{1,...,N\}} D_j$ as

$$Q(A_1, ..., A_N)(\psi_1 \otimes \cdots \otimes \psi_N) := \sum_{j_1,...,j_N=1}^{M_1,...,M_N} c_{j_1,...,j_N} (A_1^{j_1} \psi_1) \otimes \cdots \otimes (A_N^{j_N} \psi_N) \tag{B.16}$$

and extended linearly to the rest of $\otimes^{\mathscr{F}}_{j\in\{1,...,N\}} D_j$, is an *essentially self-adjoint* operator (on $\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_N$). We will denote

$$\sum_{j_1,...,_N=1}^{M_1,...,M_N} c_{j_1,...,j_N} A_1^{j_1} \otimes \cdots \otimes A_N^{j_N} := Q(A_1, ..., A_N).$$

◆

Proof: See Theorem VIII.33 in (Reed and Simon, 1981). □

We proceed to "lift" Prop. 72 to the ITP.

Lemma 46. For each dense subspace $D \subseteq \mathcal{H}$, $Id \restriction_D$ is *essentially self-adjoint.* ◆

Proof: Let $\Psi \in \mathcal{H}$. By density, $\exists (\psi_n)_{n\in\mathbb{N}} \subseteq D$ such that $\psi_n \xrightarrow[n\to\infty]{} \Psi$. But then, $Id\psi_n = \psi_n \xrightarrow[n\to\infty]{} \Psi = Id\Psi$, so $\Psi \in D(\overline{Id \restriction_D})$ and $\overline{Id \restriction_D}\Psi = Id\Psi$. Hence, $\overline{Id \restriction_D} = Id$. Since Id is self-adjoint, the lemma is proven. ***o.ε.δ.***

Proposition 73. For each $k \in \mathbb{N}$ let $\mathcal{H}_k$ be a Hilbert space and let $D_k \subseteq \mathcal{H}_k$ be a dense subspace. For some $N \in \mathbb{N}$ and each $k \in \{1, ..., N\}$, let $A_k \in \mathfrak{L}(\mathcal{H}_k)$ be a self-adjoint operator and let $D_k \subseteq D((A_k)^{M_k})$ be such that $(A_k)^{M_k} \restriction_{D_k}$ is essentially self-adjoint. Also, let $Q \in B_{poly}(\mathbb{R}^N)$ such that

$$Q(x_1, ..., x_N) = \sum_{j_1,...,j_N=1}^{M_1,...,M_N} c_{j_1,...,j_N} x_1^{j_1} \cdots x_N^{j_N}$$

for some $M_{\ell} \in \mathbb{N}$, $c_{j_1,....,j_N} \in \mathbb{C}$. Then, given

$$\otimes^{\mathscr{F}}_{j\in\mathbb{N}} D_j := \operatorname{span}\left\{ \bigotimes_{j\in\mathbb{N}} \psi_j \in \bigotimes_{j\in\mathbb{N}} \mathcal{H}_j \;\middle|\; \psi_j \in D_j \ \forall j \in \mathbb{N} \right\},$$

the operator $\Big(Q^{\infty}(A_1, .., A_N),\ \otimes^{\mathscr{F}}_{j\in\mathbb{N}} D_j\Big)$ defined for $\otimes_{j\in\mathbb{N}}\psi_j \in \otimes^{\mathscr{F}}_{j\in\mathbb{N}} D_j$ as

$$Q^{\infty}(A_1, ..., A_N)\Big(\bigotimes_{j\in\mathbb{N}} \psi_j\Big) := \sum_{j_1,\dots,j_N=1}^{M_1,\dots,M_N} c_{j_1,\dots,j_N} (A_1^{j_1}\psi_1) \otimes \cdots \otimes (A_N^{j_N}\psi_N) \otimes \psi_{N+1} \otimes \cdots \tag{B.17}$$

and extended linearly to the rest of $\otimes^{\mathscr{F}}_{j\in\mathbb{N}} D_j$, is an *essentially self-adjoint* operator (on $\bigotimes_{j\in\mathbb{N}}\mathcal{H}_j$). We denote

$$\sum_{j_1,\dots,j_N=1}^{M_1,\dots,M_N} c_{j_1,\dots,j_N} A_1^{j_1} \otimes \cdots \otimes A_N^{J_N} \otimes Id \otimes \cdots := Q(A_1, ..., A_N) \otimes Id \otimes \cdots := Q^{\infty}(A_1, ..., A_N).$$

◆

Proof: By Prop. 72, $(Q(A_1, ..., A_N),\ \otimes^{\mathscr{F}}_{j\in\{1,\dots,N\}} D_j)$ is an essentially self-adjoint operator on $\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_N$. By Lemma 46 and Proposition 71, $(Id,\ \otimes^{\mathscr{F}}_{j\in\{N+1,N+2,\dots\}} D_j)$ is an essentially self-adjoint operator on $\bigotimes_{j=N+1}^{\infty} \mathcal{H}_j$. But applying again Prop. 72 now for the polynomial $(x_1, x_2) \in \mathbb{R}^2 \mapsto x_1 \cdot x_2$ and the space $(\otimes_{j=1}^{N}\mathcal{H}_j) \otimes (\otimes_{j=N+1}^{\infty}\mathcal{H}_j)$, on the domain

$$(\otimes^{\mathscr{F}}_{j\in\{1,\dots,N\}} D_j) \otimes^{\mathscr{F}} (\otimes^{\mathscr{F}}_{j\in\{N+1,N+2,\dots\}} D_j) :=$$

$$= \operatorname{span}\left\{ \Psi_1 \otimes \Psi_2 \;\middle|\; \begin{array}{c} \Psi_1 = \sum_{j=1}^{M_1} c_j \psi_1^j \otimes \cdots \otimes \psi_N^j \text{ and } \Psi_2 = \sum_{j=1}^{M_2} d_j \psi_{N+1}^j \otimes \psi_{N+2}^j \otimes \cdots \\ \text{for some } c_j, d_j \in \mathbb{C},\ M_j \in \mathbb{N},\ \psi_k^j \in D_k \end{array} \right\} =$$

$$\overset{\left(\substack{\text{linearity of } \otimes \\ \text{as in Prop. 53}}\right)}{=} \operatorname{span}\Big\{ (\psi_1 \otimes \cdots \otimes \psi_N) \otimes (\psi_{N+1} \otimes \cdots) \;\Big|\; \psi_k \in D_k \ \ \forall k \in \mathbb{N} \Big\},$$

the operator $Q(A_1, ..., A_N) \otimes Id$ is an essentially self-adjoint operator (on $(\otimes_{j=1}^{N}\mathcal{H}_j) \otimes (\otimes_{j=N+1}^{\infty}\mathcal{H}_j)$).

Finally, using the unitary given by the associativity Theorem 26, we get what we claimed.

ο.ε.δ.

Next, since we will be interested in the question of reducibility for the CCR representation operators, before delving with them, we check the following generality.

Proposition 74. Any $Q^{\infty}(A_1, ..., A_N)$ as in Prop. 73 is *reduced* by every layer $\otimes^{\mathfrak{C}}_{j\in\mathbb{N}}\mathcal{H}_j$ and the reduced part $Q^{\infty}(A_1, ..., A_N)|_{\mathfrak{C}}$ is an essentially self-adjoint operator (on $\otimes^{\mathfrak{C}}_{j\in\mathbb{N}}\mathcal{H}_j$). In particular, using $\otimes_{j\in\mathbb{N}}\mathcal{H}_j = \oplus_{\mathfrak{C}\in\Gamma}(\otimes^{\mathfrak{C}}_{j\in\mathbb{N}}\mathcal{H}_j)$,

$$Q^{\infty}(A_1, ..., A_N) = \oplus_{\mathfrak{C}\in\Gamma}\Big(Q^{\infty}(A_1, ..., A_N)|_{\mathfrak{C}}\Big)$$

with domain $\otimes^{\mathscr{F}}_{j\in\mathbb{N}} D_j = \oplus^{\mathscr{F}}_{\mathfrak{C}\in\Gamma} D\Big(Q^{\infty}(A_1, ..., A_N)|_{\mathfrak{C}}\Big)$. Moreover, the closure $\overline{Q^{\infty}(A_1, ..., A_N)}$ is also reduced by each $\otimes^{\mathfrak{C}}_{j\in\mathbb{N}}\mathcal{H}_j$ and

$$\overline{Q^{\infty}(A_1, ..., A_N)|_{\mathfrak{C}}} = \overline{Q^{\infty}(A_1, ..., A_N)}|_{\mathfrak{C}}. \tag{B.18}$$

◆

Proof: • Let $\Psi \in \otimes^{\mathscr{F}}_{j\in\mathbb{N}} D_j = D(Q^{\infty}(A_1, ..., A_N))$. Then, $\Psi = \sum_{\ell=1}^{n} \alpha_\ell (\otimes_{k\in\mathbb{N}}\psi_k^\ell)$ for some $n \in \mathbb{N}$, $(\alpha_\ell)_\ell \subseteq \mathbb{C}$ and $\psi_k^\ell \in D_k$. Given $P^{\mathfrak{C}_0}$ is the orthogonal projector to an arbitrary $\mathfrak{C}_0 \in \Gamma$ layer,

$$P^{\mathfrak{C}_0}\Psi = \sum_{\ell\in J_{\mathfrak{C}_0}} \alpha_\ell \bigotimes_{k\in\mathbb{N}} \psi_k^\ell \quad \text{for} \quad J_{\mathfrak{C}_0} := \left\{ \ell\in\{1,...,n\} \;\middle|\; \bigotimes_{k\in\mathbb{N}} \psi_k^\ell \in \mathfrak{C}_0 \right\}.$$

As such, by definition, $P^{\mathfrak{C}_0}\Psi \in \otimes^{\mathscr{F}}_{j\in\mathbb{N}} D_j$. Thus, $P^{\mathfrak{C}_0}(\otimes^{\mathscr{F}}_{j\in\mathbb{N}} D_j) \subseteq \otimes^{\mathscr{F}}_{j\in\mathbb{N}} D_j$. Moreover, this also shows that in particular, $P^{\mathfrak{C}_0}(\otimes^{\mathscr{F}}_{j\in\mathbb{N}} D_j) = \otimes^{\mathscr{F}}_{j\in\mathbb{N}} D_j \cap \otimes^{\mathfrak{C}_0}_{j\in\mathbb{N}} \mathcal{H}_j$.

- In addition,

$$P^{\mathfrak{C}_0} Q^\infty(A_1,...,A_N)\Psi = P^{\mathfrak{C}_0}\Big(\sum_{\ell=1}^{n} \sum_{j_1,...,j_N=1}^{M_1,...,M_N} c_{j_1,...,j_N} \alpha_\ell (A_1^{j_1}\psi_1^\ell) \otimes \cdots \otimes (A_N^{j_N}\psi_N^\ell) \otimes \psi_{N+1}^\ell \otimes \cdots \Big) \overset{(\star)}{=}$$

$$= \sum_{\ell\in J_{\mathfrak{C}_0}} \sum_{j_1,...,j_N=1}^{M_1,...,M_N} c_{j_1,...,j_N} \alpha_\ell (A_1^{j_1}\psi_1^\ell) \otimes \cdots \otimes (A_N^{j_N}\psi_N^\ell) \otimes \psi_{N+1}^\ell \otimes \cdots = Q^\infty(A_1,...,A_N) P^{\mathfrak{C}_0}\Psi.$$

In $(\star)$ we used that the belonging to an $\approx$-class only depends on the asymptotic tail of an elementary tensor product, namely, that $\{\ell\in\{1,...,n\} \mid Q^\infty(A_1,...,A_N)(\otimes_{k\in\mathbb{N}}\psi_k^\ell) \in \mathfrak{C}_0\} = J_{\mathfrak{C}_0}$. This proves that $Q^\infty(A_1,...,A_N)$ is reduced by $\otimes^{\mathfrak{C}_0}_{j\in\mathbb{N}}\mathcal{H}_j$. Since $\mathfrak{C}_0 \in \Gamma$ was arbitrary, then $Q^\infty(A_1,...,A_N)$ is reduced by every layer $\otimes^{\mathfrak{C}}_{j\in\mathbb{N}}\mathcal{H}_j$.

- Now, noting that

$$D\Big(Q^\infty(A_1,...,A_N)|_{\mathfrak{C}}\Big) := D\Big(Q^\infty(A_1,...,A_N)\Big) \cap \otimes^{\mathfrak{C}}_{j\in\mathbb{N}}\mathcal{H}_j \overset{\text{(by def)}}{=} \otimes^{\mathscr{F}}_{j\in\mathbb{N}} D_j \cap \otimes^{\mathfrak{C}}_{j\in\mathbb{N}}\mathcal{H}_j \overset{\text{(shown)}}{=} P^{\mathfrak{C}} \otimes^{\mathscr{F}}_{j\in\mathbb{N}} D_j,$$

we get that $\otimes^{\mathscr{F}}_{j\in\mathbb{N}} D_j = \oplus^{\mathscr{F}}_{\mathfrak{C}\in\Gamma} D\Big(Q^\infty(A_1,...,A_N)|_{\mathfrak{C}}\Big)$. This, together with the orthogonal decomposition $\otimes_{j\in\mathbb{N}}\mathcal{H}_j \cong \oplus_{\mathfrak{C}\in\Gamma}(\otimes^{\mathfrak{C}}_{j\in\mathbb{N}}\mathcal{H}_j)$ (see the unitary identification of Prop. 15) trivially yields the following block-diagonal decomposition:

$$Q^\infty(A_1,...,A_N) = \oplus_{\mathfrak{C}\in\Gamma}\Big(Q^\infty(A_1,...,A_N)|_{\mathfrak{C}}\Big)$$

—where, recall that the action of the direct sum of operators was defined as the sector-wise action (see Prop. 18).

- By Thm. 1.47.(v) in (Arai, 2018), $\overline{Q^\infty(A_1,...,A_N)}$ is reduced by all $\otimes^{\mathfrak{C}}_{j\in\mathbb{N}}\mathcal{H}_j$ such that $\overline{Q^\infty(A_1,...,A_N)|_{\mathfrak{C}}} = \overline{Q^\infty(A_1,...,A_N)}|_{\mathfrak{C}}$. Moreover, because $\overline{Q^\infty(A_1,...,A_N)}$ is self-adjoint, by Thm 1.47.(ix) in (Arai, 2018), $\overline{Q^\infty(A_1,...,A_N)}|_{\mathfrak{C}}$ is self-adjoint. Hence, $\overline{Q^\infty(A_1,...,A_N)|_{\mathfrak{C}}}$ is self-adjoint and thus, $Q^\infty(A_1,...,A_N)|_{\mathfrak{C}}$ is essentially self-adjoint. ***o.ε.δ.***

Finally, let us employ all the above results for our particular case study.

Proposition 75. For each $\alpha,\beta\in\mathbb{R}$, B_{Her} is a set of *analytic vectors* of the operator $(\alpha\,\hat{q} + \beta\,\hat{p},\ B_{Her})$, which is defined for $\psi\in B_{Her}$ as

$$((\alpha\,\hat{q} + \beta\,\hat{p})\,\psi)(x) := \alpha\; x\psi(x) - \beta\; i\hbar\frac{d\psi}{dx}(x) \qquad \forall x\in\mathbb{R}. \tag{B.19}$$

In particular, $(\alpha\,\hat{q} + \beta\,\hat{p},\ B_{Her})$ (and any of its restrictions to a symmetric densely defined operator that leaves its domain invariant) is *essentially self-adjoint.* ♦

Proof: To prove the claims, we only need to modify conveniently the proof given for $\hat{q}$ in Example 2 of §X.6 in (Reed and Simon, 1975). Consider the operators

$$\hat{a} := \frac{1}{\sqrt{2}}\Big(x + \frac{d}{dx}\Big) \quad \text{and} \quad \hat{a}^{\dagger} := \frac{1}{\sqrt{2}}\Big(x - \frac{d}{dx}\Big), \tag{B.20}$$

defined on B_{Her} as in Lemma 43 (for $\omega = 1$). Let $(\phi_n)_{n\in\mathbb{N}_0} \subseteq L^2(\mathbb{R}, dx)$ be the ONB of $L^2(\mathbb{R}, dx)$ given by Lemma 43 (for $\omega = 1$). As proven in that lemma, $\forall n \in \mathbb{N}_0$, $\hat{a}\phi_n$, $\hat{a}^{\dagger}\phi_n \in B_{Her}$, so, $\phi_n \in D(\hat{a}^k) \cap D(\hat{a}^{\dagger k})$ for all $k \in \mathbb{N}$. As such, it is meaningful to apply $\hat{a}, \hat{a}^{\dagger}$ arbitrarily often on ϕ_n. In particular, by equation (X.61) in (Reed and Simon, 1975), for any $k, n \in \mathbb{N}_0$

$$\Big\|\hat{a}^{\#_1} \cdots \hat{a}^{\#_k}\phi_n\Big\| \leq \sqrt{(n+k)!} \qquad \forall \#_j \in \{1, \dagger\}. \tag{B.21}$$

Now, defining $(\hat{q},\ B_{Her})$, $(\hat{p},\ B_{Her})$ as the $(\alpha, \beta) = (1, 0)$ and $(0, 1)$ cases of (B.19), it is immediate to see that $\hat{q} = \frac{1}{\sqrt{2}}(\hat{a} + \hat{a}^{\dagger})$ and $\hat{p} = \frac{-i\hbar}{\sqrt{2}}(\hat{a} - \hat{a}^{\dagger})$. Plugging them in $\alpha\,\hat{q} + \beta\,\hat{p}$ we get $(\alpha\ \hat{q} + \beta\ \hat{p}) = \frac{1}{\sqrt{2}}(\overline{z}\,\hat{a} + z\,\hat{a}^{\dagger})$ for $z := \alpha + i\hbar\,\beta$. Next, developing $(\overline{z}\hat{a} + z\hat{a}^{\dagger})^k$ we get 2^k terms like $c_\ell\hat{a}^{\#_1^\ell} \cdots \hat{a}^{\#_k^\ell}$ with $\#_j^\ell \in \{1, \dagger\}$ such that $c_\ell = z^r\ \overline{z}^j$ for some $r, j \in \mathbb{N}_0 : r + j = k$. In particular, $|c_\ell| = |z|^k$. Altogether,

$$\Big\|(\alpha\ \hat{q} + \beta\hat{p})^k\phi_n\Big\| = \frac{1}{2^{k/2}}\Big\|(\overline{z}\hat{a} + z\hat{a}^{\dagger})^k\phi_n\Big\| = \frac{1}{2^{k/2}}\left\|\sum_{\ell=1}^{2^k} c_\ell\hat{a}^{\#_1^\ell} \cdots \hat{a}^{\#_k^\ell}\phi_n\right\| \overset{\text{(trgl.ineq.)}}{\leq}$$

$$\leq \frac{1}{2^{k/2}}\sum_{\ell=1}^{2^k}|z|^k\Big\|\hat{a}^{\#_1^\ell} \cdots \hat{a}^{\#_k^\ell}\phi_n\Big\| \overset{\text{(B.21)}}{\leq} 2^{k/2}|z|^k\sqrt{(n+k)!}.$$

Hence, for all $t > 0$,

$$\sum_{k=0}^{\infty}\frac{\Big\|(\alpha\hat{q} + \beta\hat{p})^k\phi_n\Big\|}{k!}t^k \leq \sum_{k=0}^{\infty}\underbrace{\frac{|z|^k 2^{k/2}\sqrt{(n+k)!}}{k!}t^k}_{\gamma_k}. \tag{B.22}$$

Finally, since

$$\lim_{k\to\infty}\frac{\gamma_{k+1}}{\gamma_k} = \lim_{k\to\infty}\frac{|z|\ 2^{1/2}\sqrt{(n+k+1)}}{k+1}t \overset{\text{(for some } C \geq 0)}{=} \lim_{k\to\infty}\frac{C}{\sqrt{k}} = 0,$$

by the well-known "ratio test" the r.h.s series in (B.22) converges and thus, so does the l.h.s one. By definition, this implies that ϕ_n is an analytic vector of $\alpha\,\hat{x} + \beta\,\hat{p}$. As a consequence, applying the triangle inequality, any linear combination of $(\phi_n)_{n\in\mathbb{N}_0}$ is also analytic. Hence, $\text{span}\{\phi_n\}_{n\in\mathbb{N}} = B_{Her}$ are all analytic vectors of $\alpha\hat{q} + \beta\hat{p}$.

- We miss the last claim. Note that $(\alpha\,\hat{q} + \beta\,\hat{p},\ B_{Her})$ is symmetric: for $\psi, \varphi \in B_{Her}$,

$$\langle\varphi,\ (\alpha\,\hat{q} + \beta\,\hat{p})\psi\rangle = \int_{x\in\mathbb{R}}\overline{\varphi(x)}\,\alpha x\,\psi(x)dx - i\hbar\beta\int_{x\in\mathbb{R}}\overline{\varphi(x)}\,\frac{d}{dx}\,\psi(x)dx \overset{\left(\substack{\text{integrate by parts} \\ \text{\& use } \varphi,\psi\in\mathcal{S}(\mathbb{R})}\right)}{=}$$

$$= \int_{x\in\mathbb{R}}\overline{\alpha x\varphi(x)}\,\psi(x)dx + \int_{x\in\mathbb{R}}\overline{(-\beta\,i\hbar)\frac{d}{dx}\varphi(x)}\,\psi(x)dx = \langle(\alpha\,\hat{q} + \beta\,\hat{p})\varphi, \psi\rangle.$$

Then, since B_{Her} is dense in $L^2(\mathbb{R}, dx)$, by Nelson's theorem (see Theorem X.39 in (Reed and Simon, 1975)), the operator $(\alpha\,\hat{q} + \beta\,\hat{p},\ B_{Her})$ is essentially self-adjoint. The statement in parenthesis about the restrictions is a literal application of Corollary 2 after X.39 in (Reed and Simon, 1975). ***o.ε.δ.***

Definition 56. Let $\mathcal{H}$ be a *real* separable Hilbert space and let $(e_n)_{n\in\mathbb{N}} \subseteq \mathcal{H}$ be an ONB. Define $\mathcal{W} := \text{span}\{e_n\}_{n\in\mathbb{N}}$ and for each $f \in \mathcal{W}$ (for which there always $\exists M_f \in \mathbb{N}$ such that $f = \sum_{n=1}^{M_f}\langle e_n, f\rangle e_n$), define in Prop. 73's notation the following $\otimes_{k\in\mathbb{N}} L^2(\mathbb{R}, dx)$ operators:

$$\left(\widehat{q}(f) := \sum_{n=1}^{M_f}\langle e_n, f\rangle\ \hat{q}_n,\ \otimes^{\mathscr{F}}_{k\in\mathbb{N}} B_{Her}\right) \quad \text{and} \quad \left(\widehat{p}(f) := \sum_{n=1}^{M_f}\langle e_n, f\rangle\ \hat{p}_n,\ \otimes^{\mathscr{F}}_{k\in\mathbb{N}} B_{Her}\right),$$

where $\hat{q}_n := \underbrace{Id \otimes \cdots \otimes \hat{q}}_{n} \otimes Id \otimes \cdots$ and $\hat{p}_n := \underbrace{Id \otimes \cdots \otimes \hat{p}}_{n} \otimes Id \otimes \cdots$ such that $(\hat{q}\psi_n)(x) := x\psi_n(x)$ and $(\hat{p}\psi_n)(x) := -i\hbar\frac{d\psi_n}{dx}(x)$ for $\psi_n \in B_{Her}$. That is, consider the operators such that for $\otimes_{j\in\mathbb{N}}\psi_j \in \otimes^{\mathscr{F}}_{j\in\mathbb{N}} B_{Her}$,

$$\hat{q}(f)\Big(\bigotimes_{j\in\mathbb{N}}\psi_j\Big) := \sum_{n=1}^{M_f}\langle e_n, f\rangle \psi_1 \otimes \cdots \otimes (\hat{q}\psi_n) \otimes \psi_{n+1} \otimes \cdots$$

$$\text{and} \quad \hat{p}(f)\Big(\bigotimes_{j\in\mathbb{N}}\psi_j\Big) := \sum_{n=1}^{M_f}\langle e_n, f\rangle \psi_1 \otimes \cdots \otimes (\hat{p}\psi_n) \otimes \psi_{n+1} \otimes \cdots ,$$

linearly extended to the rest of $\otimes^{\mathscr{F}}_{j\in\mathbb{N}} B_{Her}$. ♦

Corollary 26. Both $\hat{q}(f)$ and $\hat{p}(f)$ are *essentially self-adjoint* operators on $\bigotimes_{k\in\mathbb{N}} L^2(\mathbb{R}, dx)$ and they are *reduced* by every layer $\otimes^{\mathfrak{C}}_{k\in\mathbb{N}} L^2(\mathbb{R}, dx)$, $\mathfrak{C} \in \Gamma$. ♦

Proof: $(\hat{q}, B_{Her})$ and $(\hat{p}, B_{Her})$ are the $(\alpha, \beta) = (1, 0)$ and $(0, 1)$ cases of Prop. 75. Then, given the polynomial $Q(x_1, ..., x_N) := \sum_{n=1}^{M_f}\langle e_n, f\rangle\ x_n$, the claim on self-adjointness is given by Prop. 73, while that of reducibility is given by Prop. 74. ***o.ε.δ.***

Theorem 29. $\left(\bigotimes_{k\in\mathbb{N}} L^2(\mathbb{R}, dx),\ \otimes^{\mathscr{F}}_{k\in\mathbb{N}} B_{Her},\ \{\hat{q}(f),\ \hat{p}(f)\}_{f\in\mathcal{W}}\right)$ is a *reducible representation of the Heisenberg CCR*, reduced by every layer $\otimes^{\mathfrak{C}}_{j\in\mathbb{N}} L^2(\mathbb{R}, dx)$, $\mathfrak{C} \in \Gamma$. ♦

Proof: We check all the items of Def. 46.

- **Item (i):** First, by definition, $\forall f \in \mathcal{W}$, $\otimes^{\mathscr{F}}_{j\in\mathbb{N}} B_{Her} = D(\hat{q}(f)) = D(\hat{p}(f))$.

On the other hand, since trivially, both $\hat{q}B_{Her} \subseteq \otimes^{\mathscr{F}}_{j\in\mathbb{N}} B_{Her}$ and $\hat{p}B_{Her} \subseteq \otimes^{\mathscr{F}}_{j\in\mathbb{N}} B_{Her}$, the application of $\hat{q}_j$ and $\hat{p}_j$ on $\psi_1 \otimes \psi_2 \otimes \cdots \in \otimes^{\mathscr{F}}_{j\in\mathbb{N}} B_{Her}$ keeps their belonging to $\otimes^{\mathscr{F}}_{j\in\mathbb{N}} B_{Her}$. Linear combinations of such applications constitute $\hat{q}(f)$ and $\hat{p}(f)$, so, because $\otimes^{\mathscr{F}}_{j\in\mathbb{N}} B_{Her}$ is a vector space, $\hat{q}(f)\Big(\otimes^{\mathscr{F}}_{j\in\mathbb{N}} B_{Her}\Big) \subseteq \otimes^{\mathscr{F}}_{j\in\mathbb{N}} B_{Her}$ and $\hat{p}(f)\Big(\otimes^{\mathscr{F}}_{j\in\mathbb{N}} B_{Her}\Big) \subseteq \otimes^{\mathscr{F}}_{j\in\mathbb{N}} B_{Her}$.

Finally, because $\hat{q}(f), \hat{p}(f)$ were proven in Cor. 26 to be essentially self-adjoint, they must be symmetric as well.

- **Item (ii):** Let there be arbitrary $f, g \in \mathcal{W}$, $\alpha, \beta \in \mathbb{R}$, and $\Psi \in \otimes^{\mathscr{F}}_{j\in\mathbb{N}} B_{Her}$, such that $\Psi = \sum_{\ell=1}^{M} c_\ell \psi_1^\ell \otimes \psi_2^\ell \otimes \cdots$ for some $M \in \mathbb{N}$, $c_\ell \in \mathbb{C}$ and $\psi_j^\ell \in B_{Her}$. Then,

$$\hat{q}(\alpha f + \beta g)\Psi = \sum_{\ell=1}^{M} c_\ell \sum_{n=1}^{\max(M_f, M_g)} \langle e_n, \alpha f + \beta g\rangle\ \psi_1^\ell \otimes \cdots \otimes (\hat{q}\psi_n^\ell) \otimes \psi_{n+1} \otimes \cdots \overset{(\langle e_n,\cdot\rangle \text{ is linear})}{=}$$

$$= \alpha \sum_{\ell=1}^{M} c_\ell \sum_{n=1}^{\max(M_f, M_g)} \langle e_n, f\rangle\ \psi_1^\ell \otimes \cdots \otimes (\hat{q}\psi_n^\ell) \otimes \psi_{n+1} \otimes \cdots +$$

$$+\beta \sum_{\ell=1}^{M} c_\ell \sum_{n=1}^{\max(M_f,M_g)} \langle e_n, g\rangle \; \psi_1^\ell \otimes \cdots \otimes (\hat{q}\psi_n^\ell) \otimes \psi_{n+1} \otimes \cdots \overset{\text{(by def)}}{=}$$

$$= (\alpha \; \hat{q}(f) + \beta \; \hat{p}(g))\Psi.$$

The same holds for $\hat{p}(\cdot)$ if we change q by p in the last computation.

- **Item (iii):** For arbitrary $f, g \in \mathcal{W}$ and $\Psi \in \otimes^{\mathscr{F}}_{j\in\mathbb{N}} B_{Her}$ (as above), note that

$$[\hat{q}_n, \hat{p}_m](\psi_1^\ell \otimes \psi_2^\ell \otimes \cdots) = \begin{cases} \text{if } n \neq m: & \begin{pmatrix} \psi_1^\ell \otimes \cdots \otimes (\hat{p}\psi_m^\ell) \otimes \cdots \otimes (\hat{q}\psi_n^\ell) \otimes \psi_{n+1} \otimes \cdots - \\ -\psi_1^\ell \otimes \cdots \otimes (\hat{p}\psi_m^\ell) \otimes \cdots \otimes (\hat{q}\psi_n^\ell) \otimes \psi_{n+1} \otimes \cdots \end{pmatrix} = 0 \\ \text{if } n = m: & \Big(\psi_1^\ell \otimes \cdots \otimes (\underbrace{[\hat{q}, \hat{p}]\psi_n^\ell}_{i\hbar\psi_n^\ell}) \otimes \psi_{n+1} \otimes \cdots\Big) = i(\psi_1^\ell \otimes \psi_2^\ell \otimes \cdots) \end{cases} =$$

$$= i\hbar\delta_{nm}\psi_1^\ell \otimes \psi_2^\ell \otimes \cdots .$$

Therefore, we get, considering $\hbar = 1$,

$$[\hat{q}(f), \hat{p}(g)]\Psi = \hat{q}(f)\hat{p}(g)\Psi - \hat{p}(f)\hat{q}(g)\Psi = \sum_{\ell=1}^{M} c_\ell \sum_{n=1}^{M_f} \sum_{m=1}^{M_g} \langle e_n, f\rangle\langle e_m, g\rangle [\hat{q}_j, \hat{p}_j](\psi_1^\ell \otimes \psi_2^\ell \otimes \cdots) =$$

$$= \sum_{\ell=1}^{M} c_\ell \underbrace{\sum_{n=1}^{\min(M_f,M_g)} \langle e_n, f\rangle\langle e_m, g\rangle}_{\langle f,g\rangle} \; i \; \psi_1^\ell \otimes \psi_2^\ell \otimes \cdots = i\langle f, g\rangle\Psi.$$

Next, note that trivially, $[\hat{q}_n, \hat{q}_m](\psi_1^\ell \otimes \psi_2^\ell \otimes \cdots) = 0 = [\hat{p}_n, \hat{p}_m](\psi_1^\ell \otimes \psi_2^\ell \otimes \cdots)$. Hence, by linearity $[\hat{q}_n, \hat{q}_m]\Psi = [\hat{p}_n, \hat{p}_m]\Psi = 0$, such that, $[\hat{q}(f), \hat{q}(g)]\Psi = 0 = [\hat{p}(f), \hat{p}(g)]\Psi$.

- Finally, it is a *reducible* representation because by Cor. 26, each $\otimes^{\mathfrak{C}}_{j\in\mathbb{N}} L^2(\mathbb{R}, dx)$ reduces $\hat{q}(f)$ and $\hat{p}(f)$ for all $f \in \mathcal{W}$. ***o.ε.δ.***

B.2.2 The Weyl CCR Representation

In order to learn that the above Heisenberg CCR representation generates (via the complex exponential) also a Weyl representation, we first need to understand the interplay between SCOPUGs, their generators and their lifts to the ITP. In particular, to prove part of the Weyl relations for the SCOPUGs generated by $\overline{\hat{q}(f)}$ and $\overline{\hat{p}(f)}$, we will harness the commutation relations of the generators (essentially given by the previous section). But, for that, one needs to prove first that the generators commute strongly, and this in turn requires us to lift analytical vectors to the ITP. With that purpose, we start with the following technical result.

Lemma 47. Let $(a_n(k))_{n\in\{1,\dots,N\},\, k\in\mathbb{N}} \subseteq \mathbb{C}$ for some $N \in \mathbb{N}$. Then, for all $K \in \mathbb{N}$,

$$\prod_{n=1}^{N} \Big(\sum_{k_n=0}^{K} a_n(k_n)\Big) = \sum_{k=0}^{N\cdot K} \sum_{\substack{k_1+\cdots+k_N=k \\ 0\le k_1,\dots,k_N\le K}} \prod_{n=1}^{N} a_n(k_n). \tag{B.23}$$

If for all $n \in \{1, \dots, N\}$, $\exists \lim_{K\to\infty} \sum_{k=0}^{K} a_n(k) = b_n \in \mathbb{C}$, then

$$\lim_{K\to\infty} \prod_{n=1}^{N} \Big(\sum_{k_n=0}^{K} a_n(k_n)\Big) = \prod_{n=1}^{N} b_n.$$

By (B.23), this is the same as saying that

$$\lim_{K\to\infty}\sum_{k=0}^{N\cdot K}\sum_{\substack{k_1+\cdots+k_N=k\\0\leq k_1,\ldots,k_N\leq K}}\prod_{n=1}^{N}a_n(k_n)=\prod_{n=1}^{N}b_n. \tag{B.24}$$

In particular, if $a_n(k)$ are non-negative real numbers, then (B.24) implies that changing NK by K the limit is preserved, such that

$$\sum_{k=0}^{\infty}\sum_{\substack{k_1+\cdots+k_N=k\\k_1,\ldots,k_N\in\mathbb{N}_0}}\prod_{n=1}^{N}a_n(k_n)=\prod_{n=1}^{N}b_n. \tag{B.25}$$

♦

Proof: We first prove equation (B.23):

$$\prod_{n=1}^{N}\Big(\sum_{k_n=0}^{K}a_n(k_n)\Big)=(a_1(0)+\cdots+a_1(K))\cdots(a_N(0)+\cdots+a_N(K))\overset{\text{(develop prod.)}}{=}$$

$$=\sum_{k_1,\ldots,k_N\in\{0,\ldots,K\}}a_1(k_1)\cdots a_N(k_N)\overset{\text{(rewrite }\sum)}{=}\sum_{k=0}^{N\cdot K}\sum_{\substack{k_1+\cdots+k_N=k\\0\leq k_1,\ldots,k_N\leq K}}\prod_{n=1}^{N}a_n(k_n). \tag{B.26}$$

Next, by definition of $\lim_{K\to\infty}\sum_{k=0}^{K}a_n(k)=b_n$ for each $n\in\{1,\ldots,N\}$, given any $\delta_n>0$, there exists $K^{\delta_n}\in\mathbb{N}$ such that $|b_n-\sum_{k=0}^{K}a_n(k)|\leq\delta_n$ for all $K\geq K^{\delta_n}$. Now, for an arbitrary fixed $\varepsilon>0$, choose recursively first $\delta_1:=\frac{\varepsilon}{N|b_2\cdots b_N|}$ and then, for $r\in\{2,\ldots,N-1\}$,

$$\delta_r:=\frac{\varepsilon}{N\prod_{n=1}^{r-1}|b_n+\delta_n|\,|\prod_{n=r+1}^{N}b_n|}$$

until $\delta_N=\frac{\varepsilon}{N\prod_{n=1}^{N-1}|b_n+\delta_n|}$. Then, we get that $\forall K\geq\max\{K^{\delta_1},\ldots,K^{\delta_N}\}$,

$$\Big|\prod_{n=1}^{N}b_n-\prod_{n=1}^{N}\Big(\sum_{k=0}^{K}a_n(k)\Big)\Big|=\Big|b_1\cdots b_N-\Big(\sum_{k_1=0}^{K}a_1(k_1)\Big)\cdots\Big(\sum_{k_N=0}^{K}a_N(k_N)\Big)\Big|\overset{(\star)}{\leq}$$

$$\leq\Big|b_1-\Big(\sum_{k_1=0}^{K}a_1(k_1)\Big)\Big|\cdot\Big|b_2\cdots b_N\Big|+\sum_{r=2}^{N-1}\left(\prod_{n=1}^{r-1}\Big|\sum_{k_n=0}^{K}a_n(k_n)\Big|\cdot\Big|b_r-\Big(\sum_{k_r=0}^{K}a_r(k_r)\Big)\Big|\cdot\prod_{n=r+1}^{N}|b_n|\right)+$$

$$+\prod_{n=1}^{N-1}\Big|\sum_{k_r=0}^{K}a_r(k_r)\Big|\cdot\Big|b_N-\Big(\sum_{k_N=0}^{K}a_N(k_N)\Big)\Big|\overset{\text{(by def. of }\delta_r)}{\leq}\varepsilon.$$

In $(\star)$ we added and subtracted first $(\sum_{k_1=0}^{K}a_1(k_1))\cdot b_2\cdots b_N$, then $(\sum_{k_1=0}^{K}a_1(k_1))\cdot(\sum_{k_2=0}^{K}a_2(k_2))\cdot b_3\cdots b_N$, and so on, until $(\sum_{k_1=0}^{K}a_1(k_1))\cdots(\sum_{k_{N-1}=0}^{K}a_{N-1}(k_{N-1}))\cdot b_N$. Then we applied the triangle inequality finitely many times.

o.ε.δ.

Proposition 76. Let there be for each $j\in\mathbb{N}$ a densely defined operator (A_j,D_j) acting on the Hilbert space $\mathcal{H}_j$, with D_j made of analytic vectors of A_j. Then, for any $(c_n)_{n=1}^{N}\subset\mathbb{C}$, the set $\otimes_{j\in\mathbb{N}}^{\mathscr{F}}D_j$ are all *analytic vectors* of the operator $(\sum_{n=1}^{N}c_nA_n,\ \otimes_{j\in\mathbb{N}}^{\mathscr{F}}D_j)$ —defined as in Prop. 73 for the polynomial $Q(x_1,\ldots,x_N):=\sum_{n=1}^{N}c_nx_n$.

♦

Proof: Let there be a generic elementary tensor product $\Psi \in \otimes^{\mathscr{F}}_{j\in\mathbb{N}} D_j \setminus \{0\}$, say, $\Psi = \otimes_{j\in\mathbb{N}}\psi_j$ with $\psi_j \in D_j$. Then, using the multinomial formula in $(\star)$,

$$\left\|\Big(\sum_{n=1}^{N} c_n A_n\Big)^k \Psi\right\| \overset{(\star)}{=} \left\| \sum_{\substack{k_1+\cdots+k_N=k\\ k_1,\dots,k_N\in\mathbb{N}_0}} \frac{k!}{k_1!\cdots k_N!} c_1^{k_1}\cdots c_N^{k_N}\,(A_1^{k_1}\psi_1)\otimes\cdots\otimes(A_N^{k_N}\psi_N)\otimes\psi_{N+1}\otimes\cdots \right\| \overset{\text{(trgl.ineq)}}{\le}$$

$$\le \sum_{\substack{k_1+\cdots+k_N=k\\ k_1,\dots,k_N\in\mathbb{N}_0}} \frac{k!}{k_1!\cdots k_N!} |c_1|^{k_1}\cdots|c_N|^{k_N} \left\|(A_1^{k_1}\psi_1)\otimes\cdots\otimes(A_N^{k_N}\psi_N)\otimes\psi_{N+1}\otimes\cdots\right\| =$$

$$= \underbrace{\frac{\|\Psi\|}{\|\psi_1\otimes\cdots\otimes\psi_N\|}}_{=:C_\Psi} \sum_{\substack{k_1+\cdots+k_N=k\\ k_1,\dots,k_N\in\mathbb{N}_0}} k! \prod_{n=1}^{N} \frac{|c_n|^{k_n}}{k_n!}\left\|A_n^{k_n}\psi_n\right\|. \tag{B.27}$$

This leaves for $t>0$,

$$\sum_{k=0}^{\infty} \frac{\left\|\Big(\sum\limits_{n=1}^{N} c_n A_n\Big)^k \Psi\right\|}{k!} t^k \overset{\text{(B.27)}}{\le} C_\Psi \sum_{k=0}^{\infty} \sum_{\substack{k_1+\cdots+k_N=k\\ k_1,\dots,k_N\in\mathbb{N}_0}} k! \prod_{n=1}^{N} \frac{(|c_n|t)^{k_n}}{k_n!}\left\|A_n^{k_n}\psi_n\right\|. \tag{B.28}$$

- $\psi_n \in D_n$ is an analytic vector for A_n by hypothesis and therefore, $\exists t_n \in (0,+\infty)$ such that

$$\sum_{k=0}^{\infty} \frac{\left\|A_n^k\psi_n\right\|}{k!} t^k \quad \text{converges for all } t\in(0,t_n).$$

Hence, fixing $t_0 := \min_{n\in\{1,\dots,N\}}\{\frac{t_n}{|c_n|}\}$, for each $n\in\{1,\dots,N\}$,

$$\sum_{k_n=0}^{\infty} \underbrace{\frac{(|c_n|t)^{k_n}}{k!}\left\|A_n^{k_n}\psi_n\right\|}_{=:a_n(k_j)} \quad \text{converges for all } t\in(0,t_0).$$

But then, by Lemma 47 (in particular by (B.25)), this implies that (B.28) converges. As such, Ψ is an analytic vector of $(\sum_{n=1}^{N} c_n A_n)$.

- Finally, using the triangle inequality it is immediate to see that any vector in the span of elementary tensor products like Ψ, namely, any vector in $\otimes^{\mathscr{F}}_{j\in\mathbb{N}} D_j$ is analytic as well. ***o.ε.δ.***

Then, as mentioned, we get the following strong commutativity result:

Proposition 77. Let there be for each $j\in\mathbb{N}$ a densely defined symmetric operator (A_j, D_j) acting on the Hilbert space $\mathcal{H}_j$, with D_j made of analytic vectors of A_j.[4] Then, for every $N, M\in\mathbb{N}$ and $(\alpha_n)_{n=1}^N, (\beta_n)_{n=1}^M \subseteq \mathbb{R}$, the closures of $(\sum_{n=1}^N \alpha_n A_n,\ \otimes^{\mathscr{F}}_{j\in\mathbb{N}} D_j)$ and $(\sum_{m=1}^M \beta_m A_m,\ \otimes^{\mathscr{F}}_{j\in\mathbb{N}} D_j)$ *commute strongly* with each other. Moreover, the reduced parts by $\mathfrak{C}\in\Gamma$ (see Prop. 74), $\Big(\sum_{n=1}^N \alpha_n A_n\Big)|_{\mathfrak{C}}$ and $\Big(\sum_{n=1}^M \beta_m A_m\Big)|_{\mathfrak{C}}$ also commute strongly. ♦

Proof: First, we note that $\sum_{n=1}^N \alpha_n A_n$ and $\sum_{m=1}^M \beta_m A_m$ are symmetric because they are es-

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[4] This implies, by Nelson's theorem, that (A_j, D_j) is essentially self-adjoint.

sentially self-adjoint. Second, by Prop. 76, $\otimes^{\mathscr{F}}_{j\in\mathbb{N}} D_j$ are all analytic vectors of both $\sum_{n=1}^{N}\alpha_n A_n$ and $\sum_{m=1}^{M}\beta_m A_m$. And third, for any $\Psi \in \otimes^{\mathscr{F}}_{j\in\mathbb{N}} D_j$, there exist $c_\ell \in \mathbb{C}$, $r\in\mathbb{N}$, $\psi^\ell_j \in D_j$ such that $\Psi = \sum_{\ell=1}^{r} c_\ell \otimes_{j\in\mathbb{N}} \psi^\ell_j$, causing

$$\Big(\sum_{n=1}^{N}\alpha_n A_n\Big)\Big(\sum_{m=1}^{M}\beta_m A_m\Big)\Psi =$$

$$=\sum_{\ell=1}^{r}\sum_{n=1}^{N}\sum_{m=1}^{M} c_\ell\alpha_n\beta_m \begin{cases}\text{if } n\neq m, & \psi^\ell_1\otimes\cdots\otimes(A_n\psi_n)\otimes\psi_{n+1}\otimes\cdots\otimes(A_m\psi_m)\otimes\psi_{m+1}\otimes\cdots\\ \text{if } n=m, & \psi^\ell_1\otimes\cdots\otimes(A_n^2\psi_n)\otimes\psi_{n+1}\otimes\cdots\end{cases} =$$

$$=\Big(\sum_{m=1}^{M}\beta_m A_m\Big)\Big(\sum_{n=1}^{N}\alpha_n A_n\Big)\Psi.$$

That is, the two operators commute in their common domain. The three together imply, by Thm. 7.18 in (Schmüdgen, 2012) that the closures of $\Big(\sum_{n=1}^{N}\alpha_n A_n\Big)$ and $\Big(\sum_{m=1}^{M}\beta_m A_m\Big)$ commute strongly.

- The claim on the strong commutativity of the reduced parts is a trivial application of Theorem 1.47.(xii) and (v) of (Arai, 2018). ***o.ε.δ.***

Corollary 27. For every $f\in\mathcal{W}$, *all* vectors in $\otimes^{\mathscr{F}}_{j\in\mathbb{N}} B_{Her}$ are *analytic* for $\hat{q}(f)$ and $\hat{p}(f)$. Moreover, $\{\overline{\hat{q}(f)}\}_{f\in\mathcal{W}}$ and $\{\overline{\hat{p}(f)}\}_{f\in\mathcal{W}}$ are each a family of *strongly commuting* self-adjoint operators reduced by every $\mathfrak{C}\in\Gamma$. In particular, the reduced parts $\{\overline{\hat{q}(f)}|_{\mathfrak{C}}\}_{f\in\mathcal{W}}$ and $\{\overline{\hat{p}(f)}|_{\mathfrak{C}}\}_{f\in\mathcal{W}}$ also make each a strongly commuting family. ♦

Proof: By Prop. 75 and Prop. 76, $\otimes^{\mathscr{F}}_{j\in\mathbb{N}} B_{Her}$ is a set of analytic vectors for any $\hat{q}(f)$, $\hat{p}(f)$. As such, the claims on strong commutativity are immediately given by Prop. 77. ***o.ε.δ.***

(Note that this constitutes an alternative way to prove that the lifted position operators commute strongly! —recall that we proved it in the main text in a different way.)

Now, using the strong commutativity of the generators, we can prove the commutation of the generated SCOPUGs as follows (essential for part of the Weyl relations).

Corollary 28. For any pair of operators as in Prop. 77,

$$\forall t,s\in\mathbb{R},\quad e^{it\overline{\left(\sum_{n=1}^{N}\alpha_n A_n\right)}}\, e^{is\overline{\left(\sum_{m=1}^{M}\beta_m A_m\right)}} = e^{is\overline{\left(\sum_{m=1}^{M}\beta_m A_m\right)}}\, e^{it\overline{\left(\sum_{n=1}^{N}\alpha_n A_n\right)}}. \tag{B.29}$$

Assuming without loss of generality that $N=M$,[5]

$$\overline{\Big(\sum_{n=1}^{\max(N,M)}(\alpha_n+\beta_n)A_n\Big)} \equiv \overline{\Big(\sum_{n=1}^{N}\alpha_n A_n\Big)+\Big(\sum_{m=1}^{M}\beta_m A_n\Big)} = \overline{\overline{\Big(\sum_{n=1}^{N}\alpha_n A_n\Big)}+\overline{\Big(\sum_{m=1}^{M}\beta_m A_n\Big)}} \tag{B.30}$$

and

$$e^{i\overline{\left(\sum_{n=1}^{\max(N,M)}(\alpha_n+\beta_n)A_n\right)}} = e^{i\overline{\left(\sum_{n=1}^{N}\alpha_n A_n\right)}}\, e^{i\overline{\left(\sum_{m=1}^{M}\beta_m A_m\right)}}. \tag{B.31}$$

♦

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[5] We could otherwise put $\alpha_n = 0$ for $n\in\{N,...,M\}$ if $N\le M$ or put $\beta_n=0$ for $n\in\{M,...,N\}$ if $M\le N$.

Proof: Equation (B.29) holds by combining Thm. VIII.13 in (Reed and Simon, 1981) and Prop. 77. As a consequence, (B.30) and (B.31) hold by Rem. 5.33 in (Arai, 2018). □

The only part of the Weyl relations that cannot be proven using this last result (namely, the crossed commutation relation for the exponentials of $\hat{q}(f)$ and $\hat{p}(g)$), can be proven by lifting the Weyl relation of $\hat{q}$ and $\hat{p}$ from $L^2(\mathbb{R}, dx)$. But for that we first need to prove that taking the lift to the ITP and applying the functional calculus commute.

Lemma 48. Given an arbitrary family of Hilbert spaces $(\mathcal{H}_j)_{j\in I}$, let there be a fixed $k \in \mathbb{N}$ and a SCOPUG $\{U(t)\}_{t\in\mathbb{R}}$ on $\mathcal{H}_k$. Then, the lifted family of operators $\{\widehat{U(t)}\}_{t\in\mathbb{R}}$ (in the notation of Prop. 62) is a SCOPUG on $\bigotimes_{j\in I} \mathcal{H}_j$. ♦

Proof: • **(i)** We check that for all $t \in \mathbb{R}$, $\widehat{U(t)}$ is a **unitary** operator. First, denoting the set of elementary tensor products in $\bigotimes_{j\in I} \mathcal{H}_j$ by S, for all $\otimes_{j\in I}\psi_j,\ \bigotimes_{j\in I} \phi_j \in S$,

$$\left\langle \widehat{U(t)}\Big(\bigotimes_{j\in I} \psi_j,\Big)\ \widehat{U(t)}\Big(\bigotimes_{j\in I} \phi_j\Big)\right\rangle = \prod_{j\in I\setminus\{k\}} \langle \psi_j,\, \phi_j\rangle\cdot\langle U(t)\psi_k,\, U(t)\phi_k\rangle = \prod_{j\in I}\langle\psi_j,\phi_j\rangle = \Big\langle \bigotimes_{j\in I} \psi_j,\, \bigotimes_{j\in I} \phi_j\Big\rangle.$$

Using this in $(\star)$, for the linear combination of two elementary tensor products $\otimes_{j\in I}\psi_j^1$, $\otimes_{j\in I}\psi_j^2$ $(c_\ell \in \mathbb{C},\ \psi_j^\ell \in \mathcal{H}_j)$

$$\left\|\widehat{U(t)}\Big(c_1 \bigotimes_{j\in I} \psi_j^1 + c_2 \bigotimes_{j\in I} \psi_j^2\Big)\right\|^2 = \left\|\widehat{U(t)}\Big(c_1 \bigotimes_{j\in I} \psi_j^1\Big)\right\|^2 + \left\|\widehat{U(t)}\Big(c_2 \bigotimes_{j\in I} \psi_j^2\Big)\right\|^2 +$$

$$+2\,\mathrm{Re}\left\{\overline{c_1}c_2\Big\langle\widehat{U(t)}\Big(\bigotimes_{j\in I} \psi_j^1\Big),\ \widehat{U(t)}\Big(\bigotimes_{j\in I} \psi_j^2\Big)\Big\rangle\right\} \overset{(\star)}{=}$$

$$= \left\|c_1 \bigotimes_{j\in I} \psi_j^1\right\|^2 + \left\|c_2 \bigotimes_{j\in I} \psi_j^2\right\|^2 + 2\,\mathrm{Re}\left\{\overline{c_1}c_2\Big\langle \bigotimes_{j\in I} \psi_j^1,\ \bigotimes_{j\in I} \psi_j^2\Big\rangle\right\} = \left\|c_1 \bigotimes_{j\in I} \psi_j^1 + c_2 \bigotimes_{j\in I} \psi_j^2\right\|^2.$$

Iteratively applying this result, we get that $\widehat{U(t)}$ is an isometry in the span of S. Hence, $\widehat{U(t)}$ is also injective in $\mathrm{span}(S)$ and its restriction to $\mathrm{span}(S)$ has operator norm 1. By Thm. 8, the restriction has a unique bounded extension to the rest of $\otimes_{j\in I}\mathcal{H}_j$ and it has operator norm 1. Thus, $\widehat{U(t)}$ must have operator norm 1 in $\bigotimes_{j\in I} \mathcal{H}_j$.

Let us prove that $\widehat{U(t)}$ is also surjective in $\mathrm{span}(S)$. Let $\Psi \in \mathrm{span}(S)$, i.e., $\Psi = \sum_{\ell=1}^M c_\ell \bigotimes_{j\in I} \psi_j^\ell$ for some $M \in \mathbb{N}$, $c_\ell \in \mathbb{C}$, $\psi_j^\ell \in \mathcal{H}_j$. Then, defining $\Phi := \sum_{\ell=1}^M c_\ell \bigotimes_{j\in I\setminus\{k\}} \psi_j^\ell \otimes \Big(U(-t)\psi_k^\ell\Big)$, we trivially get that $\widehat{U(t)}\Phi = \Psi$.

Finally, let $\Psi \in \bigotimes_{j\in I} \mathcal{H}_j$ be arbitrary. By density of $\mathrm{span}(S)$, there exists a sequence $(\Phi_n)_{n\in\mathbb{N}} \subseteq \mathrm{span}(S)$ converging to Ψ. Hence,

$$\left\|\widehat{U(t)}\Psi\right\| \overset{(\widehat{U(t)}\text{ conts})}{=} \lim_{n\to\infty} \left\|\widehat{U(t)}\Phi_n\right\| \overset{(\Phi_n\in\mathrm{span}(S))}{=} \lim_{n\to\infty} \|\Phi_n\| = \|\Psi\|.$$

and thus, $\widehat{U(t)}$ is an isometry in the whole $\otimes_{j\in I}\mathcal{H}_j$.

We only miss proving surjectivity for arbitrary $\Psi \in \otimes_{j\in I}\mathcal{H}_j$. Since $\Phi_n \in \mathrm{span}(S)$, by the above, there exists $\varphi_n \in \mathrm{span}(S)$ such that $\widehat{U(t)}\varphi_n = \Phi_n$. Then, $(\varphi_n)_{n\in\mathbb{N}}$ is Cauchy because $(\Phi_n)_{n\in\mathbb{N}}$ is Cauchy (it converges to Ψ) and $\|\varphi_n - \varphi_m\| = \left\|\widehat{U(t)}(\varphi_n - \varphi_m)\right\| = \|\Phi_n - \Phi_m\|$.

Thereby, there exists an η such that $\varphi_n \xrightarrow[n\to\infty]{} \eta$. But then, by Thm. 8, $\widehat{U(t)}\eta = \lim_{n\to\infty}\widehat{U(t)}\varphi_n = \lim_{n\to\infty}\phi_n = \Psi$.

- **(ii)** We check **strong continuity.** Let $t_0 \in \mathbb{R}$ and $(t_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}$ a sequence converging to t_0. By strong continuity of $U(t)$, $\forall\psi_k \in \mathcal{H}_k$, $U(t_n)\psi_k \xrightarrow[n\to\infty]{\|\cdot\|_{\mathcal{H}_k}} U(t_0)\psi_k$. Then, for any $\otimes_{j\in I}\psi_j \in \otimes_{j\in I}\mathcal{H}_j \setminus \{0\}$,

$$\left\|\widehat{U(t_n)}\Big(\bigotimes_{j\in I}\psi_j\Big) - \widehat{U(t_0)}\Big(\bigotimes_{j\in I}\psi_j\Big)\right\| = \left\|\bigotimes_{j\in I\setminus\{k\}}\psi_j\otimes\Big(U(t_n)\psi_k\Big) - \bigotimes_{j\in I\setminus\{k\}}\psi_j\otimes\Big(U(t_0)\psi_k\Big)\right\| \overset{\left(\substack{\otimes\text{ is linear}\\ \text{by Prop. 53}}\right)}{=}$$

$$= \left\|\bigotimes_{j\in I\setminus\{k\}}\psi_j\otimes\Big(U(t_n)\psi_k - U(t_0)\psi_k\Big)\right\| = \frac{\left\|\bigotimes_{j\in I}\psi_j\right\|}{\|\psi_k\|}\|U(t_n)\psi_k - U(t_0)\psi_k\| \xrightarrow[n\to\infty]{} 0.$$

By linearity, this also proves strong continuity of $\widehat{U(t)}$ for any vector in $\operatorname{span}(S)$. Now, let there be a general vector $\Psi \in \otimes_{j\in I}\mathcal{H}_j$ and an arbitrary $\varepsilon > 0$. Since by definition, $\operatorname{span}(S)$ is dense in $\otimes_{j\in I}\mathcal{H}_j$, there exists $\Phi \in \operatorname{span}(S)$ such that $\|\Phi - \Psi\| \leq \varepsilon/3$. In particular, by the above, $\exists N^\varepsilon \in \mathbb{N}$ such that $\left\|\widehat{U(t_n)}\Phi - \widehat{U(t_0)}\Phi\right\| \leq \varepsilon/3\ \forall n \geq N^\varepsilon$. Then,

$$\left\|\widehat{U(t_n)}\Psi - \widehat{U(t_0)}\Psi\right\| \leq \left\|\widehat{U(t_n)}(\Psi - \Phi)\right\| + \left\|\widehat{U(t_n)}\Phi - \widehat{U(t_0)}\Phi\right\| + \left\|\widehat{U(t_0)}(\Phi - \Psi)\right\| \leq$$

$$\leq \cancelto{1}{\left\|\widehat{U(t_n)}\right\|_{op}}\|\Psi - \Phi\| + \frac{\varepsilon}{3} + \cancelto{1}{\left\|\widehat{U(t_0)}\right\|_{op}}\|\Phi - \Psi\| \leq \varepsilon \quad \forall n \geq N^\varepsilon \quad \Longrightarrow \quad \widehat{U(t_n)}\Psi \xrightarrow[n\to\infty]{\|\cdot\|} \widehat{U(t_0)}\Psi.$$

- **(iii)** Regarding the **group property**, let $\bigotimes_{j\in I}\psi_j \in S$ arbitrary,

$$\widehat{U(t+s)}\Big(\bigotimes_{j\in I}\psi_j\Big) = \bigotimes_{j\in I\setminus\{k\}}\psi_j\otimes\Big(U(t+s)\psi_k\Big) = \bigotimes_{j\in I\setminus\{k\}}\psi_j\otimes\Big(U(t)U(s)\psi_k\Big) = \widehat{U(t)}\widehat{U(s)}\Big(\bigotimes_{j\in I}\psi_j\Big).$$

By linearity, $\widehat{U(t+s)} = \widehat{U(t)}\widehat{U(s)}$ also holds in the span of S. But then the equality must hold everywhere in $\otimes_{j\in I}\mathcal{H}_j$ because two continuous maps between Hausdorff spaces that agree on a dense set must agree everywhere (Lemma 32). ***o.ε.δ.***

Lemma 49. Given an arbitrary family of Hilbert spaces $(\mathcal{H}_j)_{j\in I}$, let there be a fixed $k \in \mathbb{N}$ and a PVM $\{P(\Omega)\}_{\Omega\in\mathfrak{B}(\mathbb{R})} \subseteq \mathcal{L}(\mathcal{H}_k)$ on $\mathcal{H}_k$. Then, $\{\widehat{P(\Omega)}\}_{\Omega\in\mathfrak{B}(\mathbb{R})} \subseteq \mathcal{L}(\otimes_{j\in I}\mathcal{H}_j)$ is a PVM on $\otimes_{j\in I}\mathcal{H}_j$. ♦

Proof: We check all the points of Definition 17.

- **Item (i)** They are orthogonal projectors:

$$(a) \qquad \widehat{P(\Omega)}^2 \overset{[a]}{=} \widehat{P(\Omega)^2} \overset{(P(\Omega)\text{ is projector})}{=} \widehat{P(\Omega)}.$$

(b) The lift of a self-adjoint bounded operator is still bounded by Prop. 62 and so, is self-adjoint by Proposition 63.

- **Item (ii)** $\widehat{P(\mathbb{R})} = \widehat{Id} = Id$.

- **Item (iii)** Let $\{\Omega_n\}_{n\in\mathbb{N}} \subseteq \mathfrak{B}(\mathbb{R})$ be pairwise disjoint and $\Omega := \cup_{n\in\mathbb{N}}\Omega_n$. Since, $P(\cdot)$ is a PVM, $\forall\psi_k \in \mathcal{H}_k$, $\sum_{n=1}^N P(\Omega_n)\psi_k \xrightarrow[N\to\infty]{\|\cdot\|_{\mathcal{H}_k}} P(\Omega)\psi_k$. Then, for any $\otimes_{j\in I}\psi_j \in \otimes_{j\in I}\mathcal{H}_j \setminus \{0\}$,

$$\left\|\sum_{n=1}^{N}\widehat{P(\Omega_n)}\Big(\bigotimes_{j\in I}\psi_j\Big)-\widehat{P(\Omega)}\Big(\bigotimes_{j\in I}\psi_j\Big)\right\| = \left\|\sum_{n=1}^{N}\bigotimes_{j\in I\setminus\{k\}}\psi_j\otimes\Big(P(\Omega_n)\psi_k\Big)-\bigotimes_{j\in I\setminus\{k\}}\psi_j\otimes\Big(P(\Omega)\psi_k\Big)\right\| \overset{\left(\substack{\otimes\text{ is linear}\\ \text{by Prop. }53}\right)}{=}$$

$$=\left\|\bigotimes_{j\in I\setminus\{k\}}\psi_j\otimes\Big(\sum_{n=1}^{N}P(\Omega_n)\psi_k-P(\Omega)\psi_k\Big)\right\| = \frac{\left\|\bigotimes_{j\in I}\psi_j\right\|}{\|\psi_k\|}\left\|\sum_{n=1}^{N}P(\Omega_n)\psi_k-P(\Omega)\psi_k\right\| \xrightarrow[N\to\infty]{} 0.$$

By linearity, this also proves (iii) for any vector in the span of elementary tensor products.

- Before we prove it for general vectors, we need to note that, still for elementary tensor products,

$$\sum_{n=1}^{N}\widehat{P(\Omega_n)}(\otimes_{j\in I}\psi_j)=\sum_{n=1}^{N}\bigotimes_{j\in I\setminus\{k\}}\psi_j\otimes\Big(P(\Omega_n)\psi_k\Big)=\bigotimes_{j\in I\setminus\{k\}}\psi_j\otimes\Big(\underbrace{\sum_{n=1}^{N}P(\Omega_n)\psi_k}_{P(\cup_{n=1}^{N}\Omega_n)\psi_k}\Big)=\widehat{P(\cup_{n=1}^{N}\Omega_n)}(\otimes_{j\in I}\psi_j).$$

By linearity, this implies that in the span of elementary products (which is a dense set in $\otimes_{j\in I}\mathcal{H}_j$), $\sum_{n=1}^{N}\widehat{P(\Omega_n)}=\widehat{P(\cup_{n=1}^{N}\Omega_n)}$. Since both sides are continuous maps between Hausdorff spaces the equality holds everywhere (Lemma 32).

- Now, let there be a general vector $\Psi\in\otimes_{j\in I}\mathcal{H}_j$ and an arbitrary $\varepsilon>0$. By density, there exists a linear combination of elementary tensor products Φ such that $\|\Phi-\Psi\|\leq\varepsilon/3$. In particular, by the two above items, $\exists N^{\varepsilon}\in\mathbb{N}$ such that $\varepsilon/3\geq\left\|\sum_{n=1}^{N}\widehat{P(\Omega_n)}\Phi-\widehat{P(\Omega)}\Phi\right\|$ $\forall N\geq N^{\varepsilon}$. But then,

$$\left\|\sum_{n=1}^{N}\widehat{P(\Omega_n)}\Psi-\widehat{P(\Omega)}\Psi\right\|=\left\|\widehat{P(\cup_{n=1}^{N}\Omega_n)}\Psi-\widehat{P(\Omega)}\Psi\right\|\leq\left\|\widehat{P(\cup_{n=1}^{N}\Omega_n)}(\Psi-\Phi)\right\|+$$

$$+\left\|\widehat{P(\cup_{n=1}^{N}\Omega_n)}\Phi-\widehat{P(\Omega)}\Phi\right\|+\left\|\widehat{P(\Omega)}(\Phi-\Psi)\right\|\leq$$

$$\leq\left\|\widehat{P(\cup_{n=1}^{N}\Omega_n)}\right\|_{op}\|\Psi-\Phi\|+\frac{\varepsilon}{3}+\left\|\widehat{P(\Omega)}\right\|_{op}\|\Phi-\Psi\|\overset{(\star)}{\leq}\varepsilon\quad\forall n\geq N^{\varepsilon}.$$

That is, $\sum_{n=1}^{N}\widehat{P(\Omega_n)}\Psi\xrightarrow[N\to\infty]{\|\cdot\|}\widehat{P(\Omega)}\Psi$. In $(\star)$ we used that orthogonal projectors have operator norm smaller than 1.

o.ε.δ.

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[a] Let $\otimes_{j\in I}\psi_j\in\otimes_{j\in I}\mathcal{H}_j$. Then, for any $B\in\mathcal{L}(\mathcal{H}_k)$, $(\widehat{B})^2(\otimes_{j\in I}\psi_j)=\bigotimes_{j\in I\setminus\{k\}}\psi_j\otimes(B^2\psi_k)=(\widehat{B^2})(\otimes_{j\in I}\psi_j)$. Using linearity, we get the equality in the span of elementary tensors and thus by continuity everywhere else too.

Proposition 78. Given an arbitrary family of Hilbert spaces $(\mathcal{H}_j)_{j\in I}$, let there be a fixed $k\in\mathbb{N}$ and an essentially self-adjoint operator $(A_k,\ D_k)$. Consider the lift $(\widehat{A_k},\ D_0(\widehat{A_k}))$ as in Lemma 24,[6] which we proved to be essentially self-adjoint (see Prop. 32). Then,

(i) *("Exponential and lift commute"):* For all $t\in\mathbb{R}$, in the notation of Prop. 62 (see the uniqueness characterization for lifts of bounded operators given there)

$$e^{it\overline{\widehat{A_k}}}=\widehat{e^{it\overline{A_k}}},\qquad\text{i.e.,}\qquad\forall\bigotimes_{j\in I}\psi_j\in\bigotimes_{j\in I}\mathcal{H}_j,\quad e^{it\overline{\widehat{A_k}}}\Big(\bigotimes_{j\in I}\psi_j\Big)=\bigotimes_{j\in I\setminus\{k\}}\psi_j\otimes(e^{it\overline{A_k}}\psi_k)$$

---

[6] Note that for $I=\mathbb{N}$, this "lift" equals the operator $Id\otimes\cdots\otimes A_k\otimes Id\otimes\cdots$ of Prop. 73 given by the polynomial $p(x_1,...,x_k)=x_k$ and for $\ell<k$, $A_\ell=Id,\ D_\ell=\mathcal{H}_\ell$.

(ii) *("Lift and PVM commute"):* Given that $P^{\overline{\widehat{A_k}}} : \mathfrak{B}(\mathbb{R}) \to \mathcal{L}(\otimes_{j\in I}\mathcal{H}_j)$ and $P_k : \mathfrak{B}(\mathbb{R}) \to \mathcal{L}(\mathcal{H}_k)$ denote respectively, the spectral PVMs of $\overline{\widehat{A_k}}$ and $\overline{A_k}$, then,

$$P^{\overline{\widehat{A_k}}} = \widehat{P_k} \quad \text{i.e.,} \quad \forall \bigotimes_{j\in I}\psi_j \in \bigotimes_{j\in I}\mathcal{H}_j, \quad P^{\overline{\widehat{A_k}}}\Big(\bigotimes_{j\in I}\psi_j\Big) = \bigotimes_{j\in I\setminus\{k\}}\psi_j \otimes (P_k\psi_k).$$

(iii) *("Lift and functional calculus commute"):* Given that $\Phi^{\overline{\widehat{A_k}}}(\cdot)$ and $\Phi^{\overline{A_k}}(\cdot)$ denote respectively the functional calculus of $\overline{\widehat{A_k}}$ and $\overline{A_k}$, for all measurable and bounded $f : \mathbb{R} \to \mathbb{C}$,[7]

$$\Phi^{\overline{\widehat{A_k}}}(f) = \widehat{\Phi^{\overline{A_k}}(f)} \quad \text{i.e.,} \quad f(\overline{\widehat{A_k}}) = \widehat{f(\overline{A_k})}.$$

♦

Proof: • **Item (i):** Let $U(t) := e^{it\overline{A_k}}$ be the SCOPUG on $\mathcal{H}_k$ generated by $\overline{A_k}$ and consider its lift $\widehat{U(t)}$, which by Lemma 48 is an SCOPUG on $\otimes_{j\in I}\mathcal{H}_j$. Let $\otimes_{j\in I}\psi_j \in D_0(\widehat{A_k})$. Then,

$$\frac{\widehat{U(t)}(\bigotimes_{j\in I}\psi_j) - \bigotimes_{j\in I}\psi_j}{t} = \frac{\bigotimes_{j\in I\setminus\{k\}}\psi_j \otimes (U(t)\psi_k) - \bigotimes_{j\in I}\psi_j}{t} \overset{\left(\substack{\otimes \text{ is linear} \\ \text{by Prop. 53}}\right)}{=} \bigotimes_{j\in I\setminus\{k\}}\psi_j \otimes \left(\frac{U(t)\psi_k - \psi_k}{t}\right). \tag{B.32}$$

By definition of generator,

$$\lim_{t\to 0}\left(\frac{U(t)\psi_k - \psi_k}{t}\right) = iA_k\psi_k.$$

Then, using that $\otimes$ is continuous —thus, sequentially continuous— in the sense of Prop. 53, retaking (B.32), we get,

$$\lim_{t\to 0}\frac{\widehat{U(t)}(\bigotimes_{j\in I}\psi_j) - \bigotimes_{j\in I}\psi_j}{t} = \bigotimes_{j\in I\setminus\{k\}}\psi_j \otimes (iA_k\psi_k) \overset{\text{(by def.)}}{=} i\widehat{A_k}\Big(\otimes_{j\in I}\psi_j\Big).$$

Hence, the generator of $\widehat{U(t)}$ coincides with $\overline{\widehat{A_k}}$ in $D_0(\widehat{A_k})$. Since the image of $\overline{\widehat{A_k}}$ on $D(\overline{\widehat{A_k}}) \setminus D_0(\widehat{A_k})$ is determined by the images of points in $D_0(\widehat{A_k})$ —more precisely, the graph of $\overline{\widehat{A_k}}$ is the closure of the graph of $(\widehat{A_k}, D_0(\widehat{A_k}))$—, and because every SCOPUG has a unique self-adjoint generator (see Theorem 6.2 in (Schmüdgen, 2012)), a self-adjoint operator admitting no proper self-adjoint extension (Prop. 1.21 in (Arai, 2018))), $\overline{\widehat{A_k}}$ must be the generator of $\widehat{U(t)}$, i.e., it must be that $\widehat{U(t)} = e^{it\overline{\widehat{A_k}}}$.

• **Item (ii):** Define $E(\Omega) := \widehat{P_k(\Omega)}$ for each $\Omega \in \mathfrak{B}(\mathbb{R})$. By Lemma 49, it is a PVM. Let $f : \mathbb{R} \to \mathbb{C}$ be measurable and bounded. By Thm. 2.10 in (Folland, 1999), there is a sequence of simple functions $(\sum_{\ell=1}^{N_n} c_\ell^n \mathbb{1}_{\Omega_\ell^n})_{n\in\mathbb{N}}$ (with $N_n \in \mathbb{N}$, $c_\ell \in \mathbb{C}$, $\Omega_\ell^n \in \mathfrak{B}(\mathbb{R})$) converging in supremum-norm to f. As such, given we denote by $\Phi^E(\cdot)$ the functional calculus of E, for all $\bigotimes_{j\in I}\psi_j \in \otimes_{j\in I}\mathcal{H}_j$,

$$\Phi^E(f)\Big(\bigotimes_{j\in I}\psi_j\Big) = \lim_{n\to\infty}\sum_{\ell=1}^{N_n} c_\ell^n E(\Omega_\ell^n)\Big(\bigotimes_{j\in I}\psi_j\Big) = \lim_{n\to\infty}\sum_{\ell=1}^{N_n} c_\ell^n \bigotimes_{j\in I\setminus\{k\}}\psi_j \otimes (P_k(\Omega_\ell^n)\psi_k) \overset{\left(\substack{\otimes \text{ is linear} \\ \text{by Prop. 53}}\right)}{=}$$

$$= \lim_{n\to\infty}\bigotimes_{j\in I\setminus\{k\}}\psi_j \otimes \Big(\sum_{\ell=1}^{N_n} c_\ell^n P_k(\Omega_\ell^n)\psi_k\Big) \overset{\left(\substack{\otimes \text{ is conts.} \\ \text{by Prop. 53}}\right)}{=} \bigotimes_{j\in I\setminus\{k\}}\psi_j \otimes \Big(\Phi^{\overline{A_k}}(f)\psi_k\Big) = \widehat{\Phi^{\overline{A_k}}(f)}\Big(\otimes_{j\in I}\psi_j\Big).$$

---

[7] A similar statement holds also for unbounded f.

By linearity, the equality holds also for linear combinations of elementary tensors, which are dense in $\otimes_{j\in I}\mathcal{H}_j$. Since both $\Phi^E(f)$, $\widehat{\Phi^{\overline{A_k}}(f)}$ are continuous, this implies they are the equal elsewhere too (Lemma 32). But then, taking $f(x) := e^{itx}$, this implies that $\Phi^E(f) = \widehat{\Phi^{\overline{A_k}}(f)} = \widehat{e^{it\overline{A_k}}}$, which by the previous item equals $e^{it\widehat{\overline{A_k}}} = \Phi^{\widehat{\overline{A_k}}}(f)$. That is, the SCOPUG generated by $\Phi^E(x\in\mathbb{R}\mapsto x)$ and $\Phi^{\widehat{\overline{A_k}}}(x\in\mathbb{R}\mapsto x)$ are the same. By uniqueness of generator and uniqueness of spectral PVM for a self-adjoint operator, this implies that $E = P^{\widehat{\overline{A_k}}}$.

- **Item (iii):** We just proved that for all bounded f, $\Phi^E(f) = \widehat{\Phi^{\overline{A_k}}(f)}$ and that $E = P^{\widehat{\overline{A_k}}}$, hence, $\Phi^{\widehat{\overline{A_k}}}(f) = \widehat{\Phi^{\overline{A_k}}(f)}$.

o.ε.δ.

Lemma 50. Let $(\mathcal{H}_j)_{j\in\mathbb{N}}$ be Hilbert spaces. For a fixed $N\in\mathbb{N}$ and each $j\in\{1,...,N\}$, let $A_j, B_j \in \mathcal{L}(\mathcal{H}_j)$. Then,

(i) *There is a unique bounded linear operator* on $\bigotimes_{j\in\mathbb{N}}\mathcal{H}_j$ —denoted $B_1\otimes\cdots\otimes B_N\otimes Id\otimes\cdots$— that acts on $\bigotimes_{j\in\mathbb{N}}\psi_j \in \bigotimes_{j\in I}\mathcal{H}_j$ as

$$(B_1\otimes\cdots\otimes B_N\otimes Id\otimes\cdots)\Big(\bigotimes_{j\in\mathbb{N}}\psi_j\Big) = (B_1\psi_1)\otimes\cdots\otimes(B_N\psi_N)\otimes Id\otimes\cdots\,. \tag{B.33}$$

(ii) $(A_1\otimes\cdots\otimes A_N\otimes Id\otimes\cdots)(B_1\otimes\cdots\otimes B_N\otimes Id\otimes\cdots) = (A_1B_1)\otimes\cdots\otimes(A_NB_N)\otimes Id\otimes\cdots$

(iii) The operator norm of $(B_1\otimes\cdots\otimes B_N\otimes Id\otimes\cdots)$ is $\|B_1\|_{op}\cdots\|B_N\|_{op}$. ⧫

Proof: • **Item (i):** By linearity, the action of any such operator on the span of elementary tensor products (which we denote by $\mathrm{span}(S)$) is fixed. As such, we can start by proving (ii) for $\Psi\in\mathrm{span}(S)$, for which there are $M\in\mathbb{N}$, $c_\ell\in\mathbb{C}$, $\psi_j^\ell\in\mathcal{H}_j$ such that $\Psi = \sum_{\ell=1}^M c_\ell \otimes_{j\in\mathbb{N}}\psi_j^\ell$. By linearity,

$$(A_1\otimes\cdots\otimes A_N\otimes Id\otimes\cdots)(B_1\otimes\cdots\otimes B_N\otimes Id\otimes\cdots)\Psi = \tag{B.34}$$

$$= (A_1\otimes\cdots\otimes A_N\otimes Id\otimes\cdots)\left(\sum_{\ell=1}^M c_\ell\;(B_1\psi_1^\ell)\otimes\cdots\otimes(B_N\psi_N^\ell)\otimes Id\otimes\cdots\right) =$$

$$= \sum_{\ell=1}^M c_\ell\;(A_1B_1\psi_1^\ell)\otimes\cdots\otimes(A_NB_N\psi_N^\ell)\otimes Id\otimes\cdots = \Big((A_1B_1)\otimes\cdots(A_NB_N)\otimes Id\otimes\cdots\Big)\Psi.$$

In particular, this implies that

$$(B_1\otimes\cdots\otimes B_N\otimes Id\otimes\cdots)\restriction_{\mathrm{span}(S)} = (B_1\otimes\ Id\otimes\cdots)\cdots(Id\otimes\cdots\otimes B_N\otimes Id\otimes\cdots)\restriction_{\mathrm{span}(S)}\,.$$

But each $(Id\otimes\cdots\otimes B_k\otimes\ Id\otimes\cdots)\restriction_{\mathrm{span}(S)}$ equals $\widehat{B_k}\restriction_{\mathrm{span}(S)}$ from Prop. 62, so $(B_1\otimes\cdots\otimes B_N\otimes Id\otimes\cdots)$ coincides in $\mathrm{span}(S)$ with the product of bounded operators $\widehat{B_1}\cdots\widehat{B_N}$ —which is itself a bounded operator. Thus, $(B_1\otimes\cdots\otimes B_N\otimes Id\otimes\cdots)$ is a bounded linear operator on $\mathrm{span}(S)$ and by Thm. 8 it has a unique bounded linear extension. As such, its extension must be equal to $\widehat{B_1}\cdots\widehat{B_N}$ everywhere.

- **Item (ii):** In equation (B.34) we proved the equality of the desired operators (which we now know are well-defined everywhere) in $\mathrm{span}(S)$. Two continuous maps between Hausdorff spaces that agree in a dense subspace agree everywhere. Hence, the equality holds everywhere.

- **Item (iii):** We proved in (i) that $(B_1 \otimes \cdots \otimes B_N \otimes Id \otimes \cdots) = \widehat{B_1} \cdots \widehat{B_N}$. Hence,

$$\|(B_1 \otimes \cdots \otimes B_N \otimes Id \otimes \cdots)\|_{op} \overset{\text{(op. norm submultiplicative)}}{\leq} \left\|\widehat{B_1}\right\|_{op} \cdots \left\|\widehat{B_N}\right\|_{op} \overset{\text{(Prop. 62)}}{=} \|B_1\|_{op} \cdots \|B_N\|_{op}.$$

To get the equality, we find a sequence of vectors realizing this upper bound. For each $j \in \{1, ..., N\}$, let the sequence $(\psi_j^n)_{n\in\mathbb{N}} \subseteq \mathcal{H}_j$ be unit norm vectors realizing the operator norm of B_j, i.e., such that $\left\|B_j \psi_j^n\right\| \xrightarrow{n\to\infty} \|B_j\|_{op}$.[a] Then, given $\bigotimes_{j\in\mathbb{N}} \phi_j \in \bigotimes_{j\in\mathbb{N}} \mathcal{H}_j \setminus \{0\}$ with $\|\phi_j\| = 1$ for all $j \in \mathbb{N}$,

$$\left\| \Big(B_1 \otimes \cdots \otimes B_N \otimes Id \otimes \cdots\Big) \underbrace{(\psi_1^n \otimes \cdots \otimes \psi_N^n \otimes \phi_{N+1} \otimes \cdots)}_{=:\Psi_n} \right\| = \|(B_1\psi_1^n) \otimes \cdots \otimes (B_N\psi_N^n) \otimes \phi_{N+1} \otimes \cdots\| =$$

$$= \|B_1\psi_1^n\| \cdots \|B_N\psi_N^n\| \xrightarrow[n\to\infty]{(\star)} \|B_1\|_{op} \cdots \|B_N\|_{op},$$

where $(\star)$ is easily proven following the proof-strategy given for Lemma 47. With all, Ψ_n is a sequence of unit norm vectors such that

$$\sup_{n\in\mathbb{N}} \|(B_1 \otimes \cdots \otimes B_N \otimes Id \otimes \cdots)\Psi_n\| = \|B_1\|_{op} \cdots \|B_N\|_{op},$$

just as we needed.

o.ε.δ.

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[a] Such a sequence must exist because by definition $\|B\|_{op} := \sup_{\psi\in\mathcal{H}_j,\ \|\psi\|=1} \|B_j\psi\|$.

With all the above, we finally get what we claimed:

Theorem 30. $\left(\bigotimes_{k\in\mathbb{N}} L^2(\mathbb{R}, dx),\ \{\overline{\hat{q}(f)},\ \overline{\hat{p}(f)}\}_{f\in\mathcal{W}}\right)$ is a *reducible representation of the Weyl CCR*, reduced by every layer $\otimes_{j\in\mathbb{N}}^{\mathfrak{C}} L^2(\mathbb{R}, dx)$, $\mathfrak{C} \in \Gamma$. In particular,

$$e^{i\overline{\hat{q}(f)}} = e^{i\langle e_1, f\rangle \overline{\hat{q}}} \otimes \cdots \otimes e^{i\langle e_{N_f}, f\rangle \overline{\hat{q}}} \otimes Id \otimes \cdots \quad \text{and} \quad e^{i\overline{\hat{p}(f)}} = e^{i\langle e_1, f\rangle \overline{\hat{p}}} \otimes \cdots \otimes e^{i\langle e_{N_f}, f\rangle \overline{\hat{p}}} \otimes Id \otimes \cdots.$$

♦

Proof: We check each item of Def. 47.

- **Item (i):** By Corollary 26, $\overline{\hat{q}(f)}$, $\overline{\hat{p}(f)}$ are self-adjoint $\forall f \in \mathcal{W}$.

- **Item (ii):** First, we check for arbitrary $f \in \mathcal{W}$ and $t, s \in \mathbb{R}\setminus\{0\}$ that $D(\overline{\hat{q}(sf)}) = D(\overline{\hat{q}(tf)})$. Let $\Psi \in D(\overline{\hat{q}(sf)})$. By definition, there exists a sequence $(\Psi_n)_{n\in\mathbb{N}} \subseteq \otimes_{j\in\mathbb{N}}^{\mathscr{F}} B_{Her} = D(\hat{q}(sf))$ such that $\lim_{n\to\infty} \Psi_n = \Psi$ and $\lim_{n\to\infty} \hat{q}(sf)\Psi_n = \overline{\hat{q}(sf)}\Psi$. Now, we claim that $(\hat{q}(tf)\Psi_n)_{n\in\mathbb{N}}$ is also a convergent sequence. We prove it is a Cauchy sequence:

$$\|\hat{q}(tf)\Psi_n - \hat{q}(tf)\Psi_m\| \overset{(\star)}{=} |t| \|\hat{q}(f)\Psi_n - \hat{q}(f)\Psi_m\| \overset{(\star)}{=} \frac{|t|}{|s|} \|\hat{q}(sf)\Psi_n - \hat{q}(sf)\Psi_m\| \xrightarrow[n,m\to\infty]{(\star\star)} 0.$$

In $(\star)$ we used that for $\varphi \in \otimes_{j\in\mathbb{N}}^{\mathscr{F}} B_{Her}$, for all $t \in \mathbb{R}$,

$$\hat{q}(tf)\varphi = \sum_{n=1}^{M_f} \langle e_n, tf\rangle \hat{q}_n \varphi \overset{(\langle e_n, \cdot\rangle \text{ linear})}{=} t \sum_{n=1}^{M_f} \langle e_n, f\rangle \hat{q}_n \varphi = t\hat{q}(f)\varphi.$$

In $(\star\star)$ we used that $(\hat{q}(sf)\Psi_n)_{n\in\mathbb{N}}$ is a Cauchy sequence because we found it to be convergent. With all, $\exists \eta \in \otimes_{j\in\mathbb{N}} L^2(\mathbb{R}, dx)$ such that $\hat{q}(tf)\Psi_n \xrightarrow[n\to\infty]{} \eta$ and thus, $\Psi \in D(\overline{\hat{q}(tf)})$ with $\eta = \overline{\hat{q}(tf)}\Psi$. Since s, t were arbitrary, this proves that $D(\overline{\hat{q}(sf)}) = D(\overline{\hat{q}(tf)})$.

- But then, since (keeping the above notation) $\hat{q}(tf)\Psi_n = t\hat{q}(f)\Psi_n$ for all $n \in \mathbb{N}$, using the above for $s = 1$ and $t \in \mathbb{R}$ in $(\star\star\star)$,

$$\lim_{n\to\infty} \hat{q}(tf)\Psi_n = t \lim_{n\to\infty} \hat{q}(f)\Psi_n \overset{(\star\star\star)}{=} t\,\overline{\hat{q}(f)}\Psi.$$

This proves that $\overline{\hat{q}(tf)} = t\,\overline{\hat{q}(f)}$. Changing q by p in all this reasoning proves that $\overline{\hat{p}(tf)} = t\,\overline{\hat{p}(f)}$.

- By Corollary 28, for every $f, g \in \mathcal{W}$, $\overline{\hat{q}(f) + \hat{q}(g)} = \overline{\overline{\hat{q}(f)} + \overline{\hat{q}(g)}}$. Thus, trivially, $\overline{\hat{q}(f)} + \overline{\hat{q}(f)} \subseteq \overline{\hat{q}(f) + \hat{q}(g)}$. The same thing holds for $\hat{p}(\cdot)$.

- **Item (iii):** Let $f, g \in \mathcal{W}$ and $t \in \mathbb{R}$ arbitrary. By Corollary 28,

$$e^{i\overline{\hat{q}(f)}}e^{i\overline{\hat{q}(g)}} = e^{i\overline{\hat{q}(g)}}e^{i\overline{\hat{q}(f)}} \quad \text{and} \quad e^{i\overline{\hat{p}(f)}}e^{i\overline{\hat{p}(g)}} = e^{i\overline{\hat{p}(g)}}e^{i\overline{\hat{p}(f)}}.$$

By the same Corollary, $e^{i\overline{\hat{q}(f+g)}} = e^{i\overline{\hat{q}(f)}}e^{i\overline{\hat{q}(g)}}$, which, used iteratively in $(\star)$ and denoting $\hat{q}_n := \hat{q}(e_n)$ leaves

$$e^{i\overline{\hat{q}(f)}} = e^{\overline{i\hat{q}\left(\sum_{n=1}^{M_f}\langle e_n,f\rangle e_n\right)}} \overset{(\star)}{=} e^{i\overline{\hat{q}(\langle e_1,f\rangle e_1)}}\cdots e^{i\overline{\hat{q}(\langle e_N,f\rangle e_N)}} \overset{\text{(ii)}}{=} e^{i\langle e_n,f\rangle\;\overline{\hat{q}_1}}\cdots e^{i\langle e_n,f\rangle\;\overline{\hat{q}_N}} \overset{\text{(Prop. 78)}}{=}$$

$$= (e^{i\langle e_1,f\rangle\overline{\hat{q}}} \otimes Id \otimes \cdots)(Id \otimes e^{i\langle e_2,f\rangle\overline{\hat{q}}} \otimes Id \otimes \cdots)\cdots(Id \otimes \cdots \otimes Id \otimes e^{i\langle e_{N_f},f\rangle\overline{\hat{q}}} \otimes Id \otimes \cdots) \overset{\text{(Lem. 50)}}{=}$$

$$= e^{i\langle e_1,f\rangle\overline{\hat{q}}} \otimes \cdots \otimes e^{i\langle e_{N_f},f\rangle\overline{\hat{q}}} \otimes Id \otimes \cdots . \tag{B.35}$$

Likewise, $e^{i\overline{\hat{p}(g)}} = e^{i\langle e_1,g\rangle\overline{\hat{p}}} \otimes \cdots \otimes e^{i\langle e_{N_g},g\rangle\overline{\hat{p}}} \otimes Id \otimes \cdots$. Now, without loss of generality, we can assume that $N_f = N_g =: N$ because we can always consider explicitly $\langle e_n, f\rangle = 0$ for the $n \in \{N_f, ..., N_g\}$ (if $N_f < N_g$ and similar if $N_g < N_f$). Then,

$$e^{i\overline{\hat{q}(f)}}e^{i\overline{\hat{p}(g)}} = (e^{i\langle e_1,f\rangle\overline{\hat{q}}} \otimes \cdots \otimes e^{i\langle e_N,f\rangle\overline{\hat{q}}} \otimes Id \otimes \cdots)(e^{i\langle e_1,g\rangle\overline{\hat{p}}} \otimes \cdots \otimes e^{i\langle e_N,g\rangle\overline{\hat{p}}} \otimes Id \otimes \cdots) \overset{\text{(Lem. 50)}}{=}$$

$$= (e^{i\langle e_1,f\rangle\overline{\hat{q}}}e^{i\langle e_1,g\rangle\overline{\hat{p}}}) \otimes \cdots \otimes (e^{i\langle e_N,f\rangle\overline{\hat{q}}}e^{i\langle e_N,g\rangle\overline{\hat{p}}}) \otimes Id \otimes \cdots \overset{(\star\star)}{=}$$

$$= (e^{-i\langle e_1,f\rangle\langle e_1,g\rangle}\; e^{i\langle e_1,g\rangle\overline{\hat{p}}}e^{i\langle e_1,f\rangle\overline{\hat{q}}}) \otimes \cdots \otimes (e^{-i\langle e_N,f\rangle\langle e_N,g\rangle}\; e^{i\langle e_N,g\rangle\overline{\hat{p}}}e^{i\langle e_N,f\rangle\overline{\hat{q}}}) \otimes Id \otimes \cdots =$$

$$= e^{-i\sum_{n=1}^{M}\langle e_n,f\rangle\langle e_n,g\rangle}\Big[(e^{i\langle e_1,g\rangle\overline{\hat{p}}}e^{i\langle e_1,f\rangle\overline{\hat{q}}}) \otimes \cdots \otimes (e^{i\langle e_N,g\rangle\overline{\hat{p}}}e^{i\langle e_N,f\rangle\overline{\hat{q}}}) \otimes Id \otimes \cdots\Big] \overset{\left(\substack{\text{“undo steps”} \\ \text{and } (\star\star\star)}\right)}{=} e^{-i\langle f,g\rangle}e^{i\overline{\hat{p}(g)}}e^{i\overline{\hat{q}(f)}},$$

In $(\star\star)$ we used the well-known Weyl relations of position and momentum in $L^2(\mathbb{R}, dx)$, i.e., that for all $a, b \in \mathbb{R}$, $e^{ia\,\overline{\hat{q}}}e^{ib\,\overline{\hat{p}}} = e^{-iab}e^{ib\,\overline{\hat{p}}}e^{ia\,\overline{\hat{q}}}$ —see for instance Thm. 4.1.3.(b) in (Weaver, 2001). In $(\star\star\star)$ we used that $\sum_{n=1}^{M}\langle e_n, f\rangle\langle e_n, g\rangle = \langle f, g\rangle$.

- Finally, by Cor. 27, for all $f \in \mathcal{W}$, both $\overline{\hat{q}(f)}$ and $\overline{\hat{q}(f)}$ are reduced by every $\otimes_{j\in\mathbb{N}}^{\mathfrak{C}} L^2(\mathbb{R}, dx)$ layer, $\mathfrak{C} \in \Gamma$. Consequently, by Theorem 1.49 in (Arai, 2018), both $e^{i\overline{\widehat{q}(f)}}$ and $e^{i\overline{\widehat{p}(f)}}$ are reduced by every $\otimes_{j\in\mathbb{N}}^{\mathfrak{C}} L^2(\mathbb{R}, dx)$ layer. So, by definition, the representation is reducible as we claimed.

o.ε.δ.

B.2.3 The Creation-Annihilation CCR Representation

As shown by Arai (2018) in Proposition 5.27, one can extract a creation-annihilation CCR from a Heisenberg CCR by going through an analogue of the “Segal field operator” of Fock space (to be defined later). In this section we adapt this route to our case.

Corollary 29. Given any $N \in \mathbb{N}$, $(c_n)_{n=1}^N \subset \mathbb{C}$ and $(a_n,\ b_n)_{n=1}^N \subset \mathbb{R}^2$, the operator

$$\left(\sum_{n=1}^N c_n \Big(a_n \hat{q}_n + b_n \hat{p}_n \Big),\ \otimes_{j\in\mathbb{N}}^{\mathscr{F}} B_{Her} \right) \tag{B.36}$$

(defined as in Prop. 73 for $A_n := a_n \hat{q} + b_n \hat{p}$ and $Q(x_1, ..., x_N) := \sum_{n=1}^N c_n x_n$) is an *essentially self-adjoint* operator on $\bigotimes_{j\in\mathbb{N}} \mathcal{H}_j$.

♦

Proof: By Prop. 75, $(a\hat{q} + b\hat{p},\ B_{Her})$ is essentially self-adjoint, so the self-adjointness of (B.36) follows by Prop. 73. ***o.ε.δ.***

Corollary 30. Let $\mathcal{W}_{\mathbb{C}}$ be the complexification of $\mathcal{W}$.

(i) For each $f \in \mathcal{W}_{\mathbb{C}}$,

$$\Big(\widehat{\Phi}(f) := \hat{q}(\operatorname{Re} f) + \hat{p}(\operatorname{Im} f),\ \otimes_{j\in\mathbb{N}}^{\mathscr{F}} B_{Her} \Big) \tag{B.37}$$

is an essentially self-adjoint operator on $\bigotimes_{j\in\mathbb{N}} L^2(\mathbb{R}, dx)$. We call it an *ITP-Segal field operator* (of test-function f).

(ii) $\Big[\widehat{\Phi}(f),\ \widehat{\Phi}(g) \Big] = i \operatorname{Im}\langle f, g\rangle$.

(iii) $\widehat{\Phi}(f)$ is reduced by all $\otimes_{j\in\mathbb{N}}^{\mathfrak{C}} L^2(\mathbb{R}, dx)$, $\mathfrak{C} \in \Gamma$, and $\widehat{\Phi}(f)|_{\mathfrak{C}}$ is essentially self-adjoint.

(iv) $e^{i\overline{\widehat{\Phi}(f)}} = e^{i\frac{\langle \operatorname{Re} f, \operatorname{Im} f\rangle}{2}} e^{i\overline{\widehat{q}(\operatorname{Re} f)}} e^{i\overline{\widehat{p}(\operatorname{Im} f)}}$ and $e^{i\overline{\widehat{\Phi}(f+g)}} = e^{i \operatorname{Im} \frac{\langle f, f\rangle}{2}} e^{i\overline{\widehat{\Phi}(f)}} e^{i\overline{\widehat{\Phi}(f)}}$.

♦

Proof: **(i):** (B.37) is the particular case of Cor. 29 when $c_n \equiv 1$, $a_n = \langle e_n, \operatorname{Re} f\rangle$ and $b_n = \langle e_n, \operatorname{Im} f\rangle$.

(ii): It is given by Prop. 5.27 in (Arai, 2018) because we proved that $\{\hat{q}(f),\ \hat{p}(f)\}_{f\in\mathcal{W}}$ is a Heisenberg CCR representation.

(iii): It follows by Proposition 74.

(iv): Trivially $D(\hat{q}(f)) \cap D(\hat{p}(f)) \subseteq D(\overline{\hat{q}(f)}) \cap D(\overline{\hat{p}(f)})$, so,

$$\overline{\hat{q}(\operatorname{Re} f) + \hat{p}(\operatorname{Im} f)} \subseteq \overline{\overline{\hat{q}(\operatorname{Re} f)} + \overline{\hat{p}(\operatorname{Im} f)}}. \tag{B.38}$$

By (i), the l.h.s of (B.38) is self-adjoint, but by Thm. 5.48.(ii) in (Arai, 2018), its r.h.s is also self-adjoint. Thus, the r.h.s of (B.38) is a self-adjoint extension of the self-adjoint operator in the l.h.s. Because a self-adjoint operator has no proper symmetric extension (see Thm. 1.13 in (Arai, 2018)), the l.h.s must equal the r.h.s. With that, Thm. 5.48.(ii) in (Arai, 2018) gives the rest of the claims. ***o.ε.δ.***

Lemma 51. Let $\mathcal{H}$ be a Hilbert space and let $(A, D(A))$ be a densely defined operator reduced by the closed subspace $M \subseteq \mathcal{H}$. Then, $(A^*,\ D(A^*))$ is also reduced by M and $(A^*)|_M = (A|_M)^*$.

♦

Proof: • We check reducibility first. Let $\psi \in D(A^*)$. By definition $\exists \eta \in \mathcal{H} : \langle \psi, A\varphi\rangle = \langle \eta, \varphi\rangle\ \ \forall \varphi \in D(A)$ and $A^*\psi := \eta$. But then, denoting by P^M the orthogonal projector to M, for any $\varphi \in D(A)$,

$$\langle P^M\psi,\ A\varphi\rangle \overset{(P^M = (P^M)^*)}{=} \langle \psi, P^M A\varphi\rangle \overset{(P^M A \subseteq AP^M)}{=} \langle \psi, AP^M\varphi\rangle \overset{(P^M\varphi \in D(A))}{=} \langle \eta, P^M\varphi\rangle = \langle P^M\eta, \varphi\rangle.$$

Hence, by definition, $P^M\psi \in D(A^*)$ and $A^*(P^M\psi) = P^M\eta = P^M(A^*\psi)$, which is to say that $(A^*, D(A^*))$ is reduced by M.

- Now, let $\psi \in D((A^*)|_M) \equiv D(A^*) \cap M$. Then, for all $\varphi \in D(A|_M) = D(A) \cap M$,

$$\langle (A^*)|_M\psi,\ \varphi\rangle = \langle A^*\psi,\ \varphi\rangle = \langle \psi, A\varphi\rangle = \langle \psi,\ A|_M\varphi\rangle.$$

Hence, $\psi \in D((A|_M)^*)$ and $(A|_M)^*\psi = (A^*)|_M\psi$. That is, $(A^*)|_M \subseteq (A|_M)^*$.

- Now we prove the reverse inclusion. Let $\psi \in D((A|_M)^*) \subseteq M$. By definition, $\exists \eta \in M$ such that $\langle \psi,\ A|_M\varphi\rangle = \langle \psi, A\varphi\rangle = \langle \eta,\ \varphi\rangle$ for all $\varphi \in D(A) \cap M$ and $(A|_M)^*\psi := \eta$. To prove that $\psi \in D(A^*)$ as well, we need to extend this relation to any $\varphi \in D(A)$. Let $\varphi \in D(A)$ arbitrary. Then, $\varphi = P^M\varphi + (P^M)^\perp\varphi$. Using in $(\star)$ that since A is reduced by M it is also reduced by $M^\perp$ (see Def. 44),

$$\langle \psi,\ A\varphi\rangle = \langle \psi,\ A\,P^M\varphi\rangle + \langle \psi,\ A\,(P^M)^\perp\varphi\rangle \overset{(\star)}{=} \langle \psi,\ A|_M\,P^M\varphi\rangle + \underbrace{\langle \psi,\ (P^M)^\perp A\,\varphi\rangle}_{=0 \quad (\text{since } \psi\in M)} =$$

$$\overset{(P^M\varphi\in D(A)\cap M)}{=} \langle \eta,\ P^M\varphi\rangle = \langle \eta,\ P^M\varphi\rangle + \underbrace{\langle \eta,\ (P^M)^\perp\varphi\rangle}_{=0 \quad (\text{since } \eta\in M)} = \langle \eta,\ \varphi\rangle.$$

So, by definition, $\psi \in D(A^*)$ and $A^*\psi := \eta$. Since $\psi \in M$, then, $A^*\psi = (A^*)|_M\psi = \eta$. But by definition η was $(A|_M)^*\psi$, so, $(A|_M)^* \subseteq (A^*)|_M$. ***o.ε.δ.***

Theorem 31. Define for each $f \in \mathcal{W}_\mathbb{C}$ the operator

$$\hat{a}(f) := \frac{1}{\sqrt{2}}\overline{\Big(\widehat{\Phi}(f) + i\widehat{\Phi}(if)\Big)}. \tag{B.39}$$

Then, $\Big(\bigotimes_{k\in\mathbb{N}} L^2(\mathbb{R}, dx),\ \otimes^F_{k\in\mathbb{N}} B_{Her},\ \{\hat{a}(f)\}_{f\in\mathcal{W}_\mathbb{C}}\Big)$ is a *reducible representation of the Creation-Annihilation CCR*, reduced by every layer $\otimes^{\mathfrak{C}}_{j\in\mathbb{N}} L^2(\mathbb{R}, dx)$, $\mathfrak{C} \in \Gamma$. Moreover, for every $f \in \mathcal{W}_\mathbb{C}$,

(i) $\hat{a}(f)^* \supseteq \frac{1}{\sqrt{2}}\overline{\Big(\widehat{\Phi}(f) - i\widehat{\Phi}(if)\Big)}$ —very likely it is an equality.

(ii) $\overline{\widehat{\Phi}(f)} = \frac{1}{\sqrt{2}}\overline{\Big(\hat{a}(f)^* + \hat{a}(f)\Big)}$ and $\overline{\widehat{\Phi}(if)} = \frac{i}{\sqrt{2}}\overline{\Big(\hat{a}(f)^* - \hat{a}(f)\Big)}$.

(iii) For each $f \in \mathcal{W}$, $\widehat{\Phi}(f) = \hat{q}(f)$, $\widehat{\Phi}(if) = \hat{p}(f)$,

$$\overline{\hat{q}(f)} = \frac{1}{\sqrt{2}}\overline{\Big(\hat{a}(f)^* + \hat{a}(f)\Big)} \quad \text{and} \quad \overline{\hat{p}(f)} = \frac{i}{\sqrt{2}}\overline{\Big(\hat{a}(f)^* - \hat{a}(f)\Big)}.$$

◆

Proof: By Prop. 5.27 of (Arai, 2018) $\widehat{\Phi}(f) + i\widehat{\Phi}(if)$ is closable on $\otimes_{j\in\mathbb{N}} B_{Her}$ for all $f \in \mathcal{W}_\mathbb{C}$ and $\hat{a}(\cdot)$ gives a representation of the creation-annihilation CCR.

Now, by Corollary 30, all $\widehat{\Phi}(f)$ and $\widehat{\Phi}(if)$ are reduced by $\otimes^{\mathfrak{C}}_{j\in\mathbb{N}} L^2(\mathbb{R}, dx)$ $\forall \mathfrak{C} \in \Gamma$. Then, by Prop. 1.53 in (Arai, 2018), the same subspaces reduce $\Phi(f) + i\Phi(if)$ and by Thm. 1.47.(v), also $\overline{\widehat{\Phi}(f) + i\widehat{\Phi}(if)} = \hat{a}(f)$. In particular, these Prop. and Thm. also give us that

$$\hat{a}(f)|_{\mathfrak{C}} = \overline{\widehat{\Phi}(f)|_{\mathfrak{C}} + i\widehat{\Phi}(if)|_{\mathfrak{C}}}.$$

By Lemma 51, all this implies that $\hat{a}(f)^*$ is also reducible by $\otimes^{\mathfrak{C}}_{j\in\mathbb{N}} L^2(\mathbb{R}, dx)$. Therefore, by Def. 49 it is a *reducible* creation-annihilation CCR representation.

- **Item (i):** For all $\varphi \in D(\hat{a}(f))$, by definition, there exists a sequence $(\varphi_n)_{n\in\mathbb{N}} \subseteq \otimes^{\mathscr{F}}_{j\in\mathbb{N}} B_{Her}$ such that

$$\left(\varphi_n,\ \frac{1}{\sqrt{2}}\Big(\widehat{\Phi}(f) + i\widehat{\Phi}(if)\Big)\varphi_n\right) \xrightarrow[n\to\infty]{} \Big(\varphi,\ \hat{a}(f)\varphi\Big).$$

Consequently, for any $\psi \in \otimes^{\mathscr{F}}_{j\in\mathbb{N}} B_{Her}$ it holds that for all $\varphi \in D(\hat{a}(f))$,

$$\langle\psi, \hat{a}(f)\varphi\rangle = \Big\langle \psi,\ \lim_{n\to\infty}\frac{1}{\sqrt{2}}\Big(\widehat{\Phi}(f) + i\widehat{\Phi}(if)\Big)\varphi_n\Big\rangle \overset{(\langle\psi,\cdot\rangle \text{ is conts.})}{=} \lim_{n\to\infty}\Big\langle \psi,\ \frac{1}{\sqrt{2}}\Big(\widehat{\Phi}(f) + i\widehat{\Phi}(if)\Big)\varphi_n\Big\rangle =$$

$$= \lim_{n\to\infty}\frac{1}{\sqrt{2}}\left(\langle\psi,\ \widehat{\Phi}(f)\varphi_n\rangle + i\langle\psi, \widehat{\Phi}(if)\varphi_n\rangle\right) \overset{(\widehat{\Phi}(g)\text{ ess.s.a.}\Rightarrow sym.)}{=} \lim_{n\to\infty}\frac{1}{\sqrt{2}}\left(\langle\widehat{\Phi}(f)\psi,\ \varphi_n\rangle + i\langle\widehat{\Phi}(if)\psi, \varphi_n\rangle\right) =$$

$$= \lim_{n\to\infty}\Big\langle\frac{1}{\sqrt{2}}\Big(\widehat{\Phi}(f) - i\widehat{\Phi}(if)\Big)\psi,\ \varphi_n\Big\rangle = \Big\langle\frac{1}{\sqrt{2}}\Big(\widehat{\Phi}(f) - i\widehat{\Phi}(if)\Big)\psi,\ \varphi\Big\rangle.$$

Hence, $\psi \in D(\hat{a}(f)^*)$ and $\hat{a}(f)^*\psi = \frac{1}{\sqrt{2}}\Big(\widehat{\Phi}(f) - i\widehat{\Phi}(if)\Big)\psi$, namely,

$$\hat{a}(f)^* \restriction_{\otimes^{\mathscr{F}}_{j\in\mathbb{N}} B_{Her}} = \frac{1}{\sqrt{2}}\Big(\widehat{\Phi}(f) - i\widehat{\Phi}(if)\Big).$$

Since $\hat{a}(f)^*$ is a closed operator (because it is the adjoint of an operator), $\frac{1}{\sqrt{2}}\Big(\widehat{\Phi}(f) - i\widehat{\Phi}(if)\Big)$ is closable (we just proved the existence of a closed extension), and our claim is proven.

- **Item (ii):** For $\Psi \in \otimes^{\mathscr{F}}_{j\in\mathbb{N}} B_{Her}$,

$$\Big(\hat{a}(f)^* + \hat{a}(f)\Big)\Psi = \frac{1}{\sqrt{2}}\Big(\widehat{\Phi}(f) - i\widehat{\Phi}(if)\Big)\Psi + \frac{1}{\sqrt{2}}\Big(\widehat{\Phi}(f) + i\widehat{\Phi}(if)\Big)\Psi = \sqrt{2}\widehat{\Phi}(f)\Psi,$$

Hence,

$$\widehat{\Phi}(f) = \frac{1}{\sqrt{2}}\Big(\hat{a}(f)^* + \hat{a}(f)\Big) \restriction_{\otimes^{\mathscr{F}}_{j\in\mathbb{N}} B_{Her}}. \tag{B.40}$$

Similarly,

$$(\hat{a}(f)^* - \hat{a}(f))\Psi = \frac{1}{\sqrt{2}}\Big(\widehat{\Phi}(f) - i\widehat{\Phi}(if)\Big)\Psi - \Big(\widehat{\Phi}(f) + i\widehat{\Phi}(if)\Big)\Psi = \sqrt{2}i\widehat{\Phi}(if)\Psi.$$

implying that

$$\widehat{\Phi}(if) = \frac{i}{\sqrt{2}}\Big(\hat{a}(f)^* - \hat{a}(f)\Big) \restriction_{\otimes^{\mathscr{F}}_{j\in\mathbb{N}} B_{Her}}. \tag{B.41}$$

- Next, we prove that $\frac{1}{\sqrt{2}}(\hat{a}(f) + \hat{a}(f)^*)$ (defined, as usual, in $D(\hat{a}(f)) \cap D(\hat{a}(f)^*)$) is a symmetric operator: let $\psi, \varphi \in D(\hat{a}(f)) \cap D(\hat{a}(f)^*)$. Then,

$$\langle\psi,\ (\hat{a}(f)^* + \hat{a}(f))\varphi\rangle = \langle\psi,\ \hat{a}(f)^*\varphi\rangle + \langle\psi,\ \hat{a}(f)\varphi\rangle = \langle\underbrace{(\hat{a}(f)^*)^*}_{(\star)}\psi,\ \varphi\rangle + \langle\hat{a}(f)^*\psi,\ \varphi\rangle = \langle(\hat{a}(f)^* + \hat{a}(f))\psi,\ \varphi\rangle,$$

where in $(\star)$ we used that since $\hat{a}(f)$ is a closed operator $(\hat{a}(f)^*)^* = \hat{a}(f)$ (see e.g., 3.81 in (Teufel, 2021)).

- Now, since by Cor. 30, $\widehat{\Phi}(f)$ is essentially self-adjoint, by 3.93 in (Teufel, 2021), range$(\widehat{\Phi}(f) \pm i)$ is dense in the full space $\otimes_{j\in\mathbb{N}} L^2(\mathbb{R}, dx)$. But by equation (B.40), range$\Big(\widehat{\Phi}(f) \pm i\Big) \subseteq$range$\Big(\frac{1}{\sqrt{2}}\Big(\hat{a}(f)^* + \hat{a}(f)\Big) \pm i\Big)$ and thus, range$\Big(\frac{1}{\sqrt{2}}\Big(\hat{a}(f)^* + \hat{a}(f)\Big) \pm i\Big)$ must be dense as well. This, together with the afore-proven symmetry imply (by 3.93 in (Teufel, 2021)) that $\frac{1}{\sqrt{2}}\Big(\hat{a}(f)^* + \hat{a}(f)\Big)$ is essentially self-adjoint.

- This leaves that both, $\overline{\frac{1}{\sqrt{2}}\Big(\hat{a}(f)^* + \hat{a}(f)\Big)}$ and $\overline{\widehat{\Phi}(f)}$ are self-adjoint extensions of $\widehat{\Phi}(f)$. But then, the uniqueness of self-adjoint extension for an essentially self-adjoint operator (Prop. 1.21 in (Arai, 2018)) implies that $\overline{\frac{1}{\sqrt{2}}\Big(\hat{a}(f)^* + \hat{a}(f)\Big)} = \overline{\widehat{\Phi}(f)}$.

- The exact same arguments allows to lift equation (B.41) to $\overline{\frac{i}{\sqrt{2}}\Big(\hat{a}(f)^* - \hat{a}(f)\Big)} = \overline{\widehat{\Phi}(if)}$.

- **Item (iii):** For $f \in \mathcal{W}$, $\operatorname{Re} f = f$ and $\operatorname{Im} f = 0$, so $\widehat{\Phi}(f) = q(\operatorname{Re} f) + \cancelto{0}{\hat{p}(\operatorname{Im} f)} = \hat{q}(f)$. Then, (ii) yields what we claimed. Likewise, $\operatorname{Re}(if) = 0$ and $\operatorname{Im}(if) = f$ so, $\widehat{\Phi}(if) = \hat{p}(f)$, such that (ii) yields our claim. ***o.ε.δ.***

Proposition 79. Let there be the "ladder operators" of $L^2(\mathbb{R}, dx)$ $\Big(\hat{a} := \frac{1}{\sqrt{2}}(\hat{q} + i\hat{p}),\ B_{Her}\Big)$ and $\Big(\hat{a}^\dagger := \frac{1}{\sqrt{2}}(\hat{q} - i\hat{p}),\ B_{Her}\Big)$ —as in Lemma 43 ($\omega = 1$). Then,

$$\hat{a}_n := \hat{a}(e_n) = \overline{\underbrace{Id \otimes \cdots \otimes \hat{a}}_{n} \otimes Id \otimes \cdots} \quad \text{and} \quad \hat{a}_n^\dagger := \hat{a}(e_n)^* \supseteq \overline{\underbrace{Id \otimes \cdots \otimes \hat{a}_n^\dagger}_{n} \otimes Id \otimes \cdots},$$

—where one would get an equality in the second relation if proven so for Thm. 31.(i). ♦

Proof: Let $\Psi \in \otimes^{\mathscr{F}}_{j\in\mathbb{N}} B_{Her}$, so that for some $M \in \mathbb{N}$, $c_\ell \in \mathbb{C}$, $\psi_j^\ell \in B_{Her}$,

$$\hat{a}(e_n)\Psi = \frac{1}{\sqrt{2}}\Big(\widehat{\Phi}(e_n) \pm i\widehat{\Phi}(ie_n)\Big)\Psi = \frac{1}{\sqrt{2}}\Big(\hat{q}(e_n) \pm i\hat{p}(e_n)\Big)\Psi =$$

$$= \sum_{\ell=1}^{M} c_\ell \psi_1^\ell \otimes \cdots \otimes \left(\frac{1}{\sqrt{2}}(\hat{q} \pm i\hat{p})\psi_n^\ell\right) \otimes \psi_{n+1}^\ell \otimes \cdots = \left(Id \otimes \cdots \otimes \frac{1}{\sqrt{2}}(\hat{q} \pm i\hat{p}) \otimes Id \otimes \cdots\right)\Psi.$$

Thus, $\hat{a}_n \upharpoonright_{\otimes^{\mathscr{F}}_{j\in\mathbb{N}} B_{Her}} = (Id \otimes \cdots \otimes \hat{a} \otimes Id \otimes \cdots) \upharpoonright_{\otimes^{\mathscr{F}}_{j\in\mathbb{N}} B_{Her}}$ and $\hat{a}_n^\dagger \upharpoonright_{\otimes^{\mathscr{F}}_{j\in\mathbb{N}} B_{Her}} = (Id \otimes \cdots \otimes \hat{a}^\dagger \otimes Id \otimes \cdots) \upharpoonright_{\otimes^{\mathscr{F}}_{j\in\mathbb{N}} B_{Her}}$. This proves that both $Id \otimes \cdots \otimes \hat{a} \otimes Id \otimes \cdots$ and $Id \otimes \cdots \otimes \hat{a}^\dagger \otimes Id \otimes \cdots$ are closable (because $\hat{a}_n$ and $\hat{a}_n^\dagger$ are respectively closed extensions). Now, regarding the former, since $\otimes^{\mathscr{F}}_{j\in\mathbb{N}} B_{Her}$ is a core for both $\hat{a}_n$ and $\overline{(Id \otimes \cdots \otimes \hat{a} \otimes Id \otimes \cdots)}$, they must agree everywhere. Regarding the latter, we just have $\hat{a}_n^\dagger \supseteq \overline{\underbrace{Id \otimes \cdots \otimes \hat{a}_n^\dagger}_{n} \otimes Id \otimes \cdots}$. ***o.ε.δ.***

Hence, the creation-annihilation CCR we found for the ITP is precisely the lift of the one described in §1.3 for $L^2(\mathbb{R}, dx)$.

B.3 The Irreducible CCR Representations Induced in each Layer

In this section we will prove that the reduced parts (to each layer $\mathfrak{C} \in \Gamma$) of the CCR representations found in the last section, still yield CCR representations. Moreover, we will prove they are *irreducible* and we will provide *necessary and sufficient* conditions for their *equivalence*. For the latter sentence (which is the non-trivial part of our claim) we will follow the laudably clear proof-strategy by Streit (1967) —although we will leave much less work to the reader.[8]

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[8] Streit compactified the whole strategy to a single page —with a corresponding lack of exhaustivity, of course.

As usual when dealing with reducibility, we will need to use the theory of commutants. In particular, the following results are the key facts we need for our later proofs.

Definition 57. Given a Hilbert space $\mathcal{H}$ and a set of bounded operators $\mathcal{A} \subseteq \mathcal{L}(\mathcal{H})$, we denote its *commutant* by

$$\mathcal{A}' := \Big\{ B \in \mathcal{L}(\mathcal{H}) \;\Big|\; AB = BA \quad \forall A \in \mathcal{A} \Big\}. \tag{B.42}$$

Its *double commutant* will be denoted by $\mathcal{A}'' := (\mathcal{A}')'$. Similarly, we define $\mathcal{A}''' := ((\mathcal{A}')')'$ etc.♦

Definition 58. Given a Hilbert space $\mathcal{H}$, a set $\mathcal{A} \subseteq \mathcal{L}(\mathcal{H})$ is a $*$*-algebra* if it is a vector subspace, it is closed under adjoint ($A \in \mathcal{A} \Rightarrow A^* \in \mathcal{A}$) and closed under composition ($A, B \in \mathcal{A} \Rightarrow AB \in \mathcal{A}$). Then, by Def. 41, a W^**-algebra* is a $*$-algebra that is closed in the *weak operator topology*.♦

Lemma 52. Let $\mathcal{H}$ be a Hilbert space and $\mathcal{B}, \mathcal{C} \subseteq \mathcal{L}(\mathcal{H})$. Then,

(i) $\mathcal{B} \subseteq \mathcal{B}''$.

(ii) If $\mathcal{B} \subseteq \mathcal{C} \Longrightarrow \mathcal{C}' \subseteq \mathcal{B}' \Longrightarrow \mathcal{B}'' \subseteq \mathcal{C}''$.

(iii) $\mathcal{B}' = \mathcal{B}'''$ and thus, $(\mathcal{B}'')'' = \mathcal{B}''$. ♦

Proof: • **(i):** Let $A \in \mathcal{B}$. By definition, A commutes with every element of $\mathcal{B}'$, and hence $A \in (\mathcal{B}')' = \mathcal{B}''$.

• **(ii):** $A \in \mathcal{C}'$ if it commutes with every element of $\mathcal{C}$ and that includes every element of $\mathcal{B}$ by hypothesis. Hence $A \in \mathcal{B}'$.

• **(iii):** By (i) $\mathcal{B} \subseteq \mathcal{B}''$, so by (ii), $\mathcal{B}' \supseteq (\mathcal{B}'')' = \mathcal{B}'''$. But also by (i), $\mathcal{B}' \subseteq (\mathcal{B}')'' = \mathcal{B}'''$. Hence, $\mathcal{B}' = \mathcal{B}'''$. Applying this fact to $\mathcal{B}'$ we get $(\mathcal{B}')' = (\mathcal{B}')'''$, i.e., $\mathcal{B}'' = (\mathcal{B}'')''$. **o.ε.δ.**

Theorem 32. Let $\mathcal{H}$ be a Hilbert space and let $\mathcal{B} \subseteq \mathcal{L}(\mathcal{H})$ be a *non-degenerate* subset —i.e., such that the only vector in the kernel of all its operators is 0 (trivially satisfied when $Id \in \mathcal{B}$).

(i) *(von Neumann's double commutant theorem:)* If $\mathcal{B}$ is a $*$-algebra, then, $\mathcal{B}$ is a W^*-algebra *if and only if* $\mathcal{B}'' = \mathcal{B}$.

(ii) There is a minimal[9] $*$-algebra $\mathcal{A}$ containing $\mathcal{B}$ and the *closure* of $\mathcal{A}$ in both the *weak and strong operator topologies* is $\mathcal{A}''$. That is, $\mathcal{A}''$ is the set of limit points of operator nets in $\mathcal{A}$ that converge strongly (or equivalently, weakly) in $\mathcal{L}(\mathcal{H})$.

(iii) $\mathcal{B}''$ equals the *smallest* W^**-algebra* containing $\mathcal{B}$. We refer to this as $\mathcal{B}''$ being the W^*-algebra generated by $\mathcal{B}$. In particular, $\mathcal{B}'' = \mathcal{A}''$. ♦

Proof: • **Item (i):** See Thm. 2.4.11 of (Bratteli and Robinson, 2012). Note that in the reference it is the condition $\mathcal{B}'' = \mathcal{B}$ what defines a W^*-algebra, whereas for us it is the closure in the weak topology what makes a $*$-algebra be also a W^*-algebra. Yet, their statement still yields what we wanted. For the characterization of non-degeneracy that we used in our claim, see the discussion before Def. 2.3.5 in (Bratteli and Robinson, 2012).

• **Item (ii):** By Lemma 15.1 in (Arai, 2018), there exists a (unique) minimal $*$-subalgebra of $\mathcal{L}(\mathcal{H})$ containing $\mathcal{B}$. Let us call it $\mathcal{A}$.

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[9] Minimal in the sense that any other $*$-subalgebra of $\mathcal{L}(\mathcal{H})$ containing $\mathcal{B}$ also contains $\mathcal{A}$.

Any super-set of $\mathcal{B}$, say, $\mathcal{A}$ or $\mathcal{A}''$, is non-degenerate because the intersection of the kernels of their operators must be a subset of the intersection of kernels in $\mathcal{B}$ (which by non-degeneracy is $\{0\}$). On the other hand, by Lemma 52.(iii) $(\mathcal{A}'')'' = \mathcal{A}''$, so, by item (i) $\mathcal{A}''$ is a W^*-algebra and as such, it is closed under the weak topology. Putting this two facts together, by Cor. 2.4.15 in (Bratteli and Robinson, 2012), $\mathcal{A}$ is a dense subspace of $\mathcal{A}''$ in the weak and strong topologies. Namely, $\mathcal{A}''$ is the closure of $\mathcal{A}$ in both topologies.

Lastly, the statement about nets follows from the net characterization of closure —see for instance Thm. 11.7 in (Willard, 2012).

- **Item (iii):** Since any $*$-algebra containing $\mathcal{B}$ also contains $\mathcal{A}$, then, any W^* algebra containing $\mathcal{B}$ will need to contain $\mathcal{A}$. Hence, it is enough to find the minimal W^*-algebra that contains $\mathcal{A}$. But by (ii), $\mathcal{A}''$ is a W^*-algebra and it is the closure of $\mathcal{A}$ in the weak-topology, so there can be no other W^* algebra properly contained in between $\mathcal{A}$ and $\mathcal{A}''$ and any bigger W^*-algebra must contain $\mathcal{A}''$. ***o.ε.δ.***

Lemma 53. Let $\mathcal{H}$ be a Hilbert space and for each $j \in I$, with I an arbitrary index set, let there be a set $\mathcal{B}_j \subseteq \mathcal{L}(\mathcal{H})$. Then,

$$\left(\bigcup_{j\in I} \mathcal{B}_j\right)'' = \left(\bigcup_{j\in I} \mathcal{B}_j''\right)''.$$

♦

Proof: • First we prove $\subseteq$. By Lemma 32.(i), $\mathcal{B}_j \subseteq \mathcal{B}_j''$ and hence, $\cup_{j\in I}\mathcal{B}_j \subseteq \cup_{j\in I}\mathcal{B}_j''$. But then, by Lem. 32.(ii), $(\bigcup_{j\in I} \mathcal{B}_j)'' \subseteq (\bigcup_{j\in I} \mathcal{B}_j'')''$.

- We now prove $\supseteq$. Certainly, $\mathcal{B}_j \subseteq \cup_{j\in I}\mathcal{B}_j$ so by Lem. 32.(ii), $\mathcal{B}_j'' \subseteq \left(\cup_{j\in I}\mathcal{B}_j\right)''$. In particular, $\cup_{j\in I}\mathcal{B}_j'' \subseteq \left(\cup_{j\in I}\mathcal{B}_j\right)''$ and applying Lem. 32.(ii) again, $\left(\cup_{j\in I}\mathcal{B}_j''\right)'' \subseteq \left(\left(\cup_{j\in I}\mathcal{B}_j\right)''\right)''$. But, by Lemma 52.(iii), the r.h.s equals $(\cup_{j\in I}\mathcal{B}_j)''$. Hence, $\left(\cup_{j\in I}\mathcal{B}_j''\right)'' \subseteq \left(\cup_{j\in I}\mathcal{B}_j\right)''$. ***o.ε.δ.***

Next, we check the compatibility of the lift of a set of operators and their double commutant.

Proposition 80. Let $(\mathcal{H}_j)_{j\in I}$ be Hilbert spaces with I an arbitrary index set. Let there be a fixed $k \in I$ and a non-degenerate subset $\mathcal{B}_k \subseteq \mathcal{L}(\mathcal{H}_k)$. Denoting the set of lifted bounded operators of $\mathcal{B}_k$ by $\widehat{\mathcal{B}_k}$ (as in Prop. 62), then $\widehat{(\mathcal{B}_k)''} = \left(\widehat{\mathcal{B}_k}\right)''$. ♦

Proof: First, both $(\mathcal{B}_k)''$ and $(\widehat{\mathcal{B}_k})''$ are W^*-algebras due to Thm. 32.(iii). Then, by Prop. 64.(ii), $\widehat{(\mathcal{B}_k)''}$ is also a W^*-algebra. All of them are trivially non-degenerate given that $\mathcal{B}_k$ is so.

- Let us prove $\supseteq$. By Lem. 52.(i), $\mathcal{B}_k \subseteq (\mathcal{B}_k)''$, which trivially implies that $\widehat{\mathcal{B}_k} \subseteq \widehat{(\mathcal{B}_k)''}$ —recall that (by Prop. 63) $\widehat{\cdot}$ is a $*$-isomorphism. Therefore, by Lem. 52.(ii), $(\widehat{\mathcal{B}_k})'' \subseteq (\widehat{(\mathcal{B}_k)''})''$ (denote this relation by $(\star)$). Finally, because $\widehat{(\mathcal{B}_k)''}$ is a non-degenerate W^*-algebra, by Thm. 32.(i), $(\widehat{(\mathcal{B}_k)''})'' = \widehat{(\mathcal{B}_k)''}$, which, plugged in $(\star)$, yields $(\widehat{\mathcal{B}_k})'' \subseteq \widehat{(\mathcal{B}_k)''}$.

- We prove now $\subseteq$. By Thm. 32.(ii), given $\mathcal{A}_k$ is the smallest $*$-subalgebra of $\mathcal{L}(\mathcal{H}_k)$ containing $\mathcal{B}_k$, every element $C \in (\mathcal{A}_k)''$ is the strong limit of a net of operators $(C_\alpha)_{\alpha\in K} \subseteq \mathcal{A}_k$. Now, for every $\alpha \in K$, $\widehat{C_\alpha} \in \widehat{\mathcal{A}_k}$, so, the fact (proven in $(\star\star)$ below) that $(\widehat{C}_\alpha)_{\alpha\in K}$ converges in the strong topology to $\widehat{C} \in \widehat{\mathcal{L}(\mathcal{H}_k)}$ implies that $\widehat{C} \in (\widehat{\mathcal{A}_k})''$, after all, $(\widehat{\mathcal{A}_k})''$ is also closed in the strong topology. Namely, $\widehat{(A_k)''} \subseteq (\widehat{\mathcal{A}_k})''$. But we proved in Thm. 32.(iii) that $(\mathcal{B}_k)'' = (\mathcal{A}_k)''$, so, $\widehat{(\mathcal{B}_k)''} \subseteq (\widehat{\mathcal{A}_k})''$ (call this result $(\star\star\star)$). Finally, that $\widehat{\cdot}$ is a $*$-isomorphism from $\mathcal{L}(\mathcal{H}_k)$ to the

$*$-subalgebra $\widehat{\mathcal{L}(\mathcal{H}_k)}$ of $\mathcal{L}(\otimes_{j\in I}\mathcal{H}_j)$ implies that if $\mathcal{A}_k$ is the $*$-algebra generated by $\mathcal{B}_k$, $\widehat{\mathcal{A}_k}$ is the $*$-algebra generated by $\widehat{\mathcal{B}_k}$. Hence, by Thm. 32.(iii), $(\widehat{\mathcal{A}_k})'' = (\widehat{\mathcal{B}_k})''$. Plugging this in $(\star\star\star)$ leaves $\widehat{(\mathcal{B}_k)''} \subseteq (\widehat{\mathcal{B}_k})''$.

Proof of $(\star\star)$**:** First, note that $(C_\alpha)_{\alpha\in K}$ being strongly convergent means that $\forall\psi \in \mathcal{H}_k$, $\|C_\alpha\psi - C\psi\| \xrightarrow[\alpha\in K]{} 0$, which in particular implies that $\sup_{\alpha\in K}\{\|C_\alpha\psi\|\} < +\infty$ $\forall\psi \in \mathcal{H}_k$. This implies, by the uniform boundedness principle (Sokal, 2011), that $\beta := \sup_{\alpha\in K}\|C_\alpha\|_{op} < +\infty$. In particular, $\|C\|_{op} \leq \beta$.[a]

Now, let Ψ be a non-zero vector in the span of elementary tensors, namely, $\Psi = \sum_{\ell=1}^M c_\ell \otimes_{j\in I}\psi_j^\ell$ for some $M \in \mathbb{N}$, $c_\ell \in \mathbb{C}$, $\otimes_{j\in I}\psi_j^\ell \in \otimes_{j\in I}\mathcal{H}_j \setminus \{0\}$. Then,

$$\left\|\widehat{C_\alpha}\Psi - \widehat{C}\Psi\right\| = \left\|\sum_{\ell=1}^M c_\ell \bigotimes_{j\in I\setminus\{k\}} \psi_j^\ell \otimes \left((C_\alpha - C)\psi_k^\ell\right)\right\| \leq \sum_{k=1}^M |c_\ell| \underbrace{\frac{\left\|\bigotimes_{j\in I}\psi_j^\ell\right\|}{\left\|\psi_k^\ell\right\|}}_{\text{(constant)}} \left\|(C_\alpha - C)\psi_k^\ell\right\| \xrightarrow[\alpha\in K]{(\dagger)} 0,$$

where we used in $(\dagger)$ that $C_\alpha \xrightarrow[\alpha\in K]{\text{strongly}} C$.

Now, let $\Phi \in \bigotimes_{j\in I}\mathcal{H}_j$ and let $\varepsilon > 0$, both arbitrary. Then, by density, there exists a Ψ in the span of elementary tensors such that $\|\Phi - \Psi\| \leq \varepsilon/(3\beta)$. This leaves,

$$\begin{aligned}\left\|\widehat{C_\alpha}\Phi - \widehat{C}\Phi\right\| \leq \left\|\widehat{C_\alpha}(\Phi - \Psi)\right\| + \left\|(\widehat{C}_\alpha - \widehat{C})\Psi\right\| + \left\|\widehat{C}(\Psi - \Phi)\right\| \overset{(\dagger\dagger)}{\leq} 2\beta\|\Phi - \Psi\| + \left\|(\widehat{C}_\alpha - \widehat{C})\Psi\right\| \leq \\ \leq \frac{2\varepsilon}{3} + \left\|(\widehat{C}_\alpha - \widehat{C})\Psi\right\|. \qquad \text{(B.43)}\end{aligned}$$

In $(\dagger\dagger)$ we used that by Prop. 62, $\left\|\widehat{C_\alpha}\right\|_{op} = \|C_\alpha\|_{op}$, such that $\left\|\widehat{C_\alpha}\right\|_{op} \leq \beta$ —likewise for $\widehat{C}$.

Finally, by definition of convergence, $\exists M^\varepsilon \in K$ such that $\left\|(\widehat{C}_\alpha - \widehat{C})\Psi\right\| \leq \varepsilon/3$ for all $\alpha \geq M^\varepsilon$ (where $\geq$ is the relation of the directed set K). Then, (B.43) leaves

$$\left\|\widehat{C_\alpha}\Phi - \widehat{C}\Phi\right\| \leq \varepsilon \quad \forall\alpha \geq M^\varepsilon.$$

Thus, $\widehat{C}_\alpha$ converges strongly to the operator $\widehat{C}$ (now everywhere in $\bigotimes_{j\in I}\mathcal{H}_j$). **o.ε.δ.**

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[a] The reason is that otherwise, there would exist $\varepsilon > 0$ and a unit vector $\psi \in \mathcal{H}_k$ such that $\|C\psi\| \geq \beta + \varepsilon$. But then, $\|C_\alpha\psi - C\psi\| \xrightarrow[\alpha\in K]{} 0$ implies $\exists\alpha_0 \in K$ such that $\Big|\|C_{\alpha_0}\psi\| - \|C\psi\|\Big| \leq \|C_{\alpha_0}\psi - C\psi\| \leq \varepsilon/2$, and thus, $\|C_{\alpha_0}\psi\| \geq \|C\psi\| - \varepsilon/2 \geq \varepsilon/2 + \beta$, which is absurd, because $\|C_{\alpha_0}\|_{op} \leq \beta$.

Now, let us get back to our case study.

Lemma 54. Given the position and momentum operators $(\hat{q},\ B_{Her})$, $(\hat{p},\ B_{Her})$ (which we proved to be essentially self-adjoint) the set $W := \{e^{it\hat{p}},\ e^{it\hat{q}} \mid t \in \mathbb{R}\}$ is *irreducible* and $W'' = \mathcal{L}(L^2(\mathbb{R}, dx))$. ♦

Proof: Irreducibility is given by Prop. 14.7 in (Hall, 2013). Then, because W is trivially closed under adjoints (hence, it is $*$-invariant), by Prop. 5.11 in (Arai, 2018), $W' = \mathbb{C}Id := \{zId \mid z \in \mathbb{C}\}$. Therefore,

$$W'' = (\mathbb{C}Id)' \overset{[a]}{=} \mathcal{L}(L^2(\mathbb{R}, dx)).$$

o.ε.δ.

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[a] Let $\mathcal{H}$ be a Hilbert space. Then, by definition, $B \in (\mathbb{C}Id_{\mathcal{H}})'$ *if and only if* $\forall z \in \mathbb{C}$, $zId_{\mathcal{H}}\ B = zB\ Id_{\mathcal{H}}$. But, this is true for all $B \in \mathcal{L}(\mathcal{H})$, so, $(\mathbb{C}Id)' = \mathcal{L}(\mathcal{H})$.

Corollary 31. $\left\{e^{i\overline{\hat{q}(f)}},\ e^{i\overline{\hat{p}(f)}}\ \middle|\ f\in\mathcal{W}\right\}''=\mathcal{L}^q\Big(\bigotimes_{j\in\mathbb{N}}\mathcal{H}_j\Big)$ —see Definition 42.

(In words, *the quasi-local operators are precisely the weak (or strong) limits of compositions and linear combinations of* $e^{i\overline{\hat{q}(f)}},\ e^{i\overline{\hat{q}(f)}}$.) ♦

Proof: Given Hilbert spaces $\mathcal{H},(\mathcal{H}_j)_{j\in I}$ and given for some $k\in I$, an operator $A\in\mathfrak{L}(\mathcal{H}_k)$, if $\mathcal{H}_j=\mathcal{H}$ for all $j\in I$, it is unclear "to which factor" A is lifted when we use the notation $\widehat{A}$. To be clear, we will use $\widehat{A}^{(k)}$ to denote the lift of A as an operator on $\mathcal{H}_k$.

- Denoting $\hat{q}_n:=\hat{q}(e_n)$ and $\hat{p}_n:=\hat{p}(e_n)$, by Prop. 78, $\forall n\in\mathbb{N}$ and $t\in\mathbb{R}$, $e^{it\,\overline{\hat{q}_n}}=\widehat{e^{it\,\overline{\hat{q}}}}^{(n)}$ and $e^{it\,\overline{\hat{p}_n}}=\widehat{e^{it\,\overline{\hat{p}}}}^{(n)}$. Thus, defining $W:=\{e^{it\hat{p}},\ e^{it\hat{q}}\mid t\in\mathbb{R}\}$,

$$\left\{e^{it\,\overline{\hat{q}_n}},e^{it\,\overline{\hat{p}_n}}\ \middle|\ t\in\mathbb{R}\right\}=\widehat{W}^{(n)}\tag{B.44}$$

But then,

$$\left\{e^{it\,\overline{\hat{q}_n}},e^{it\,\overline{\hat{p}_n}}\ \middle|\ t\in\mathbb{R}\right\}''=\left(\widehat{W}^{(n)}\right)''\overset{\text{(Prop. 80)}}{=}\widehat{(W'')}^{(n)}\overset{\text{(Lemma 54)}}{=}\widehat{\mathcal{L}(L^2(\mathbb{R}))}^{(n)}.$$

Using this in $(\star)$,

$$\left(\bigcup_{n\in\mathbb{N}}\left\{e^{it\,\overline{\hat{q}_n}},e^{it\,\overline{\hat{p}_n}}\ \middle|\ t\in\mathbb{R}\right\}\right)''\overset{\text{(Lem. 53)}}{=}\left(\bigcup_{n\in\mathbb{N}}\left\{e^{it\,\overline{\hat{q}_n}},e^{it\,\overline{\hat{p}_n}}\ \middle|\ t\in\mathbb{R}\right\}''\right)''\overset{(\star)}{=}\left(\bigcup_{n\in\mathbb{N}}\widehat{\mathcal{L}(L^2(\mathbb{R}))}^{(n)}\right)''=$$

$$\overset{\text{(Def. 42)}}{=}\mathcal{L}^q\Big(\bigotimes_{j\in\mathbb{N}}L^2(\mathbb{R},dx)\Big).\tag{B.45}$$

Finally, as proven in the first line of equation (B.35), for any $f:=\sum_{n=1}^{N_f}\langle e_n,f\rangle e_n\in\mathcal{W}$,

$$e^{i\overline{\hat{q}(f)}}=e^{i\langle e_1,f\rangle\overline{\hat{q}_1}}\cdots e^{i\langle e_{N_f},f\rangle\overline{\hat{q}_{N_f}}}.$$

Since a $*$-algebra is closed under finite compositions, this implies that any $*$-algebra containing $\bigcup_{n\in\mathbb{N}}\left\{e^{it\,\overline{\hat{q}_n}},e^{it\,\overline{\hat{p}_n}}\ \middle|\ t\in\mathbb{R}\right\}$ must also contain $\left\{e^{i\,\overline{\hat{q}(f)}},e^{i\,\overline{\hat{p}(f)}}\ \middle|\ f\in\mathcal{W}\right\}$ —while the reciprocal is trivial. Therefore,

$$\left\{e^{i\,\overline{\hat{q}(f)}},e^{i\,\overline{\hat{p}(f)}}\ \middle|\ f\in\mathcal{W}\right\}''=\left(\bigcup_{n\in\mathbb{N}}\left\{e^{it\,\overline{\hat{q}_n}},e^{it\,\overline{\hat{p}_n}}\ \middle|\ t\in\mathbb{R}\right\}\right)'',$$

which, by equation (B.45) leaves what we wanted to prove. ***o.ε.δ.***

It is known that heuristically, the W^*-algebra determined by a SCOPUG equals that determined by its self-adjoint generator (or more rigorously, that determined by the *spectral PVM* of the generator). If so, the last theorem implies that heuristically, only the quasi-local operators (and precisely all those) can be written in terms of $\hat{q}(f),\hat{p}(f)$'s, their compositions, adjoints, linear combinations and weak limits. If this were true, then, in order to exploit all the possible dynamics in $\otimes_{j\in\mathbb{N}}L^2(\mathbb{R},dx)$ (say, the ones given by unitaries outside the quasi-local operators), one would need to consider Hamiltonians beyond those achievable in terms of $\hat{q}(f)'s$ and $\hat{p}(f)'s$, with $f\in\mathcal{W}$. We will explore the true extent of this heuristic idea rigorously elsewhere.

As a final step before we provide the promised results, we need to check that "reduction" and "double commutant" commute.

Lemma 55. Let $\mathcal{H}$ be a Hilbert space and let $\mathcal{B} \subseteq \mathcal{L}(\mathcal{H})$ be a set closed under adjoint[10] where every operator is reduced by a common closed subspace $M \subseteq \mathcal{H}$. Then, $(\mathcal{B}'')|_M = (\mathcal{B}|_M)''$ —note that in the l.h.s the commutants are taken within $\mathcal{L}(\mathcal{H})$ while in the r.h.s they are taken in $\mathcal{L}(M)$.

♦

Proof: First, let us prove that $\mathcal{B}'$ is a $*$-algebra. It is clear that it is a vector space so we just need to check its closure under $*$. Let $C \in \mathcal{B}'$ arbitrary. Then, $\forall A \in \mathcal{B}$, $AC = CA$ and (using that $\mathcal{B}$ is $*$-invariant) $A^*C = CA^*$. Using the latter in (†), $AC^* = (CA^*)^* \overset{(\dagger)}{=} (A^*C)^* = C^*A$, so $C^* \in \mathcal{B}'$ too.

But then, $\mathcal{B}'$ is a $*$-algebra with $(\mathcal{B}')'' = \mathcal{B}''' \overset{(\star)}{=} \mathcal{B}'$ (where we used Lemma 52 in ($\star$)), so, by Theorem 32.(i), B' is also a W^*-algebra.

Now, $\mathcal{B}$ is reduced by M, so, by definition, P^M commutes with all the operators in $\mathcal{B}$, which implies that $P^M \in \mathcal{B}'$.

With all this, Prop. 1.(i) in Chapter 2 of (Dixmier, 1981) implies (putting his $\mathcal{A}$ equal to our $\mathcal{B}'$) that $(\mathcal{B}'')|_M = ((\mathcal{B}')|_M)'$. Hence, if we prove that $(\mathcal{B}')|_M = (\mathcal{B}|_M)'$ this will yield what we wanted. We prove so in the remaining.

- Let $C \in \mathcal{B}'$. Recall that since C need not be reduced by M, $C|_M := P^M C \in \mathcal{L}(M)$. Then, for all $A \in \mathcal{B}$

$$C|_M A|_M \overset{\left(\substack{\text{by Def. 44 \&}\\ \text{A reduced by } M}\right)}{=} (P^M CA)|_M \overset{(C\in\mathcal{B}')}{=} (P^M AC)|_M \overset{(P^M\in\mathcal{B}')}{=} (AP^M C)|_M \overset{\left(\substack{\text{by Def. 44 \&}\\ \text{A reduced by } M}\right)}{=} A|_M C|_M .$$

Hence, $C|_M \in (\mathcal{B}|_M)'$. This proves that $(\mathcal{B}')|_M \subseteq (\mathcal{B}|_M)'$.

- Now, let $c \in (\mathcal{B}|_M)'$ (which is an operator in $\mathcal{L}(M)$). By definition, for all $A|_M \in \mathcal{B}|_M$,

$$c\ A|_M = A|_M\ c. \tag{B.46}$$

Then, the operator on $\mathcal{L}(\mathcal{H})$ defined by $C := cP^M$ is trivially reduced by M and satisfies $C|_M = c$. But, because any $A \in \mathcal{B}$ is reduced by M, such that by Def. 44, $A = AP^M + A(P^M)^\perp$,

$$CA = cP^M AP^M + c\underbrace{P^M A}_{AP^M}(P^M)^\perp \overset{(P^M(P^M)^\perp=0)}{=} \underbrace{cAP^M}_{cA|_M} P^M \overset{(\text{B.46})}{=} AP^M cP^M = AC.$$

Therefore, $C \in \mathcal{B}'$ and $c = C|_M \in (\mathcal{B}')|_M$, proving that $(\mathcal{B}|_M)' \subseteq (\mathcal{B}')|_M$. ***o.ε.δ.***

With that we get the key result to prove the irreducibility of the Weyl CCR representations when reduced to the layers of the ITP.

Theorem 33 (*Streit*)**.** For each $\mathfrak{C} \in \Gamma$, $\left\{ e^{i\,\overline{\hat{q}(f)}}|_{\mathfrak{C}},\ e^{i\,\overline{\hat{p}(f)}}|_{\mathfrak{C}} \;\middle|\; f \in \mathcal{W} \right\}$ is *irreducible* and

$$\left\{ e^{i\,\overline{\hat{q}(f)}}|_{\mathfrak{C}},\ e^{i\,\overline{\hat{p}(f)}}|_{\mathfrak{C}} \;\middle|\; f \in \mathcal{W} \right\}'' = \mathcal{L}\Big(\otimes^{\mathfrak{C}}_{j\in\mathbb{N}} L^2(\mathbb{R}, dx) \Big).$$

Note that $e^{i\,\overline{\hat{q}(f)}}|_{\mathfrak{C}} = e^{i\,\overline{\hat{q}(f)}|_{\mathfrak{C}}} = e^{i\,\overline{\hat{q}(f)|_{\mathfrak{C}}}}$ and $e^{i\,\overline{\hat{p}(f)}}|_{\mathfrak{C}} = e^{i\,\overline{\hat{p}(f)}|_{\mathfrak{C}}} = e^{i\,\overline{\hat{p}(f)|_{\mathfrak{C}}}}$.

♦

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[10] That is, such that $\forall A \in \mathcal{B}$, $A^* \in \mathcal{B}$.

Proof: By Theorem 30 all the operators in $\left\{e^{i\overline{\hat{q}(f)}},\ e^{i\overline{\hat{p}(f)}}\ \middle|\ f\in\mathcal{W}\right\}$ are reduced by every $\otimes^{\mathfrak{C}}_{j\in\mathbb{N}}L^2(\mathbb{R},dx)$, $\mathfrak{C}\in\Gamma$. At the same time, it is closed under adjoint because $(e^{i\overline{\hat{q}(f)}})^* \overset{\text{(unitary)}}{=} e^{-i\overline{\hat{q}(f)}} \overset{\left(\substack{\text{Thm. }30\\ \text{\& Def. }47.\text{(ii)}}\right)}{=} e^{i\overline{\hat{q}(-f)}}$. Using these two facts in $(\star)$, for each fixed $\mathfrak{C}\in\Gamma$,

$$\left\{e^{i\,\overline{\hat{q}(f)}}|_{\mathfrak{C}},\ e^{i\,\overline{\hat{p}(f)}}|_{\mathfrak{C}}\ \middle|\ f\in\mathcal{W}\right\}'' \overset{\left(\substack{(\star)\text{ and}\\ \text{Lem. }55}\right)}{=} \left(\left\{e^{i\overline{\hat{q}(f)}},\ e^{i\overline{\hat{p}(f)}}\ \middle|\ f\in\mathcal{W}\right\}''\right)\Big|_{\mathfrak{C}} \overset{\text{(Cor. 31)}}{=} \left(\mathcal{L}^q\Big(\bigotimes_{j\in\mathbb{N}}L^2(\mathbb{R},dx)\Big)\right)\Big|_{\mathfrak{C}}.$$

Now, by Thm. 28.(iii), for every $A^{\mathfrak{C}}\in\mathcal{L}\Big(\otimes^{\mathfrak{C}}_{j\in\mathbb{N}}L^2(\mathbb{R},dx)\Big)$ there exists an $A\in\mathcal{L}^q\Big(\otimes_{j\in\mathbb{N}}L^2(\mathbb{R},dx)\Big)$ reduced by $\mathfrak{C}$ and such that $A|_{\mathfrak{C}}=A^{\mathfrak{C}}$. But the converse also holds: by Thm. 28, every $A\in\mathcal{L}^q(\otimes_{j\in\mathbb{N}}L^2(\mathbb{R},dx))$ is reduced by $\mathfrak{C}$ and trivially, $A|_{\mathfrak{C}}\in\mathcal{L}(\otimes^{\mathfrak{C}}_{j\in\mathbb{N}}L^2(\mathbb{R},dx))$. Therefore,

$$\left(\mathcal{L}^q\Big(\bigotimes_{j\in\mathbb{N}}L^2(\mathbb{R},dx)\Big)\right)\Big|_{\mathfrak{C}}=\mathcal{L}\Big(\otimes^{\mathfrak{C}}_{j\in\mathbb{N}}L^2(\mathbb{R},dx)\Big),\quad \text{s.th.},\quad \left\{e^{i\,\overline{\hat{q}(f)}}|_{\mathfrak{C}},\ e^{i\,\overline{\hat{p}(f)}}|_{\mathfrak{C}}\ \middle|\ f\in\mathcal{W}\right\}''=\mathcal{L}\Big(\otimes^{\mathfrak{C}}_{j\in\mathbb{N}}L^2(\mathbb{R},dx)\Big).$$

With all,

$$\left\{e^{i\,\overline{\hat{q}(f)}}|_{\mathfrak{C}},\ e^{i\,\overline{\hat{p}(f)}}|_{\mathfrak{C}}\ \middle|\ f\in\mathcal{W}\right\}' \overset{\text{(Lem. 52.(iii))}}{=} \left\{e^{i\,\overline{\hat{q}(f)}}|_{\mathfrak{C}},\ e^{i\,\overline{\hat{p}(f)}}|_{\mathfrak{C}}\ \middle|\ f\in\mathcal{W}\right\}'''=\mathcal{L}\Big(\otimes^{\mathfrak{C}}_{j\in\mathbb{N}}L^2(\mathbb{R},dx)\Big)'=$$

$$\overset{\text{(footnote [a] in Lemma 54)}}{=} \mathbb{C}Id_{\otimes^{\mathfrak{C}}_{j\in\mathbb{N}}L^2(\mathbb{R},dx)}.$$

This means that the only orthogonal projectors that commute with every operator in $\left\{e^{i\,\overline{\hat{q}(f)}}|_{\mathfrak{C}},\ e^{i\,\overline{\hat{p}(f)}}|_{\mathfrak{C}}\ \middle|\ f\in\mathcal{W}\right\}$ are Id and 0, namely, the projectors to the whole space and $\{0\}$. As such, there is no non-trivial reducing space common to every operator: it is an irreducible set.

- About the last statement of the theorem, we found $\hat{q}(f),\hat{p}(f)$ to be reduced by $\otimes^{\mathfrak{C}}_{j\in\mathbb{N}}L^2(\mathbb{R},dx)$, so, by Thm. 1.47.(v) in (Arai, 2018), $\overline{\hat{q}(f)},\overline{\hat{p}(f)}$ are reduced by $\otimes^{\mathfrak{C}}_{j\in\mathbb{N}}L^2(\mathbb{R},dx)$ and $\overline{\hat{q}(f)}|_{\mathfrak{C}}=\overline{\hat{q}(f)|_{\mathfrak{C}}}$, $\overline{\hat{p}(f)}|_{\mathfrak{C}}=\overline{\hat{p}(f)|_{\mathfrak{C}}}$. From there, Thm. 1.49 in (Arai, 2018) with $f(x):=e^{ix}$ gives what we claimed. ***o.ε.δ.***

Similarly, after the following technical lemma, we prove the key ingredients of the necessary and sufficient conditions for the representation equivalence between layers $\mathfrak{C}\in\Gamma$.

Lemma 56. Let $\mathcal{H}$ be Hilbert spaces and let $M,N\subseteq\mathcal{H}$ be closed subspaces. Let $\mathcal{B}\subseteq\mathcal{L}(\mathcal{H})$ be such that $\mathcal{B}''$ is reduced by both M and N. Let there exist a unitary $U:M\to N$ such that $UA|_MU^{-1}=A|_N$ for all $A\in\mathcal{B}$. Then, for all $C\in\mathcal{B}''$, $UC|_MU^{-1}=C|_N$. ♦

Proof: The set $S:=\{C\in\mathcal{B}''\ :\ UC|_MU^{-1}=C|_N\}$ contains $\mathcal{B}$ by hypothesis. In particular, $\mathcal{B}\subseteq S\subseteq\mathcal{B}''$. Then, if we prove that S is a W^* algebra, because by Theorem 32.(iii) $\mathcal{B}''$ is the smallest W^*-algebra containing $\mathcal{B}$, it will imply that $S=\mathcal{B}''$, proving the lemma.

- The identity and the zero operator are always in $\mathcal{B}''$ and they satisfy the condition to be in S trivially.

- Using that if two operators are each reduced by a same subspace their sum is also reduced by that subspace (Prop. 1.53 (Arai, 2018)) and using that U is linear, S can be immediately seen to be a vector space.

- We check that S is closed under adjoint. Let $C \in S$. Since C is reduced by both N and M, Lemma 51 implies that so is C^* and that $C^*|_M = (C|_M)^*$, $C^*|_N = (C|_N)^*$. But then,

$$UC^*|_M U^{-1} = U(C|_M)^* U^{-1} = (UC|_M U^{-1})^* \overset{(C \in S)}{=} (C|_N)^* = C^*|_N.$$

Therefore, $C^* \in S$.

- We check that S is closed under composition. Let $C, D \in S$. Then, using that given two bounded operators reduced by the same space, the composition is reduced by the same space,[a] CD is reducible by both M and N. Moreover, trivially, $(CD)|_M = (C|_M D|_M)$. Hence,

$$U(CD)|_M U^{-1} = UC|_M D|_M U^{-1} = UC|_M U^{-1} UD|_M U^{-1} \overset{(C,D \in S)}{=} C|_N D|_N = (CD)|_N$$

and thus, $CD \in S$.

- Finally, we check that S is a closed set in the strong topology. Let $(C_\alpha)_{\alpha \in K} \subseteq S$ be an arbitrary convergent net in the strong topology, with limit point $C \in \mathcal{L}(\mathcal{H})$. First, because $\mathcal{B}''$ is a W^*-algebra, it is closed in the strong topology and thus, $C \in \mathcal{B}''$. As such, C is reduced by M and N by hypothesis. Now, fix an arbitrary $\psi \in N$. By definition of convergence, for any $\varepsilon > 0$, $\exists k_1^\varepsilon, k_2^\varepsilon \in K$ such that $\|(C - C_\alpha)(U^{-1}\psi)\| \leq \varepsilon/2$ if $\alpha \geq k_1^\varepsilon$ and $\|(C_\alpha - C)\psi\| \leq \varepsilon/2$ if $\alpha \geq k_2^\varepsilon$. Because K is a directed set, there exists a $k_0 \in K$ such that, $k_0 \geq K_1^\varepsilon$ and $k_0 \geq K_2^\varepsilon$. Then, for any fixed $\alpha \geq k_0$,

$$\left\|(UC|_M U^{-1})\psi - C|_N\psi\right\| \leq \left\|(UC|_M U^{-1})\psi - (UC_\alpha|_M U^{-1})\psi\right\| + \Big\|\underbrace{(UC_\alpha|_M U^{-1})\psi - C_\alpha|_N\psi}_{=0 \quad \text{because } C_\alpha \in S}\Big\| +$$

$$+\|(C_\alpha|_N - C|_N)\psi\| \overset{(U \text{ isometry})}{=} \left\|(C|_M - C_\alpha|_M)(U^{-1}\psi)\right\| + \|(C|_N - C_\alpha|_N)(\psi)\| \leq \varepsilon.$$

As $\varepsilon > 0$ can be taken to be arbitrarily small, this can only be if $\|(UC|_M U^{-1})\psi - C|_N\psi\| = 0$. Therefore, $C \in S$.

o.ε.δ.

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[a] If $P^M C = CP^M$ and $P^M D = DP^M$ then $P^M(CD) = CP^M D = CDP^M$.

Theorem 34 (*Klauder-McKenna-Woods-Streit*)**.** Given $\mathfrak{C}, \mathfrak{N} \in \Gamma$, there exists a unitary $U : \otimes_{j\in\mathbb{N}}^{\mathfrak{C}} L^2(\mathbb{R}, dx) \longrightarrow \otimes_{j\in\mathbb{N}}^{\mathfrak{N}} L^2(\mathbb{R}, dx)$ such that

$$U\, e^{i\,\overline{\hat{q}(f)}}|_{\mathfrak{C}}\, U^{-1} = e^{i\,\overline{\hat{q}(f)}}|_{\mathfrak{N}} \quad \text{and} \quad U\, e^{i\,\overline{\hat{p}(f)}}|_{\mathfrak{C}}\, U^{-1} = e^{i\,\overline{\hat{p}(f)}}|_{\mathfrak{N}} \quad \forall f \in \mathcal{W}$$

if and only if $\mathfrak{C}, \mathfrak{N}$ *belong to the same quasi equivalence class, i.e.,* $\mathfrak{C} \overset{q}{\approx} \mathfrak{N}$. ♦

Proof: We first prove ($\Longleftarrow$). The case in which $\mathfrak{C} = \mathfrak{N}$ is trivial (just take the identity as U). Thus, we can assume $\mathfrak{C} \neq \mathfrak{N}$. Let $(\rho_j^{\mathfrak{C}})_{j\in\mathbb{N}} \in \mathfrak{C}$ and $(\rho_j^{\mathfrak{N}})_{j\in\mathbb{N}} \in \mathfrak{N}$ be composed of unit vectors. Since $\mathfrak{C}, \mathfrak{N}$ belong to the same $\overset{q}{\approx}$-class, by Definition 57, $(\rho_j^{\mathfrak{C}})_{j\in\mathbb{N}} \overset{q}{\approx} (\rho_j^{\mathfrak{N}})_{j\in\mathbb{N}}$ and thus, $\exists\theta := (\theta_j)_{j\in\mathbb{N}} \subseteq [-\pi, \pi)$ with $(\rho_j^{\mathfrak{N}})_{j\in\mathbb{N}} \approx (e^{i\theta_j}\rho_j^{\mathfrak{C}})_{j\in\mathbb{N}}$. Equivalently, $(e^{i\theta_j}\rho_j^{\mathfrak{C}})_{j\in\mathbb{N}} \in \mathfrak{N}$ and by Proposition 59.(iii), U_θ (defined in that proposition) is a unitary mapping $\otimes_{j\in\mathbb{N}}^{\mathfrak{C}} L^2(\mathbb{R}, dx)$ to $\otimes_{j\in\mathbb{N}}^{\mathfrak{N}} L^2(\mathbb{R}, dx)$. Now, let $\otimes_{j\in\mathbb{N}}\psi_j \in \otimes_{j\in\mathbb{N}}^{\mathfrak{C}} L^2(\mathbb{R}, dx)$ be arbitrary. Then, for an arbitrary $f = \sum_{n=1}^{N_f} \langle e_n, f\rangle e_n \in \mathcal{W}$,

$$\Big(U_{-\theta}\; e^{i\,\overline{\hat{q}(f)}}|_{\mathfrak{N}}\; U_\theta\Big)\Big(\otimes_{j\in\mathbb{N}}\,\psi_j\Big) \overset{(\text{Thm. } 30)}{=} U_{-\theta}\Big((e^{i\theta_1} e^{i\langle e_1, f\rangle\overline{\hat{q}}}\psi_1) \otimes \cdots \otimes (e^{i\theta_{N_f}} e^{i\langle e_{N_f}, f\rangle\overline{\hat{q}}}\psi_{N_f}) \otimes$$

$$\otimes (e^{i\theta_{N_f+1}}\,\psi_{N_f+1}) \otimes \cdots \Big) = (e^{i\langle e_1, f\rangle \overline{\hat{q}}}\psi_1) \otimes \cdots \otimes (e^{i\langle e_{N_f}, f\rangle \overline{\hat{q}}}\psi_{N_f}) \otimes (\psi_{N_f+1}) \otimes \cdots = e^{i\,\overline{\hat{q}(f)}}|_{\mathfrak{C}}\Big(\otimes_{j\in\mathbb{N}}\,\psi_j\Big).$$

By linearity, the equality is immediate also for the span of those simple tensor products inside $\otimes^{\mathfrak{C}}_{j\in\mathbb{N}} L^2(\mathbb{R}, dx)$. But these are dense in $\otimes^{\mathfrak{C}}_{j\in\mathbb{N}} L^2(\mathbb{R}, dx)$ by definition, so, the continuous operators $\Big(U_{-\theta}\ e^{i\,\overline{\hat{q}(f)}}|_{\mathfrak{N}}\ U_\theta\Big)$ and $e^{i\,\overline{\hat{q}(f)}}|_{\mathfrak{C}}$ agree on a dense subspace. Consequently, by Lemma 32 they agree everywhere in $\otimes^{\mathfrak{C}}_{j\in\mathbb{N}} L^2(\mathbb{R}, dx)$.

In the same exact way, just changing q by p, we get that $\Big(U_{-\theta}\ e^{i\,\overline{\hat{p}(f)}}|_{\mathfrak{N}}\ U_\theta\Big) = e^{i\,\overline{\hat{p}(f)}}|_{\mathfrak{C}}$. Hence, U_θ is a unitary like the U we claimed that exists.

- Now we prove ($\Longrightarrow$). This is the key thing proven by Streit (1967) (and originally proven —in a different way— by Klauder et al. (1966)), although, as we mentioned already, Streit provided considerably few details and left most claims for the reader to prove (such as the following one).

- **Claim:** $UA|_{\mathfrak{C}}U^{-1} = A|_{\mathfrak{N}}$ for every $A \in \mathcal{L}^q\Big(\otimes_{j\in\mathbb{N}} L^2(\mathbb{R}, dx)\Big)$.

Check: The claim holds by hypothesis for all $A \in \Big\{e^{i\,\overline{\hat{q}(f)}},\ e^{i\,\overline{\hat{p}(f)}}\ \Big|\ f \in \mathcal{W}\Big\} =: \mathcal{B}$. But then, by Lemma 56, the claim must hold for all $A \in \mathcal{B}''$, which by Lemma 31 equals $\mathcal{L}^q\Big(\otimes_{j\in\mathbb{N}} L^2(\mathbb{R}, dx)\Big)$.

- And lastly the key trick: by Theorem 28.(iii), if $\mathfrak{C}$ and $\mathfrak{N}$ belonged to different quasi-equivalence classes, then there would exist a bounded operator $A \in \mathcal{L}^q\Big(\otimes_{j\in\mathbb{N}} L^2(\mathbb{R}, dx)\Big)$ such that $A|_{\mathfrak{C}} = Id$ and $A|_{\mathfrak{N}} = 0$. But then, the claim above would imply $Id = UA|_{\mathfrak{C}}U^{-1} = A|_{\mathfrak{N}} = 0$, which is absurd! Hence, it must be that $\mathfrak{C}$ and $\mathfrak{N}$ belong to the same quasi-equivalence class.

$\boldsymbol{o.\varepsilon.\delta.}$

Proposition 81. Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces and let $(A, D(A))$ and $(B, D(B))$ be closable operator, on $\mathcal{H}$ and $\mathcal{K}$, respectively. Then, for a unitary $U : \mathcal{H} \to \mathcal{K}$,

$$UAU^{-1} = B \qquad \overset{\Longrightarrow}{\underbrace{\overset{(i)}{\Longleftarrow}}_{(\text{if } U^{-1}D(B) = D(A))}} \qquad U\overline{A}U^{-1} = \overline{B} \qquad \underbrace{\overset{(ii)}{\Longleftrightarrow}}_{(\text{if } \overline{A},\ \overline{B} \text{ self-adjoint})} \qquad Ue^{it\overline{A}}U^{-1} = e^{it\overline{B}}\ \ \forall t \in \mathbb{R}.$$

♦

Proof: • $(i).(\Longrightarrow)$: Let $\Psi \in D(\overline{B})$. Then, by definition, there exists a sequence $(\Phi_n)_{n\in\mathbb{N}} \subseteq D(B)$ such that $\|\Phi_n - \Psi\| \xrightarrow[n\to\infty]{} 0$ and $\|B\Phi_n - B\Psi\| \xrightarrow[n\to\infty]{} 0$. In particular, using that U, U^{-1} are isometries, $\|U^{-1}\Phi_n - U^{-1}\Psi\| = \|\Phi_n - \Psi\| \xrightarrow[n\to\infty]{} 0$ and $\|A(U^{-1}\Phi_n) - A(U^{-1}\Phi_m)\| = \|UA(U^{-1}\Phi_n) - UA(U^{-1}\Phi_m)\| = \|B\Phi_n - B\Phi_m\|$, which converges uniformly to 0 in n, m (and thus $(A(U^{-1}\Phi_n))_{n\in\mathbb{N}}$ is Cauchy) because $(B\Phi_n)_{n\in\mathbb{N}}$ is convergent and thus Cauchy. Therefore, because $\mathcal{H}$ is complete, $(A(U^{-1}\Phi_n))_{n\in\mathbb{N}}$ must converge to some $\eta \in \mathcal{H}$ such that $U^{-1}\Psi \in D(\overline{A})$ and $\overline{A}(U^{-1}\Psi) = \eta$. But then,

$$U\overline{A}U^{-1}\Psi = U\eta = U\Big(\lim_{n\to\infty} AU^{-1}\Phi_n\Big) \overset{(U \text{ conts.})}{=} \lim_{n\to\infty} UAU^{-1}\Phi_n \overset{(\Phi_n \in D(B))}{=} \lim_{n\to\infty} B\Phi_n = \overline{B}\Psi.$$

Hence, $\overline{B} \subseteq U\overline{A}U^{-1}$. Lastly, using that trivially, $UAU^{-1} = B \iff U^{-1}BU = A$, exchanging A and B in the above argument, we obtain $\overline{B} \supseteq U\overline{A}U^{-1}$, proving that $\overline{B} = U\overline{A}U^{-1}$.

- $(i).(\Longleftarrow)$ After the assumption that $U^{-1}D(B) = D(A)$, using that $\overline{B}\restriction_{D(B)} = B$ and $\overline{A}\restriction_{D(A)} = A$, the restriction of the hypothesis: $U\overline{A}U^{-1}\restriction_{D(B)} = \overline{B}\restriction_{D(B)}$ reads $UAU^{-1} = B$.

- $(ii).(\Longrightarrow)$ Given by Thm. 1.32 in (Arai, 2018) after choosing $f(x) := e^{itx}$.
- $(ii).(\Longleftarrow)$ By hypothesis, $V_t := Ue^{it\overline{A}}U^{-1}$, $t \in \mathbb{R}$, is a SCOPUG with generator $\overline{B}$. But by Cor. 1.6 in (Arai, 2018) $Ue^{it\overline{A}}U^{-1} = e^{itU\overline{A}U^{-1}}$, so it is also generated by $U\overline{A}U^{-1}$ (which is still self-adjoint by Prop. 1.20.(iii) (Arai, 2018)). Hence, by the uniqueness of self-adjoint generator for a SCOPUG (Stone's Thm. 6.2 (Schmüdgen, 2012)), $\overline{B} = U\overline{A}U^{-1}$. ***o.ε.δ.***

At last, as a corollary of all the collected results we get:

Theorem 35. For *each* $\mathfrak{C} \in \Gamma$, defining

$$\otimes^{\mathscr{F},\mathfrak{C}}_{k\in\mathbb{N}} B_{Her} := \operatorname{span}\Big\{ \otimes_{k\in\mathbb{N}} \psi_k \in \otimes^{\mathfrak{C}}_{j\in\mathbb{N}} L^2(\mathbb{R}, dx) \;\Big|\; \psi_k \in B_{Her} \Big\} = \otimes^{F}_{k\in\mathbb{N}} B_{Her} \bigcap \otimes^{\mathfrak{C}}_{j\in\mathbb{N}} L^2(\mathbb{R}, dx),$$

respectively,

$$\Big(\otimes^{\mathfrak{C}}_{k\in\mathbb{N}} L^2(\mathbb{R}, dx),\ \{ \overline{\hat{q}(f)}|_{\mathfrak{c}},\ \overline{\hat{p}(f)}|_{\mathfrak{c}} \}_{f\in\mathcal{W}} \Big),$$

$$\Big(\otimes^{\mathfrak{C}}_{k\in\mathbb{N}} L^2(\mathbb{R}, dx),\ \otimes^{\mathscr{F},\mathfrak{C}}_{k\in\mathbb{N}} B_{Her},\ \{\hat{q}(f)|_{\mathfrak{c}},\ \hat{p}(f)|_{\mathfrak{c}}\}_{f\in\mathcal{W}} \Big)$$

$$\text{and} \quad \Big(\otimes^{\mathfrak{C}}_{k\in\mathbb{N}} L^2(\mathbb{R}, dx),\ \otimes^{\mathscr{F},\mathfrak{C}}_{k\in\mathbb{N}} B_{Her},\ \{ \hat{a}(f)|_{\mathfrak{c}} \}_{f\in\mathcal{W}_{\mathbb{C}}} \Big)$$

are *irreducible representations of the Weyl CCR, Heisenberg CCR and Creation-Annihilation CCR.* Moreover, given two such representations for $\mathfrak{C}, \mathfrak{N} \in \Gamma$, they are equivalent *if and only if* $\mathfrak{C}, \mathfrak{N}$ belong to the *same quasi-equivalence* class, $\mathfrak{C} \overset{q}{\approx} \mathfrak{N}$. ♦

Proof: **Representations:** We proved in Theorems 30, 29 and 31 that $\Big(\otimes_{k\in\mathbb{N}} L^2(\mathbb{R}, dx),\ \{ \overline{\hat{q}(f)},\ \overline{\hat{p}(f)} \}_{f\in\mathcal{W}}\Big)$, $\Big(\otimes_{k\in\mathbb{N}} L^2(\mathbb{R}, dx),\ \otimes^{\mathscr{F}}_{k\in\mathbb{N}} B_{Her},\ \{\hat{q}(f),\ \hat{p}(f)\}_{f\in\mathcal{W}}\Big)$ and $\Big(\otimes_{k\in\mathbb{N}} L^2(\mathbb{R}, dx), \otimes^{\mathscr{F}}_{k\in\mathbb{N}} B_{Her},\ \{ \hat{a}(f) \}_{f\in\mathcal{W}_{\mathbb{C}}}\Big)$ are respectively, representations of the Weyl CCR, Heisenberg CCR and Creation-Annihilation CCR, and that all their operators are reduced by $\otimes^{\mathfrak{C}}_{j\in\mathbb{N}} L^2(\mathbb{R}, dx)$. By definition, the reduced parts of the operators $\hat{q}(f), \hat{p}(f)$ have domain $\bigotimes^{\mathscr{F}}_{k\in\mathbb{N}} B_{Her} \cap \otimes^{\mathfrak{C}}_{j\in\mathbb{N}} L^2(\mathbb{R}, dx)$, which trivially equals $\bigotimes^{\mathscr{F},\mathfrak{C}}_{k\in\mathbb{N}} B_{Her}$. Hence, the only missing thing is to check that the reduced parts of the involved operators still satisfy the conditions of each representation (given we know that the unrestricted operators satisfy them). For that, one merely needs to employ the following facts: given a Hilbert space $\mathcal{H}$, a closed subspace $M \subseteq \mathcal{H}$ and $A, B \in \mathfrak{L}(\mathcal{H})$ reduced by M,

(a) if $D(A)$ is dense in $\mathcal{H}$, $D(A|_M)$ is dense in M (Thm. 1.47.(i) in (Arai, 2018)).

(b) if A is symmetric, then $A|_M$ is symmetric (Thm. 1.47.(vii) in (Arai, 2018)).

(c) $(A+B,\ D(A)\cap D(B))$ is reduced by M and $(A+B)|_M = A|_M + B|_M$ (Prop. 1.54 in (Arai, 2018)).

(d) if $A \subseteq B$ then $A|_M \subseteq B|_M$ (Prop. 1.51.(iv) in (Arai, 2018)).

(e) if A is closed then $A|_M$ is closed. Moreover, for any core D of A, $P^M D$ is a core of $A|_M$. (Thm. 1.47.(iv) in (Arai, 2018).)

(f) if A is closable $A|_M$ is closable and $\overline{A}$ is reduced by M with $(\overline{A})|_M = \overline{A|_M}$ (Thm. 1.47.(v) in (Arai, 2018)).

(g) if A is self-adjoint, then $A|_M$ is self-adjoint (Thm. 1.47.(ix) in (Arai, 2018)).

(h) AB is reduced by M and $(AB)|_M = A|_M B|_M$ (Thm. 1.48 in (Arai, 2018)).

- **(j)** if A is self-adjoint, for all measurable $f : \mathbb{R} \to \mathbb{C} \cup \{\pm\infty\}$, $f(A)$ is reduced by M and $f(A)|_M = f(A|_M)$ (Thm. 1.49 in (Arai, 2018)).
- **(k)** A^* is reducible by M and $(A^*)|_M = (A|_M)^*$ (Lemma 51).

In order to avoid the tedious but straightforward checks, we merely sketch which of the above results should be used in each case.

- Heisenberg CCR: to check Def. 46.(i) use (a), (b). For (ii) use (c). For (iii) use (c), (h).
- Weyl CCR: to check Def. 47.(i) use (f). For (ii) use (d) and (c). For (iii) use (h) and (j).
- Creation-annihilation CCR: to check Def. 48.(i) use (a), (e) and (k). For (ii) use (c). For (iii) use (c), (h) and (k).

Irreducibility:. By Theorem 33, $\{e^{i\overline{\hat{q}(f)}|_{\mathfrak{C}}},\ e^{i\overline{\hat{q}(f)}|_{\mathfrak{C}}}\}_{f\in\mathcal{W}}$ is irreducible. Thus, by definition, also the Weyl CCR representation of $\mathfrak{C}$ given above is irreducible.

Next, assume the Heisenberg CCR representation on $\mathfrak{C}$ from above was (further) *reducible*. If so, there would exist a non-trivial closed subspace $M \subseteq \otimes^{\mathfrak{C}}_{j\in\mathbb{N}} L^2(\mathbb{R}, dx)$ reducing every operator in $\{\hat{q}(f)|_{\mathfrak{C}},\ \hat{p}(f)|_{\mathfrak{C}}\}_{f\in\mathcal{W}}$, which by (f) would reduce $\{\overline{\hat{q}(f)}|_{\mathfrak{C}},\ \overline{\hat{p}(f)}|_{\mathfrak{C}}\}_{f\in\mathcal{W}}$ and hence, by (j), would also reduce $\{e^{i\overline{\hat{q}(f)}|_{\mathfrak{C}}}, e^{i\overline{\hat{q}(f)}|_{\mathfrak{C}}}\}_{f\in\mathcal{W}}$. This would contradict Theorem 33, so we must have found an irreducible Heisenberg CCR in $\mathfrak{C}$.

Similarly, assume the creation-annihilation CCR representation on $\mathfrak{C}$ was (further) reducible. Then, there would exist a non-trivial closed subspace $M \subseteq \otimes^{\mathfrak{C}}_{j\in\mathbb{N}} L^2(\mathbb{R}, dx)$ reducing every operator in $\{\hat{a}(f)|_{\mathfrak{C}}\}_{f\in\mathcal{W}_{\mathbb{C}}}$. By (k), $\{\hat{a}(f)^*|_{\mathfrak{C}}\}_{f\in\mathcal{W}_{\mathbb{C}}}$ would also be reduced by M, and by (c), even $\frac{1}{\sqrt{2}}(\hat{a}^*(f) + \hat{a}(f))|_{\mathfrak{C}}$ would be reduced for any $f \in \mathcal{W}_{\mathbb{C}}$. If so, by (f) $\overline{\frac{1}{\sqrt{2}}(\hat{a}^*(f) + \hat{a}(f))|_{\mathfrak{C}}}$ would be reduced by M for all $f \in \mathcal{W}_{\mathbb{C}}$, which by Thm. 31.(iii), would imply that for all $f \in \mathcal{W}_{\mathbb{C}}$, $\overline{\widehat{q}(f)}|_{\mathfrak{C}}$ and hence (by (j)), that $e^{i\overline{\widehat{q}(f)}|_{\mathfrak{C}}}$, would be reduced by M. Very similarly we would get that $e^{i\overline{\widehat{p}(f)}|_{\mathfrak{C}}}$ would be reduced by M for all $f \in \mathcal{W}$. This would then contradict Theorem 33. Therefore, the creation-annihilation CCR representation of $\mathfrak{C}$ found above must be irreducible.

(In)equivalence: By Theorem 34, $\mathfrak{C}$ and $\mathfrak{N}$ belong to the same quasi-equivalence class ($\mathfrak{C} \overset{q}{\approx} \mathfrak{N}$) *if and only if* there exists a unitary $U : \otimes^{\mathfrak{C}}_{j\in\mathbb{N}} L^2(\mathbb{R}, dx) \to \otimes^{\mathfrak{N}}_{j\in\mathbb{N}} L^2(\mathbb{R}, dx)$ that makes each pair $e^{i\overline{\hat{q}(f)}|_{\mathfrak{C}}}$, $e^{i\overline{\hat{q}(f)}|_{\mathfrak{N}}}$ and $e^{i\overline{\hat{p}(f)}|_{\mathfrak{C}}}$, $e^{i\overline{\hat{p}(f)}|_{\mathfrak{N}}}$ unitarily equivalent for all $f \in \mathcal{W}$. But by Proposition 81.(iii) (and the fact —proven in Theorem 30— that $\forall t \in \mathbb{R}$, $\overline{\hat{q}(tf)} = t\overline{\hat{q}(f)}$ & $\overline{\hat{p}(tf)} = t\overline{\hat{p}(f)}$), the last statement holds *if and only if* $U\overline{\hat{q}(f)}|_{\mathfrak{C}}U^{-1} = \overline{\hat{q}(f)}|_{\mathfrak{N}}$ and $U\overline{\hat{p}(f)}|_{\mathfrak{C}}U^{-1} = \overline{\hat{p}(f)}|_{\mathfrak{N}}$ for all $f \in \mathcal{W}$. This proves the inequivalence statement for the Weyl CCR.

- Now, in the proof of Thm. 34 we found that if $\mathfrak{C} \overset{q}{\approx} \mathfrak{N}$ the unitary U can be chosen such that if $\bigotimes_{j\in\mathbb{N}} \psi_j \in \otimes^{\mathfrak{C}}_{j\in\mathbb{N}} L^2(\mathbb{R}, dx)$, then $U(\otimes_{j\in\mathbb{N}}\psi_j) = \otimes_{j\in\mathbb{N}}(e^{i\theta_j}\psi_j)$ for some constants $(\theta_j)_{j\in\mathbb{N}} \subseteq [-\pi, \pi)$. In any such case $U\Big(\otimes^{\mathscr{F},\mathfrak{C}}_{j\in\mathbb{N}} B_{Her}\Big) \subseteq \otimes^{\mathscr{F},\mathfrak{N}}_{j\in\mathbb{N}} B_{Her}$ (since multiplication by a phase factor does not change the belonging of a vector ψ_j to B_{Her}). Therefore, Proposition 81.(i) proves, after the conclusion of the last paragraph, that $\mathfrak{C} \overset{q}{\approx} \mathfrak{N}$ *if and only if* there exists a unitary $U : \otimes^{\mathfrak{C}}_{j\in\mathbb{N}} L^2(\mathbb{R}, dx) \to \otimes^{\mathfrak{N}}_{j\in\mathbb{N}} L^2(\mathbb{R}, dx)$ such that[a] $U\hat{q}(f)|_{\mathfrak{C}}U^{-1} = \hat{q}(f)|_{\mathfrak{N}}$ and $U\overline{\hat{p}(f)}|_{\mathfrak{C}}U^{-1} = \overline{\hat{p}(f)}|_{\mathfrak{N}}$ for all $f \in \mathcal{W}$. This proves the inequivalence statement for the Heisenberg CCR.

- Finally, assume $\mathfrak{C} \overset{q}{\approx} \mathfrak{R}$. By definition (and the use of (c) above), for any $f \in \mathcal{W}_{\mathbb{C}}$, $\widehat{\Phi}(f)|_{\mathfrak{C}} = \hat{q}(\operatorname{Re} f)|_{\mathfrak{C}} + \hat{p}(\operatorname{Im} f)|_{\mathfrak{C}}$, so, by the last paragraph, $\widehat{\Phi}(f)|_{\mathfrak{C}}$ and $\widehat{\Phi}(f)|_{\mathfrak{N}}$ are unitarily equivalent. But then trivially, also $\widehat{\Phi}(f)|_{\mathfrak{C}} + i\widehat{\Phi}(if)|_{\mathfrak{C}}$ and $\widehat{\Phi}(f)|_{\mathfrak{N}} + i\widehat{\Phi}(if)|_{\mathfrak{N}}$ are unitarily equivalent. And by Prop. 81.(i), therefore, also their closures. After using (c) (to take the reduction out as a "common factor") this means, by definition, that the same unitary gives an equivalence between $\hat{a}(f)|_{\mathfrak{C}} = \overline{\Big(\widehat{\Phi}(f) + i\widehat{\Phi}(if)\Big)}|_{\mathfrak{C}}$ and $\hat{a}(f)|_{\mathfrak{N}} = \overline{\Big(\widehat{\Phi}(f) + i\widehat{\Phi}(if)\Big)}|_{\mathfrak{N}}$ $\forall f \in \mathcal{W}_{\mathbb{C}}$.

For the converse, assume that there exists a unitary U making $\hat{a}(f)|_{\mathfrak{C}}$ and $\hat{a}(f)|_{\mathfrak{N}}$ equivalent for all $f \in \mathcal{W}_{\mathbb{C}}$. Then, U gives a trivial unitary equivalence between their adjoints and hence between $\hat{a}(f)|_{\mathfrak{C}} + \hat{a}(f)|_{\mathfrak{C}}^*$ and $\hat{a}(f)|_{\mathfrak{N}} + \hat{a}(f)|_{\mathfrak{N}}^*$, which by (c) and (k) equal $(\hat{a}(f) + \hat{a}(f)^*)$ reduced by $\mathfrak{C}$ and $\mathfrak{N}$, respectively. But then, by Prop. 81.(i), their closures are also unitarily equivalent by U. By Thm. 31 this means that $\hat{q}(f)|_{\mathfrak{C}}$ and $\hat{q}(f)|_{\mathfrak{N}}$ are unitarily equivalent for all $f \in \mathcal{W}$. Likewise we get that $\hat{p}(f)|_{\mathfrak{C}}$ and $\hat{p}(f)|_{\mathfrak{N}}$ are unitarily equivalent. But as we just proved above, this implies that $\mathfrak{C} \overset{q}{\approx} \mathfrak{N}$. Hence, our claim holds. ***o.ε.δ.***

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[a] For this, we also use that $\overline{\hat{q}(f)|_{\mathfrak{C}}} = \overline{\hat{q}(f)}|_{\mathfrak{C}}$ by (f).

B.4 The CCR Representations Induced in $L^2(\mathbb{R}^\infty, d^\infty\mu_{\mathfrak{C}})$

In this section, we include for completeness how the CCR representation operators $\hat{q}(f)$ and $\hat{p}(f)$ (and hence, implicitly, $\widehat{\Phi}(f)$ and $\hat{a}(f)$) look from the perspective of the $L^2(\mathbb{R}^\infty, d^\infty\mu)$ spaces identified with the layers $\otimes_{j\in\mathbb{N}}^{\mathfrak{C}} L^2(\mathbb{R}, dx)$ in the main text. For that, we start by providing an ONB that is very convenient to describe the resulting operators.

B.4.1 A Polynomial ONB for each layer's $L^2(\mathbb{R}^\infty, d^\infty\mu)$ space

Lemma 57. For each $\psi \in L^2(\mathbb{R}, dx)$ such that $\|\psi\|_{L^2} = 1$ and each $\varepsilon > 0$, there exists a $\phi \in B_{Her}$ such that

(i) $\|\phi\|_{L^2} = 1$ **(ii)** $\phi(x) \neq 0$ for dx-almost every $x \in \mathbb{R}$. **(iii)** $|\langle\psi, \phi\rangle - 1| \leq \varepsilon$.

In particular, such a $\phi \in B_{Her}$ has at most finitely many zeros. ♦

Proof: Since $B_{Her} \subseteq L^2(\mathbb{R}, dx)$ is a dense subspace (see Def. 53), Lemma 44 proves the existence of a vector $\phi \in B_{Her}$ satisfying (i) and (iii). But we saw in Def. 53 that every function in B_{Her} has at most finitely many zeros and thus, ϕ is non-zero almost-everywhere. ***o.ε.δ.***

This allows us to formulate an "improved" version of Proposition 22.

Proposition 82. For each class $\mathfrak{C} \in \Gamma$ there exists an elementary tensor product $\rho_1 \otimes \rho_2 \otimes \cdots \in \otimes_{j\in\mathbb{N}}^{\mathfrak{C}} L^2(\mathbb{R}, dx)$ such that for every $j \in \mathbb{N}$,

- $\rho_j \in B_{Her} \subseteq \mathcal{S}(\mathbb{R})$ (i.e., $\rho_j(x) = q_j(x)e^{-x^2/2}$ for some polynomial q_j). Thus,
    1. $\rho_j \in D(\hat{q}) \bigcap D(-i\hbar\partial_j)$.
    2. $\rho_j(x) = 0$ only in finitely many x, so $\rho_j(x) \neq 0$ dx-almost everywhere. As such,
        (a). $|\rho_j|^2(x) > 0$ for almost every $x \in \mathbb{R}$, and thus, $d\mu_j \sim dx$ (by Cor. 4).
- $\|\rho_j\|_{L^2(\mathbb{R},dx)} = 1$. Hence,
    3. $d\mu_j := |\rho_j|^2 dx$ is a probability measure and
    4. $\rho_1 \otimes \rho_2 \otimes \cdots$ is a generator of the layer $\mathfrak{C}$ (see Theorem 9).

Therefore, there exists a WR basis $\mathfrak{R} = (\otimes_{j\in\mathbb{N}}\rho_j^{\mathfrak{C}})_{\mathfrak{C}\in\Gamma}$ in the sense of Thm. 13, purely made of polynomials times $e^{-x^2/2}$. ♦

Proof: By Prop. 9, for any fixed $\mathfrak{C} \in \Gamma$, $\exists(\psi_j)_{j\in\mathbb{N}} \in \mathfrak{C}$ with $\|\psi_j\| = 1$. Then, by Lemma 57, for each ψ_j, there exists $\rho_j \in B_{Her}$ such that $\|\rho_j\|_{L^2} = 1$, $\rho_j(x) = 0$ for finitely many $x \in \mathbb{R}$ and $|\langle\psi_j, \rho_j\rangle - 1| \leq 1/j^2$. Hence,

$$\sum_{j\in\mathbb{N}} |\langle\psi_j, \rho_j\rangle - 1| \leq \sum_{j\in\mathbb{N}} \frac{1}{j^2} = \frac{\pi^2}{6} \quad (exists) \tag{B.47}$$

By definition, this implies that $(\psi_j)_{j\in\mathbb{N}} \approx (\rho_j)_{j\in\mathbb{N}}$ and thus, $\otimes_{j\in\mathbb{N}}\rho_j$ is a generator of the same layer $\mathfrak{C}$ as $\otimes_{j\in\mathbb{N}}\psi_j$. ***o.ε.δ.***

Proposition 83. For any $\rho \in B_{Her}$,[11] $\{\rho q \mid q \in B_{poly}(\mathbb{R})\}$ is *dense* in $L^2(\mathbb{R}, dx)$. Equivalently, $B_{poly}(\mathbb{R})$ is *dense* in $L^2(\mathbb{R}, |\rho|^2 dx)$. In particular, there exists an ONB $\mathscr{O}_\rho := (r_n)_{n\in\mathbb{N}_0}$ of $L^2(\mathbb{R}, |\rho|^2 dx)$ such that $r_n \in B_{poly}(\mathbb{R})$ is a polynomial of degree n and such that $r_0(x) \equiv 1$. Furthermore, $\operatorname{span}(\mathscr{O}_\rho) = B_{poly}(\mathbb{R})$. ♦

Proof: Let $\psi \in L^2(\mathbb{R}, dx)$ be arbitrary. We proceed to prove that for an arbitrary $\varepsilon > 0$ there exists $r \in B_{poly}(\mathbb{R})$ such that $\|\psi - r\rho\|_{L^2} < \varepsilon$ —which would prove the first claimed density.

By density of the compactly supported smooth functions $\mathcal{C}_c^\infty(\mathbb{R})$ in $L^2(\mathbb{R}, dx)$ (see Theorem 0.36 in (Teschl, 2014)), there exists a $g \in \mathcal{C}_c^\infty(\mathbb{R})$ such that $\|\psi - g\|_{L^2} < \varepsilon/3$. Since it has compact support, there exists an $R_1 \in (0, +\infty)$ such that $\operatorname{supp}(g) \subset (-R_1, R_1)$. Because ρ has finitely many zeros, there also exists $R_2 \in (0, +\infty)$ such that $Z := \rho^{-1}(\{0\}) \subset (-R_2, R_2)$. Let $R := \max(R_1, R_2)$ and define the following union of balls around the zeros of ρ:

$$V_\delta := \bigcup_{z\in Z} \Big(z - \frac{\Delta_z}{2},\ z + \frac{\Delta_z}{2}\Big) \quad \text{where} \quad \Delta_z := \min\Big(\frac{\delta}{|Z|},\ \text{distance}(z, \{-R, R\})\Big). \tag{B.48}$$

All $z \in Z$ are interior points of V_δ by definition and $V_\delta \subseteq [-R, R]$. Most importantly, since $dx((z - \Delta_z/2,\ z + \Delta/2)) = \Delta_z$, then, $dx(V_\delta) \leq \delta$.

---

[11] Note that it would be a false statement if we merely imposed that $\rho \in \mathcal{S}(\mathbb{R})$, even imposing in addition that ρ is dx-almost everywhere non-zero. A counter-example is: $\rho(x) := \begin{cases} e^{-(log(|x|))^2/2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$. First, it vanishes only at $x = 0$ and it is in $\mathcal{S}(\mathbb{R})$. The latter can be checked as follows: (i) ρ is smooth because all derivatives in $x = 0$ are continuously 0, and (ii) noticing the symmetry of $|x^\alpha \partial^\beta \rho(x)|$, one can use $t(x) := log(x)$ as a change of variables in the supremum to see that $\sup_{x\in\mathbb{R}} |x^\alpha \partial^\beta \rho(x)| < +\infty\ \forall\alpha, \beta \in \mathbb{N}_0$.

Second, to check that ρ is indeed a counter-example, note first that $h(x) := \sin(2\pi \log(|x|)) \in L^2(\mathbb{R}, |\rho|^2 dx)$ (because $|\rho|^2 dx$ is a finite measure and $|h|$ is bounded), which is a non-zero vector. But (as we prove in $(\star)$) h is orthogonal to every monomial —i.e., $\int_{\mathbb{R}} x^k g(x)|\rho|^2(x)dx = 0$ for all $k \in \mathbb{N}_0$—, and thus, when normalized it is at a distance $\sqrt{2}$ of $B_{poly}(\mathbb{R})$. This proves that $B_{poly}(\mathbb{R})$ cannot be dense in $L^2(\mathbb{R}, |\rho|^2 dx)$.

$(\star)$: Let us prove the orthogonality of h and x^k in $L^2(\mathbb{R}, |\rho|^2 dx)$. (i) For odd k, $\int_{\mathbb{R}} x^k h(x)|\rho|^2(x)dx$ is the integral of an odd function and so, it is zero. (ii) For even k, it is the integral of a symmetric function about $x = 0$, so, we can equivalently integrate half of it and then double the result. In the resulting integral with $x \in (0, +\infty)$, the change of variables $t = log(x)$ can be applied to get, after a square completion, $e^{\frac{(k+1)^2}{4}} \int_{t\in\mathbb{R}} e^{-(t-\frac{n+1}{2})^2} \sin(2\pi t)dt$. Finally, after another change $s(t) := t - \frac{n+1}{2}$, one gets $e^{\frac{(n+1)^2}{4}} \int_{s\in\mathbb{R}} e^{-s^2} \sin\big(2\pi(s + \frac{n+1}{2})\big)ds$, which is the integral of an odd function and thus zero.

This counter-example was obtained feeding *ChatGPT* with the example of indeterminate Hamburger problem (due to Stieltjes) given in Ex. 16.1 of (Schmüdgen, 2012). The machine was asked to modify the given example to make the measure's density an Schwartz function and it successfully did so!

- Define the function $\eta : \mathbb{R} \to [0,1]$ such that for $\delta_0 := \varepsilon^2/(3^2\|g\|^2_\infty)$

$$\eta(x) := \begin{cases} 0 & \text{if} \quad x \in V_{\delta_0/2} \quad \text{or} \quad x \notin [-R,R] \\ 1 & \text{if} \quad x \in [-R,R] \setminus V_{\delta_0} \end{cases} \tag{B.49}$$

and it interpolates linearly over the points in $V_{\delta_0} \setminus V_{\delta_0/2}$. It is continuous, compactly supported and cuts-off the zeros of ρ. Then, $h := \eta g$ is a continuous function compactly supported inside $[-R,R]$ (because it is a product of such functions) and is zero in a neighborhood around every zero of ρ. Moreover,

$$\|g-h\|^2_{L^2} = \int_{x\in[-R,R]} |g(x)(1-\eta(x))|^2 dx \le \|g\|^2_\infty \int_{x\in V_{\delta_0}} |1-\eta(x)|^2 dx \overset{(\eta(x)\in[0,1])}{\le}$$

$$\le \|g\|_\infty dx(V_{\delta_0}) \le \|g\|^2_\infty \delta_0 \le \left(\frac{\varepsilon}{3}\right)^2 \quad \Longrightarrow \quad \|g-h\|_{L^2} \le \frac{\varepsilon}{3}.$$

- Since $\rho \in B_{Her}$, $\rho(x) = q(x)e^{-x^2/2}$ for some $q \in B_{poly}(\mathbb{R})$. That said, define $\varphi : \mathbb{R} \to \mathbb{C}$

$$\varphi(x) := \begin{cases} \dfrac{h(x)}{e^{-x^2/4}q(x)} & \text{if} \;\; x \in \mathbb{R} \setminus V_{\delta_0/2} \\ 0 & \text{if} \;\; x \in V_{\delta_0/2} \end{cases}. \tag{B.50}$$

It is a continuous function because (i) h goes continuously to zero in the boundary of $V_{\delta_0/2}$ by definition of η, and (ii) outside $V_{\delta_0/2}$ it is a ratio of two continuous functions where the denominator does not vanish. Moreover, it is compactly supported in $[-R,R]$ because h is so. Thus, φ is also bounded and consequently, it is an element of $L^2(\mathbb{R}, dx)$.

Now, take the ONB of Lemma 43 with $\omega = 1/2$, $\phi_n(x) := c_n h_n(\frac{1}{\sqrt{2}}x)e^{-\frac{x^2}{4}}$. Every element in its span is some polynomial times $e^{-x^2/4}$ and by the ONB property, such a span is dense in $L^2(\mathbb{R}, dx)$. Hence, there exists $r \in B_{poly}(\mathbb{R})$ such that

$$\left\|\varphi - re^{-(\cdot)^2/4}\right\|_{L^2} \le \frac{\varepsilon}{3\|e^{-(\cdot)^2/4}q\|_\infty}, \tag{B.51}$$

where, $|e^{-x^2/4}q(x)|$ is bounded because $(x \in \mathbb{R} \mapsto e^{-x^2/4}) \in \mathcal{S}(\mathbb{R})$ and thus, $(x \in \mathbb{R} \mapsto e^{-x^2/4}q(x)) \in \mathcal{S}(\mathbb{R})$.

- Then,

$$\|h - r\rho\|^2_{L^2} \overset{(\rho(x)=q(x)e^{-\frac{x^2}{2}})}{=} \int_{x\in\mathbb{R}} |h(x) - r(x)\, e^{-x^2/2}q(x)|^2\, dx \overset{(\text{by def})}{=} \int_{x\in\mathbb{R}} |e^{-\frac{x^2}{4}}q(x)|^2 |\varphi(x) - e^{-\frac{x^2}{4}}r(x)|^2\, dx \le$$

$$\le \left\|e^{-(\cdot)^2/4}q\right\|^2_\infty \int_{x\in\mathbb{R}} |\varphi(x) - e^{-x^2/4}r(x)|^2\, dx = \left\|e^{-(\cdot)^2/4}q\right\|^2_\infty \left\|\varphi - e^{-(\cdot)^2/4}r\right\|^2_{L^2} \le \frac{\varepsilon^2}{3^2}.$$

- With all,

$$\|\psi - r\rho\|_{L^2} \le \|\psi - g\|_{L^2} + \|g-h\|_{L^2} + \|h - r\rho\|_{L^2} \le \varepsilon.$$

- About the second claimed density, recall that $U : L^2(\mathbb{R}, dx) \to L^2(\mathbb{R}, |\rho|^2 dx)$, $U\psi = \psi/\rho$ is a unitary map, so it is in particular continuous. Moreover, $U(\{\rho r |\; r \in B_{poly}(\mathbb{R})\}) = B_{poly}(\mathbb{R})$ and by continuity (see Thm 7.2.(d) in (Willard, 2012)),

$$U\Big(\ \underbrace{\overline{\{\rho\, r|\ r\in B_{poly}(\mathbb{R})\}}}_{\text{we proved it equals } L^2(\mathbb{R},dx)}\ \Big)\subseteq \overline{B_{poly}(\mathbb{R})}.$$

Thus, $L^2(\mathbb{R},|\rho|^2dx)=\overline{B_{poly}(\mathbb{R})}$, proving that $B_{poly}(\mathbb{R})$ is dense in $L^2(\mathbb{R},|\rho|^2dx)$.

- Finally, to prove there exists an ONB of polynomials in $L^2(\mathbb{R},|\rho|^2dx)$, consider the family $B_0 := (r_k^{(0)})_{k\in\mathbb{N}_0}\subseteq L^2(\mathbb{R},|\rho|^2dx)$ with $r_k^{(0)}(x):=x^k$, the span of which is trivially $B_{poly}(\mathbb{R})$. B_0 is a family of linearly independent vectors because the only way that a linear combination of $r_k^{(0)}, r_\ell^{(0)}\in B_0$, $k\neq\ell$ yields the 0 function is if both coefficients $\alpha,\beta\in\mathbb{C}$ in $\alpha\, r_k^{(0)}+\beta\, r_\ell^{(0)}$ are 0 —likewise for arbitrary finite linear combinations. But B_0 is a countable set, namely, a sequence of vectors, so we can apply the Gram-Schmidt orthonormalization to it: let $r_0 := r_0^{(0)}$ and for $k\in\mathbb{N}$ define

$$r_k := \frac{1}{\Big\|r_k^{(0)}-\sum_{j=0}^{k-1}\langle r_j, r_k^{(0)}\rangle_{L^2(|\rho|^2dx)}\cdot r_j\Big\|_{L^2(|\rho|^2dx)}}\Big(\ r_k^{(0)}-\sum_{j=0}^{k-1}\langle r_j, r_k^{(0)}\rangle_{L^2(|\rho|^2dx)}\cdot r_j\ \Big). \tag{B.52}$$

It is elementary to check that $\mathscr{O}_\rho := (r_k)_{k\in\mathbb{N}_0}$ is an orthonormal family of vectors such that span(B_0) =span$(\mathscr{O}_\rho)$. In particular, note that r_k is a polynomial of degree k because it is the result of subtracting a polynomial of degree at most $k-1$ to a polynomial of degree k. Finally, we just proved that $B_{poly}(\mathbb{R})$ —i.e., span(B_0) and hence, span$(\mathscr{O}_\rho)$—, is dense in $L^2(\mathbb{R},|\rho|^2dx)$. Therefore, $\mathscr{O}_\rho$ is an ONB of $B_{poly}(\mathbb{R})$. ***o.ε.δ.***

Proposition 84. Let $(\rho_j)_{j\in\mathbb{N}}\subseteq B_{Her}$ be such that $\|\rho_j\|=1\ \forall j$. Define $d^\infty\mu := \odot_{j\in\mathbb{N}}|\rho_j|^2dx_j$ and

$$B_{poly}(\mathbb{R}^\infty):=\Big\{P:\mathbb{R}^\infty\to\mathbb{C}\ \Big|\ P(x_1,x_2,...)\equiv p(x_1,...,x_N)\quad\text{for some } N\in\mathbb{N},\ \ p\in B_{poly}(\mathbb{R}^N)\Big\}.$$

(i) $B_{poly}(\mathbb{R}^\infty)$ is a *subspace* of $L^2(\mathbb{R}^\infty,\ d^\infty\mu)$ —in fact, it is so for any $(\rho_j)_{j\in\mathbb{N}}\subseteq\mathcal{S}(\mathbb{R})$.

(ii) $B_{poly}(\mathbb{R}^\infty)$ is *dense* in $L^2(\mathbb{R}^\infty,d^\infty\mu)$. In particular, the following is an ONB of $L^2(\mathbb{R}^\infty,d^\infty\mu)$ made of elements in $B_{poly}(\mathbb{R}^\infty)$:

$$\mathscr{O}:=\left\{\ (x_1,x_2,...)\mapsto r_{\ell_1}^{(1)}(x_1)\cdots r_{\ell_N}^{(N)}(x_N)\ \middle|\ N\in\mathbb{N},\ \ell_k\in\mathbb{N}_0\ \text{ for } k\in\{1,...,N\}\text{ with } \ell_N\neq 0\ \right\}$$

where, for each $k\in\mathbb{N}$, $(r_n^{(k)})_{n\in\mathbb{N}_0} := \mathscr{O}_{\rho_k}$ is the polynomial ONB of $L^2(\mathbb{R},|\rho_k|^2dx)$ given by Prop. 83. Note that $r_{\ell_1}^{(1)}(x_1)\cdots r_{\ell_N}^{(n)}(x_N)$ is a polynomial of order ℓ_j for x_j, $j\in\{1,...,N\}$. As such, span$(\mathscr{O})=B_{poly}(\mathbb{R}^\infty)$.

♦

Proof: **Item (i):** Let $Q\in B_{poly}(\mathbb{R}^\infty)$ be arbitrary. By definition, $\exists q\in B_{poly}(\mathbb{R}^N)$ for some $N\in\mathbb{N}$ such that $Q(x_1,x_2,...)=q(x_1,...,x_N)$. Then,

$$\int_{x\in\mathbb{R}^\infty}|Q(x)|^2d^\infty\mu \overset{(Prop.\,20)}{=}\int_{x\in\mathbb{R}^N}|q(x)|^2\ \odot_{j=1}^N(|\rho_j|^2dx)=\int_{x\in\mathbb{R}^N}|q(x)\cdot\prod_{j=1}^N\rho_j(x_j)|^2d^Nx.$$

It is immediate to see that if $\rho_j\in\mathcal{S}(\mathbb{R})\ \forall j$, then $(x_1,...,x_N)\mapsto\rho_1(x_1)\cdots\rho_N(x_N)$ is in $\mathcal{S}(\mathbb{R}^N)$. As such, the last integral is the L^2-norm of a Schwartz function, which is finite. Hence, $Q\in L^2(\mathbb{R}^\infty,d^\infty\mu)$.

Item (ii): Because for all $k \in \mathbb{N}$, $\mathscr{O}_{\rho_k}$ is an ONB of $L^2(\mathbb{R}, |\rho_k|^2 dx)$, by Proposition 21, $\mathscr{O}$ is an ONB of $L^2(\mathbb{R}^\infty, d^\infty \mu)$. Furthermore, by definition, each of its elements is in $B_{poly}(\mathbb{R}^\infty)$. Thus, $\mathscr{O} \subseteq B_{poly}(\mathbb{R}^\infty)$, and hence, span$\mathscr{O} \subseteq B_{poly}(\mathbb{R}^\infty)$ (because $B_{poly}(\mathbb{R}^\infty)$ is a vector space). For the converse, given an arbitrary $Q \in B_{poly}(\mathbb{R}^\infty)$, $\exists q \in B_{poly}(\mathbb{R}^N) :\ Q(x_1, x_2, ...) = q(x_1, ..., x_N)$ for some $N \in \mathbb{N}$. But, $\mathscr{O}$ possesses polynomials over N variables of any combinations of degree, so, we can write Q as a finite linear combination of them. Namely, $Q \in \text{span}(\mathscr{O})$. With all, $\text{span}(\mathscr{O}) = B_{poly}(\mathbb{R}^\infty)$ and a such, $B_{poly}(\mathbb{R}^\infty)$ is dense in $L^2(\mathbb{R}^\infty, d^\infty \mu)$ because the span of an ONB is so.

o.ε.δ.

Theorem 36. There exists a WR basis $\mathfrak{R} = (\otimes_{j\in\mathbb{N}} \rho_j^{\mathfrak{C}})_{\mathfrak{C}\in\Gamma} \subseteq \otimes_{j\in\mathbb{N}} B_{Her}$ and the associated $\mathscr{W}_{\mathfrak{R}} = \oplus_{\mathfrak{C}\in\Gamma} W_{\mathfrak{C}}$ maps $\bigotimes_{k\in\mathbb{N}} L^2(\mathbb{R}, dx)$ to a "coordinate space" $\oplus_{\mathfrak{C}\in\Gamma} L^2(\mathbb{R}^\infty, d^\infty \mu_{\mathfrak{C}})$ where the two following sets are *dense* subspaces:

$$\oplus_{\mathfrak{C}\in\Gamma} B_{poly}(\mathbb{R}^\infty) := \Big\{ (\Psi_{\mathfrak{C}})_{\mathfrak{C}\in\Gamma} \mid \Psi_{\mathfrak{C}} \in B_{poly}(\mathbb{R}^\infty) \Big\} \quad \text{and} \tag{B.53}$$

$$\oplus^{\mathscr{F}}_{\mathfrak{C}\in\Gamma} B_{poly}(\mathbb{R}^\infty) := \Big\{ \ (\Psi_{\mathfrak{C}})_{\mathfrak{C}\in\Gamma} \mid \Psi_{\mathfrak{C}} \in B_{poly}(\mathbb{R}^\infty) \ \text{ for finitely many } \mathfrak{C} \in \Gamma \text{ and } \Psi_{\mathfrak{C}} = 0 \text{ else} \Big\}.$$

In particular, given that for each $\mathfrak{C} \in \Gamma$, $\mathscr{O}^{L^2}_{\mathfrak{C}}$ denotes the ONB of $L^2(\mathbb{R}^\infty, d^\infty \mu_{\mathfrak{C}})$ obtained in Prop. 84, then

$$\mathscr{O}^{L^2}_{\Gamma} := \Big\{ \ (Q_{\mathfrak{C}})_{\mathfrak{C}\in\Gamma} \mid Q_{\mathfrak{C}} = 0 \ \text{ for all } \mathfrak{C} \in \Gamma \text{ except for one } \mathfrak{C}_0 \in \Gamma, \ \text{where } Q_{\mathfrak{C}_0} \in \mathscr{O}^{L^2}_{\mathfrak{C}_0} \Big\} \tag{B.54}$$

is an ONB of $\oplus_{\mathfrak{C}\in\Gamma} L^2(\mathbb{R}^\infty, d^\infty \mu_{\mathfrak{C}})$ (exclusively made of "single-layer polynomials"). Moreover, $\text{span}(\mathscr{O}^{L^2}_{\Gamma}) = \oplus^{\mathscr{F}}_{\mathfrak{C}\in\Gamma} B_{poly}(\mathbb{R}^\infty)$.

Mapping these results back to $\bigotimes_{k\in\mathbb{N}} L^2(\mathbb{R}, dx)$ via $(\mathscr{W}_{\mathfrak{R}})^{-1} = \oplus_{\mathfrak{C}\in\Gamma} (W_{\mathfrak{C}})^{-1}$, we get that for each $\mathfrak{C} \in \Gamma$ the following is an ONB of $\otimes^{\mathfrak{C}}_{j\in\mathbb{N}} L^2(\mathbb{R}, dx)$ made of elementary tensor products in $\otimes^{F,\mathfrak{C}}_{j\in\mathbb{N}} B_{Her}$:

$$\mathscr{O}^{\otimes}_{\mathfrak{C}} := W^{-1}_{\mathfrak{C}}(\mathscr{O}^{L^2}_{\mathfrak{C}}) = \left\{ \ (r^{(1,\mathfrak{C})}_{\ell_1} \rho^{\mathfrak{C}}_1) \otimes \cdots \otimes (r^{(N,\mathfrak{C})}_{\ell_N} \rho^{\mathfrak{C}}_N) \otimes \rho^{\mathfrak{C}}_{N+1} \otimes \cdots \ \middle| \ N \in \mathbb{N}, \ \ell_k \in \mathbb{N}_0 \right\}. \tag{B.55}$$

Note that for each $k \in \mathbb{N}$, $\mathscr{O}_{\rho^{\mathfrak{C}}_k} =: (r^{(k,\mathfrak{C})}_n)_{n\in\mathbb{N}_0} \subseteq B_{poly}(\mathbb{R})$ is the polynomial ONB of $L^2(\mathbb{R}, |\rho^{\mathfrak{C}}_k|^2 dx)$ given by Prop. 83 —in particular, $r^{(k,\mathfrak{C})}_n$ has order n. Moreover, $\text{span}(\mathscr{O}^{\otimes}_{\mathfrak{C}}) \subseteq \otimes^{\mathscr{F},\mathfrak{C}}_{j\in\mathbb{N}} B_{Her}$. With all, $\mathscr{O}^{\otimes}_{\Gamma} := (\mathscr{W}_{\mathfrak{R}})^{-1}(\mathscr{O}^{L^2}_{\Gamma}) = \bigcup_{\mathfrak{C}\in\Gamma} \mathscr{O}^{\otimes}_{\mathfrak{C}}$ is an ONB of $\otimes_{j\in\mathbb{N}} L^2(\mathbb{R}, dx)$. ♦

Proof: By Prop. 82, there is a representation basis $\mathfrak{R} := (\otimes_{j\in\mathbb{N}} \rho^{\mathfrak{C}}_j)_{\mathfrak{C}\in\Gamma}$ such that each $\rho^{\mathfrak{C}}_j \in B_{Her}$. Then, by Prop. 84, for each $\mathfrak{C} \in \Gamma$ and $d^\infty \mu_{\mathfrak{C}} := \odot_{j\in\mathbb{N}} |\rho^{\mathfrak{C}}_j|^2 dx_j$, $B_{poly}(\mathbb{R}^\infty)$ is dense in $L^2(\mathbb{R}^\infty, d^\infty \mu_{\mathfrak{C}})$. As such, by Lemma 16, both $\oplus^{\mathscr{F}}_{\mathfrak{C}\in\Gamma} B_{poly}(\mathbb{R}^\infty)$ and $\oplus_{\mathfrak{C}\in\Gamma} B_{poly}(\mathbb{R}^\infty)$ are dense in $\oplus_{\mathfrak{C}\in\Gamma} L^2(\mathbb{R}^\infty, d^\infty \mu_{\mathfrak{C}})$. Finally, $\mathscr{O}^{L^2}_{\Gamma}$ is an ONB in this direct sum by Prop. 17. The rest follows from the unitarity of $\mathscr{W}^{-1}_{\mathfrak{R}}$ and its composing $W^{-1}_{\mathfrak{C}}$.

o.ε.δ.

B.4.2 The CCR Operators of $\otimes_{j\in\mathbb{N}} L^2(\mathbb{R}, dx)$ as seen from $L^2(\mathbb{R}^\infty, d^\infty \mu_{\mathfrak{C}})$

Theorem 37. For an arbitrary $\mathfrak{C} \in \Gamma$, let $\otimes_{j\in\mathbb{N}} \rho^{\mathfrak{C}}_j$ be a generator of the $\mathfrak{C}$-th layer that is in $\otimes^{\mathscr{F}}_{j\in\mathbb{N}} B_{Her}$ (they exist by Prop. 82). Let $d^\infty \mu_{\mathfrak{C}} := \odot_{j\in\mathbb{N}} |\rho^{\mathfrak{C}}_j|^2 dx_j$ and let $W_{\mathfrak{C}} : \otimes^{\mathfrak{C}}_{j\in\mathbb{N}} L^2(\mathbb{R}, dx) \to L^2(\mathbb{R}^\infty, d^\infty \mu_{\mathfrak{C}})$ be the associated unitary of Thm. 10. Also, let $\mathscr{O}^{L^2}_{\mathfrak{C}}$ denote the associated polynomial ONB of Prop. 84.(ii) and denote $\mathscr{O}^{\otimes}_{\mathfrak{C}} := W^{-1}_{\mathfrak{C}} \mathscr{O}^{L^2}_{\mathfrak{C}}$. Then,

(i) The operator $\hat{q}^{L^2}_{\mathfrak{C}}(f) := W_{\mathfrak{C}}\ \hat{q}(f) \restriction_{\mathrm{span}\mathscr{O}^{\otimes}_{\mathfrak{C}}}\ W^{-1}_{\mathfrak{C}}$ has domain $\mathrm{span}(\mathscr{O}^{L^2}_{\mathfrak{C}}) \subseteq L^2(\mathbb{R}^\infty,\, d^\infty\mu_{\mathfrak{C}})$ and action:

$$\hat{q}^{L^2}_{\mathfrak{C}}(f)\ \Big(\psi_1\cdots\psi_N\Big)(x_1,x_2,...) = \sum_{n=1}^{N} \beta_n\ x_n\ \psi_1(x_1)\cdot\cdots\cdot\psi_M(x_M)$$

for each $f := \sum_{n=1}^{N}\beta_n e_n \in \mathcal{W}$ and arbitrary $(\psi_1\cdots\psi_M)(x_1,x_2,...) \equiv \psi_1(x_1)\cdots\psi_M(x_M)$ in $\mathrm{span}(\mathscr{O}^{L^2}_{\mathfrak{C}})$. Moreover, $\hat{q}^{L^2}_{\mathfrak{C}}(f)$ is *essentially self-adjoint* and $\overline{\hat{q}^{L^2}_{\mathfrak{C}}(f)} = W_{\mathfrak{C}}\ \overline{\hat{q}(f)}|_{\mathfrak{C}}\ W^{-1}_{\mathfrak{C}}$.

(ii) The operator $\hat{p}^{L^2}_{\mathfrak{C}}(f) := W_{\mathfrak{C}}\ \hat{p}(f) \restriction_{\mathrm{span}\mathscr{O}^{\otimes}_{\mathfrak{C}}}\ W^{-1}_{\mathfrak{C}}$ has domain: $\mathrm{span}(\mathscr{O}^{L^2}_{\mathfrak{C}}) \subseteq L^2(\mathbb{R}^\infty,\, d^\infty\mu_{\mathfrak{C}})$ and action:

$$\hat{p}^{L^2}_{\mathfrak{C}}(f)\ \Big(\psi_1\cdots\psi_N\Big)(x_1,x_2,...) = \sum_{n=1}^{N}\beta_n\ (-i\hbar)\Big(\partial_n - \frac{\partial_n\rho^{\mathfrak{C}}_n(x_n)}{\rho^{\mathfrak{C}}_n(x_n)}\Big)\ \psi_1(x_1)\cdot\cdots\cdot\psi_M(x_M)$$

for each $f := \sum_{n=1}^{N}\beta_n e_n \in \mathcal{W}$ and each $(\psi_1\cdots\psi_M)(x_1,x_2,...) \equiv \psi_1(x_1)\cdots\psi_M(x_M)$ in $\mathrm{span}(\mathscr{O}^{L^2}_{\mathfrak{C}})$. Moreover, $\hat{p}^{L^2}_{\mathfrak{C}}(f)$ is *essentially self-adjoint* and $\overline{\hat{p}^{L^2}_{\mathfrak{C}}(f)} = W_{\mathfrak{C}}\ \overline{\hat{p}(f)}|_{\mathfrak{C}}\ W^{-1}_{\mathfrak{C}}$.

(iii) For every $f := \sum_{n=1}^{N}\beta_n e_n \in \mathcal{W}$ and $(\psi_1\cdots\psi_M)(x_1,x_2,...) \equiv \psi_1(x_1)\cdots\psi_M(x_M)$ in $L^2(\mathbb{R}^\infty, d^\infty\mu_{\mathfrak{C}})$:

$$e^{i\,\overline{\hat{q}^{L^2}_{\mathfrak{C}}(f)}}\Big(\psi_1\cdots\psi_M\Big)(x_1,x_2,...) = e^{-i\sum\limits_{n=1}^{N}\beta_n x_n}\quad \psi_1(x_1)\cdots\psi_M(x_M) \qquad \text{and}$$

$$e^{i\,\overline{\hat{p}^{L^2}_{\mathfrak{C}}(f)}}\Big(\psi_1\cdots\psi_M\Big)(x_1,x_2,...) = \frac{\rho^{\mathfrak{C}}_1(x_1-\beta_1)\cdots\rho^{\mathfrak{C}}_M(x_M-\beta_M)}{\rho^{\mathfrak{C}}_1(x_1)\cdots\rho^{\mathfrak{C}}_M(x_M)}\ \psi_1(x_1-\beta_1)\cdots\psi_M(x_M-\beta_M).$$

(iv) $\Big(L^2(\mathbb{R}^\infty, d^\infty\mu_{\mathfrak{C}}),\ \{\overline{\hat{q}^{L^2}_{\mathfrak{C}}(f)},\ \overline{\hat{p}^{L^2}_{\mathfrak{C}}(f)}\}_{f\in\mathcal{W}}\Big)$ is an *irreducible representation of the Weyl CCR* that is *equivalent* to $\Big(\otimes^{\mathfrak{C}}_{j\in\mathbb{N}}L^2(\mathbb{R},dx),\ \{\overline{\hat{q}(f)}|_{\mathfrak{C}},\ \overline{\hat{p}(f)}|_{\mathfrak{C}}\}_{f\in\mathcal{W}}\Big)$. It induces *irreducible Heisenberg and creation-annihilation CCRs* that are equivalent to those given in Theorem 35. Moreover, $\Psi(x_1,x_2,...) \equiv 1$ is a *quantum vacuum* for this representation (see Def. 59). ♦

Proof: **Item (i) and (ii):** By Theorem 36, $\mathrm{span}\mathscr{O}^{\otimes}_{\mathfrak{C}} \subseteq \otimes^{\mathscr{F},\mathfrak{C}}_{j\in\mathbb{N}}B_{Her}$ and it is dense in $\otimes^{\mathfrak{C}}_{j\in\mathbb{N}}L^2(\mathbb{R},dx)$. Hence, the restrictions of $\hat{q}(f), \hat{p}(f)$ to $\mathrm{span}(\mathscr{O}^{\otimes}_{\mathfrak{C}})$ are a well-defined densely-defined operators on $\otimes^{\mathfrak{C}}_{j\in\mathbb{N}}L^2(\mathbb{R},dx)$.

- Now, we check that the prescribed actions are correct. Let $\psi_1\cdots\psi_N \in \mathrm{span}\mathscr{O}^{L^2}_{\mathfrak{C}}$. Without loss of generality, we can assume that $N \le M$ because otherwise, we can put $\psi_j(x_j) \equiv 1$ $\forall j \in \{M,...,N\}$ to come to the $N \le M$ case.

$$W_{\mathfrak{C}}\ \hat{q}(f) \restriction_{\mathrm{span}\mathscr{O}^{\otimes}_{\mathfrak{C}}}\ W^{-1}_{\mathfrak{C}}(\psi_1\cdots\psi_M) = W_{\mathfrak{C}}\ \hat{q}(f)\Big((\psi_1\rho^{\mathfrak{C}}_1)\otimes\cdots\otimes(\psi_M\rho^{\mathfrak{C}}_M)\otimes\rho^{\mathfrak{C}}_{M+1}\otimes\cdots\Big) =$$

$$= W_{\mathfrak{C}}\Big(\sum_{n=1}^{N}\beta_n(\psi_1\rho^{\mathfrak{C}}_1)\otimes\cdots\otimes(\hat{q}\psi_n\rho^{\mathfrak{C}}_n)\otimes(\psi_{n+1}\rho^{\mathfrak{C}}_{n+1})\otimes\cdots\otimes(\psi_M\rho^{\mathfrak{C}}_M)\otimes\rho^{\mathfrak{C}}_{M+1}\otimes\cdots\Big) =$$

$$= \sum_{n=1}^{N}\beta_n\psi_1\cdots\Big(\frac{\hat{q}(\psi_n\rho^{\mathfrak{C}}_n)}{\rho^{\mathfrak{C}}_n}\Big)\cdots\psi_M \overset{(\star)}{=} \sum_{n=1}^{N}\beta_n\psi_1\cdots(\hat{q}\psi_n)\psi_{n+1}\cdots\psi_M,$$

where, we used in $(\star)$ that

$$\Big(\frac{\hat{q}(\psi_n\rho^{\mathfrak{C}}_n)}{\rho^{\mathfrak{C}}_n}\Big)(x_n) = \frac{x_n\psi_n(x_n)\rho^{\mathfrak{C}}_n(x_n)}{\rho^{\mathfrak{C}}_n(x_n)} = x_n\psi_n(x_n) = (\hat{q}\psi_n)(x_n).$$

Similarly,

$$W_{\mathfrak{c}}\ \hat{p}(f) \restriction_{\mathrm{span}\mathscr{O}_{\mathfrak{c}}^{\otimes}}\ W_{\mathfrak{c}}^{-1}(\psi_1\cdots\psi_M) = \sum_{n=1}^{N}\beta_n\psi_1\cdots\Big(\frac{\hat{p}(\psi_n\rho_n^{\mathfrak{c}})}{\rho_n^{\mathfrak{c}}}\Big)\cdots\psi_M.$$

In particular, $\Big(\frac{\hat{p}(\psi_n\rho_n^{\mathfrak{c}})}{\rho_n^{\mathfrak{c}}}\Big)(x_n) \overset{\text{(product rule)}}{=} \frac{-i\hbar}{\rho_n^{\mathfrak{c}}(x_n)}\Big(\rho_n^{\mathfrak{c}}(x_n)\ \partial_n\psi_n(x_n) + \psi_n(x_n)\partial_n\rho_n^{\mathfrak{c}}(x_n)\Big) =$

$$= (-i\hbar)\Big(\partial_n\psi_n(x_n) + \psi_n(x_n)\frac{\partial_n\rho_n^{\mathfrak{c}}(x_n)}{\rho_n^{\mathfrak{c}}(x_n)}\Big) = (-i\hbar)\Big(\partial_n + \frac{\partial_n\rho_n^{\mathfrak{c}}(x_n)}{\rho_n^{\mathfrak{c}}(x_n)}\Big)\psi_n(x_n),$$

and that leaves what we claimed.

- Now, by Propositions 76 and 75, all vectors in $\otimes_{j\in\mathbb{N}}^{\mathscr{F}}B_{Her}$ and hence, those in $\mathrm{span}(\mathscr{O}_{\mathfrak{c}}^{\otimes})$, are analytic vectors of both $\hat{q}(f)$ and $\hat{p}(f)$. But we saw that $\mathrm{span}(\mathscr{O}_{\mathfrak{c}}^{\otimes})$ is dense in $\otimes_{j\in\mathbb{N}}^{\mathfrak{c}}L^2(\mathbb{R},dx)$. Moreover, by Cor. 26, $\hat{q}(f)$ and $\hat{p}(f)$ are essentially self-adjoint and thus any of their restrictions are symmetric. Hence, by Nelson's theorem (Theorem X.39 in (Reed and Simon, 1975)), both $\hat{q}(f) \restriction_{\mathrm{span}(\mathscr{O}_{\mathfrak{c}}^{\otimes})}$ and $\hat{p}(f) \restriction_{\mathrm{span}(\mathscr{O}_{\mathfrak{c}}^{\otimes})}$ are essentially self-adjoint operators on $\otimes_{j\in\mathbb{N}}^{\mathfrak{c}}L^2(\mathbb{R},dx)$. As an immediate consequence, $\hat{q}_{\mathfrak{c}}^{L^2}(f)$ is closable and (by Prop. 81.(i)), $\overline{\hat{q}_{\mathfrak{c}}^{L^2}(f)} = W_{\mathfrak{c}}\,\overline{\hat{q}(f) \restriction_{\mathrm{span}(\mathscr{O}_{\mathfrak{c}}^{\otimes})}}\,W_{\mathfrak{c}}^{-1}$. Likewise for $\hat{p}$.

On the other hand, we trivially have that $\hat{q}(f) \restriction_{\mathrm{span}(\mathscr{O}_{\mathfrak{c}}^{\otimes})} \subseteq \hat{q}(f)|_{\mathfrak{c}}$ and $\hat{p}(f) \restriction_{\mathrm{span}(\mathscr{O}_{\mathfrak{c}}^{\otimes})} \subseteq \hat{p}(f)|_{\mathfrak{c}}$, and in Theorem 35 we proved that the closures of $\hat{q}(f)|_{\mathfrak{c}}$ and $\hat{p}(f)|_{\mathfrak{c}}$ are self-adjoint. Hence, by the uniqueness of self-adjoint extension of an essentially self-adjoint operator (Prop. 1.21 in (Arai, 2018)), it must be that $\overline{\hat{q}(f) \restriction_{\mathrm{span}(\mathscr{O}_{\mathfrak{c}}^{\otimes})}} = \overline{\hat{q}(f)|_{\mathfrak{c}}}$ and $\overline{\hat{p}(f) \restriction_{\mathrm{span}(\mathscr{O}_{\mathfrak{c}}^{\otimes})}} = \overline{\hat{p}(f)|_{\mathfrak{c}}}$. Plugging this in the above paragraph's conclusion, we get what we claimed.

Item (iii): By proposition 81.(ii) for any $f = \sum_{n=1}^{N}\beta_n e_n$ the above implies that $e^{i\overline{\hat{h}_{\mathfrak{c}}^{L^2}(f)}} = W_{\mathfrak{c}}\ e^{i\overline{\hat{h}(f)|_{\mathfrak{c}}}}\ W_{\mathfrak{c}}^{-1}$ for both $h\in\{q,p\}$, so, by Theorem 30, $e^{i\overline{\hat{h}_{\mathfrak{c}}^{L^2}(f)}} = W_{\mathfrak{c}}\ (e^{i\beta_1\overline{\hat{h}}}\otimes\cdots\otimes e^{i\beta_N\overline{\hat{h}}}\otimes Id\otimes\cdots)\ W_{\mathfrak{c}}^{-1}$. As such, for both $h\in\{q,p\}$,

$$e^{i\overline{\hat{h}_{\mathfrak{c}}^{L^2}(f)}}\Big(\psi_1\cdots\psi_M\Big) = W_{\mathfrak{c}}(e^{i\beta_1\overline{\hat{h}}}\otimes\cdots\otimes e^{i\beta_N\overline{\hat{h}}}\otimes Id\otimes\cdots)\Big((\psi_1\rho_1^{\mathfrak{c}})\otimes\cdots\otimes(\psi_M\rho_M^{\mathfrak{c}})\otimes\rho_{M+1}^{\mathfrak{c}}\otimes\cdots\Big) =$$

$$= \frac{e^{i\beta_1\overline{\hat{h}}}(\psi_1\rho_1^{\mathfrak{c}})}{\rho_1^{\mathfrak{c}}}\cdots\frac{e^{i\beta_N\overline{\hat{h}}}(\psi_N\rho_N^{\mathfrak{c}})}{\rho_N^{\mathfrak{c}}}.$$

Then, using that

$$\left(\frac{e^{i\beta_n\overline{\hat{q}}}(\psi_n\rho_n^{\mathfrak{c}})}{\rho_n^{\mathfrak{c}}}\right)(x_n) = \frac{e^{i\beta_n x_n}\psi_n(x_n)\rho_n^{\mathfrak{c}}(x_n)}{\rho_n^{\mathfrak{c}}(x_n)} = e^{i\beta_n x_n}\psi_n(x_n) \qquad \text{and}$$

$$\left(\frac{e^{i\beta_n\overline{\hat{p}}}(\psi_n\rho_n^{\mathfrak{c}})}{\rho_n^{\mathfrak{c}}}\right)(x_n) \overset{\left(\substack{\hat{p}\text{ generates translations}\\ \text{see Ex. 3.80 (Porta, 2019)}}\right)}{=} \frac{\psi_n(x_n-\beta_n)\rho_n^{\mathfrak{c}}(x_n-\beta_n)}{\rho_n^{\mathfrak{c}}(x_n)},$$

we get the claimed result.

- **Item (iv):** By (i) and (ii), for every $f\in\mathcal{W}$, $W_{\mathfrak{c}}$ is such that $\overline{\hat{h}_{\mathfrak{c}}^{L^2}(f)} = W_{\mathfrak{c}}\ \overline{\hat{h}(f)|_{\mathfrak{c}}}\ W_{\mathfrak{c}}^{-1}$, so by Prop. 81, $e^{i\overline{\hat{h}_{\mathfrak{c}}^{L^2}(f)}} = W_{\mathfrak{c}}\ e^{i\overline{\hat{h}(f)|_{\mathfrak{c}}}}\ W_{\mathfrak{c}}^{-1}$ for both $h\in\{q,p\}$. Using this, together with the fact that $\{\overline{\hat{q}(f)|_{\mathfrak{c}}},\ \overline{\hat{p}(f)|_{\mathfrak{c}}}\}_{f\in\mathcal{W}}$ satisfy the three conditions defining a of Weyl CCR representation (Def. 47), it is immediate to see that also $\{\overline{\hat{q}_{\mathfrak{c}}^{L^2}(f)},\ \overline{\hat{p}_{\mathfrak{c}}^{L^2}(f)}\}_{f\in\mathcal{W}}$ satisfy them. But then, $\overline{\hat{h}_{\mathfrak{c}}^{L^2}(f)} = W_{\mathfrak{c}}\ \overline{\hat{h}(f)|_{\mathfrak{c}}}\ W_{\mathfrak{c}}^{-1}$ means that the representations in question are equivalent. Finally, by Theorem 35, $\{\overline{\hat{q}(f)|_{\mathfrak{c}}},\ \overline{\hat{p}(f)|_{\mathfrak{c}}}\}_{f\in\mathcal{W}}$ is irreducible, so, by Prop. 5.10 in (Arai, 2018) $\{\overline{\hat{q}_{\mathfrak{c}}^{L^2}(f)},\ \overline{\hat{p}_{\mathfrak{c}}^{L^2}(f)}\}_{f\in\mathcal{W}}$ must be irreducible as well.

Once this is clarified, the statements on the Heisenberg and creation-annihilation CCR representations follow by just mapping the results of Theorem 35 and Proposition 85 from $\otimes_{j\in\mathbb{N}}^{\mathfrak{C}} L^2(\mathbb{R}, dx)$ to $L^2(\mathbb{R}^\infty, d^\infty\mu_{\mathfrak{C}})$ via $W_{\mathfrak{C}}$. Lastly, the claim on the quantum vacuum holds because by Prop. 85, $\otimes_{j\in\mathbb{N}}^{\mathfrak{C}}\rho_j$ is a quantum vacuum for the equivalent creation-annihilation CCR representation of $\otimes_{j\in\mathbb{N}}^{\mathfrak{C}} L^2(\mathbb{R}, dx)$ and in particular, $W_{\mathfrak{C}}(\otimes_{j\in\mathbb{N}}\rho_j^{\mathfrak{C}})(x) \equiv 1$. ***o.ε.δ.***

Note that the case $\rho_j^{\mathfrak{C}}(x) := \pi^{-1/4}e^{-x^2/2}$ yields the standard Gaussian measure $d^\infty\mu_{\mathfrak{C}} = \odot_{j\in\mathbb{N}}|\rho_j^{\mathfrak{C}}|^2 dx_j = \odot_{j\in\mathbb{N}}(\pi^{-1/2}e^{-x_j^2}dx_j)$. It turns out that the CCR representation of Theorem 37 for this particular case is described in §X.7 of (Reed and Simon, 1975) under the name *Q-space*. It is introduced as an auxiliary space "equivalent" to Fock space (to be made precise in a moment) that is convenient to prove the self-adjointness of certain QFT Hamiltonians —due to the fact that the relevant field operators become multiplication operators in this representation. Arai (2018) himself also provides a similar construction in §5.30 under the same name and with a similar spirit for its use. Here we have hinted at a more general construction, but we leave its exhaustive analysis as a future work.

B.5 Parenthesis: The Fock Representation of the CCR

In this section, we briefly remind what we mean by "Fock representation" of the CCR, because it will be a central object in the concluding section.

Lemma 58. Let $\mathcal{H}$ be a complex Hilbert space (also called *1-boson space*). Then,

$$\mathcal{F}_b(\mathcal{H}) := \bigoplus_{n\in\mathbb{N}_0} Sym(\mathcal{H}^{\otimes n})$$

is called the *(bosonic) Fock space* (see Definition 5 for the symbols). An element $\Psi \in \mathcal{F}_b(\mathcal{H})$ is a tuple $\Psi = (\psi^0, \psi^{(1)}, ...)$ with $\psi^{(n)} \in Sym(\mathcal{H}^{\otimes n})$.

- We call the multiplication operator $\widehat{N} \in \mathfrak{L}(\mathcal{F}_b(\mathcal{H}))$ of domain

$$D(\widehat{N}) := \left\{ \Psi \in \mathcal{F}_b(\mathcal{H}) \;\middle|\; \sum_{n=1}^{\infty} n^2 \left\|\psi^{(n)}\right\| < +\infty \right\}$$

and action $\hat{N}\Psi := (0, 1\psi^{(1)}, 2\psi^{(2)}, 3\psi^{(3)}, ...)$ the *number operator*. By §5.3.5 (Arai, 2018), it is the self-adjoint operator with spectral PVM $P_b(\cdot)$, such that for $B \in \mathfrak{B}(\mathbb{R})$, $P_b(B) = \sum_{n\in B\cap\mathbb{N}_0} P_b^{(n)}$ and $P_b^{(n)}\Psi := (\underbrace{0, ..., 0, \psi^{(n)}}_{n}, 0, ...)$. Hence, the Fock space is often seen as a spectral diagonalization space for the number operator.

- If $\mathcal{H}$ represents the "state space" of a quantum theory describing a single "particle", $Sym(\mathcal{H}^{\otimes n})$ represents the quantum theory of n such identical bosonic "particles" —we call it the *n boson sector*. Then, if $||\Psi|| = 1$, the spectral measure of the number operator, $\|P_b(\{n\})\Psi\|^2 = ||\psi^{(n)}||^2$, $n \in \mathbb{N}_0$, is interpreted as the probability that there are exactly n "particles" in a system described by Ψ. Hence, for such $\mathcal{H}$, Fock space describes the Hilbert space of a *variable (but finite!)*[a] *number of particles.*

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[a] Since $\sum_{n\in\mathbb{N}_0} \left\|\psi^{(n)}\right\|^2 = \|\Psi\| < +\infty$, $\mathbb{P}\Big(\#particles \geq N\Big) = \sum_{n\geq N} ||\psi^{(n)}||^2 \xrightarrow[N\to+\infty]{} 0$. Thus, the number of "particles" that can be described using a vector $\Psi \in \mathcal{F}_b(\mathcal{H})$ is unbounded but never actually infinite. See also Prop. 5.13 in (Arai, 2018).

- One calls $\Omega := (1,0,0,\dots)$ the *vacuum state* (it represents a state with no "particle"). We call $\mathcal{F}_b^{fin}(\mathcal{H}) := \bigoplus^{\mathscr{F}}_{n\in\mathbb{N}_0} Sym(\mathcal{H}_+^{(n)})$ the space of states with a *bounded number of particles.*

- For each $f \in \mathcal{H}$, the operator with domain

$$D(A^\dagger(f)) := \left\{ \Psi \in \mathcal{F}_b(\mathcal{H}) \;\middle|\; \sum_{n=0}^{\infty} \left\| \frac{1}{\sqrt{(n-1)!}}\; Sym\Big(f \otimes \psi^{(n-1)}\Big) \right\|^2 < +\infty \right\}$$

and action,

$$(A^\dagger(f)\Psi)^{(0)} := 0, \qquad (A^\dagger(f)\Psi)^{(n)} := \frac{1}{\sqrt{(n-1)!}}\; Sym\Big(f \otimes \psi^{(n-1)}\Big) \quad \forall n \in \mathbb{N}$$

for $\Psi \in D(A^\dagger(f))$, is called the *bosonic creation operator (of the test-vector f).* Up to scale factors, its action is to take the $(n-1)$ particle sector vector $\psi^{(n-1)}$, "add a particle" in the state f and then to symmetrize the result.

- For $f \in \mathcal{H}$ and $n \in \mathbb{N}$, define the auxiliary operator $B_n(f) : \mathcal{H}^{\otimes(n+1)} \to \mathcal{H}^{\otimes n}$ as the unique (see §4.14 in (Arai, 2018)) bounded operator acting on $\psi_1 \otimes \cdots \otimes \psi_{n+1} \in \mathcal{H}^{\otimes(n+1)}$ as

$$B_n(f)\Big(\psi_1 \otimes \cdots \otimes \psi_{n+1}\Big) = \langle f, \psi_1\rangle\; \psi_2 \otimes \cdots \otimes \psi_{n+1}.$$

Then, by Lemma 5.4 in (Arai, 2018), $\forall f \in \mathcal{H}$, $A^\dagger(f)$ is densely defined, closed and has an adjoint $A(f) := A^\dagger(f)^*$ of domain

$$D(A(f)) = \left\{ \Psi \in \mathcal{F}_b(\mathcal{H}) \;\middle|\; \sum_{n=0}^{\infty} \left\| \frac{\sqrt{n+1}}{\sqrt{n!}}\; Sym \circ B_n(f)\psi^{(n+1)} \right\|^2 < +\infty \right\}$$

such that, for $\Psi \in D(A(f))$,

$$(A(f)\Psi)^{(n)} = \frac{\sqrt{(n+1)}}{\sqrt{n!}}\; Sym\Big(B_n(f)\psi^{(n+1)}\Big) \quad \forall n \in \mathbb{N}.$$

We call $A(f)$ the *bosonic annihilation operator (of test-vector f).* Up to scale factors, its action is to take the $(n+1)$ particle sector vector $\psi^{(n+1)}$, "subtract a particle" in the state f (that is precisely B_n's action) and then symmetrize the result. As the adjoint of the closed operator $A^\dagger(f)$, we have that $A(f)^* = A^\dagger(f)\ \forall f \in \mathcal{H}$.

- For each $f \in \mathcal{H}$, since $A(f) + A(f)^*$ is clearly symmetric, one can define the closed symmetric operator $\widehat{\Phi}_S(f) := \frac{1}{\sqrt{2}}\overline{\Big(A(f) + A(f)^*\Big)}$, called the *Segal field operator* (of test-vector f). With them, given a conjugation C on $\mathcal{H}$ one defines for each $f \in (\mathcal{H})^C$ the *canonical conjugate field operators (associated to C)* $\widehat{\varphi}_C(f) := \widehat{\Phi}_S(f)$ and $\widehat{\pi}_C(f) := \widehat{\Phi}_S(if)$. By Thm. 5.28 in (Arai, 2018), these two last operators are self-adjoint (and essentially self-adjoint when restricted to $\mathcal{F}_b^{fin}(\mathcal{H})$).

◆

Theorem 38. Let $\mathcal{H}$ be a Hilbert space and let $\mathcal{V} \subseteq \mathcal{H}$ be an arbitrary dense subspace. Then, defining

$$\mathcal{F}_b^{fin}(\mathcal{V}) := \left\{ \Psi \in \mathcal{F}_b^{fin}(\mathcal{H}) \;\middle|\; \psi^{(n)} \in Sym\Big(\underbrace{\operatorname{span}\Big\{ \psi_1 \otimes \cdots \otimes \psi_n \;:\; \psi_j \in \mathcal{V} \Big\}}_{\otimes^{\mathscr{F}}_{j\in\{1,\dots,n\}}\mathcal{V}} \Big) \right\},$$

the triplet

$$\Big(\; \mathcal{F}_b(\mathcal{H}),\; \mathcal{F}_b^{fin}(\mathcal{V}),\; \{A(f)\}_{f\in\mathcal{V}} \;\Big),$$

is an *irreducible representation of the creation-annihilation CCR* and we call it the *Fock representation.* Moreover, given a conjugation C on $\mathcal{H}$ such that $C\mathcal{V} \subseteq \mathcal{V}$, respectively,

$$\left(\mathcal{F}_b(\mathcal{H}),\ \mathcal{F}_b^{fin}(\mathcal{H}),\ \{\widehat{\varphi}_C(f),\ \widehat{\pi}_C(f)\}_{f\in(\mathcal{V})^C} \right)$$

$$\text{and} \quad \left(\mathcal{F}_b(\mathcal{H}),\ \{\widehat{\varphi}_C(f),\ \widehat{\pi}_C(f)\}_{f\in(\mathcal{V})^C}\right)$$

are *irreducible representations of the Heisenberg and Weyl CCR.* We call them the *Fock representation of the Heisenberg and Weyl CCR*, respectively.

♦

Proof: See Examples 5.9, 5.18 together with Theorems 5.15, 5.28 and 5.46 in (Arai, 2018). ☐

B.6 The Connection Between Fock Space, the ITP, the CCR representations and the Quantum Vacuum

B.6.1 The Empty Vacuum Characterization of the Fock Representation

Let us extend the usual notion of vacuum given in Definition 5.5 of (Arai, 2018) to make it also suitable for layers other than the "Gaussian" one (to be defined later). Heuristically, we want a notion of vacuum that is not necessarily empty (one may think of a bosonic version of the "Dirac sea").

Definition 59. Let $\left(\mathscr{H},\ \mathscr{D},\ \{\widehat{C}(f)\}_{f\in\mathcal{V}}\right)$ be a creation-annihilation CCR representation. We define a *quantum vacuum* of the representation (if it exists) to be a unit vector $\Omega \in \mathscr{D}$[12] satisfying that:

(i) Ω is *cyclic* for $\{\widehat{C}(f),\ \widehat{C}(f)^*\}_{f\in\mathcal{V}}$, i.e., such that

$$\mathscr{H}_\Omega := \operatorname{span}\left\{\Omega,\ \widehat{C}(f_1)^{\#_1}\cdots\widehat{C}(f_n)^{\#_n}\Omega \;\middle|\; n\in\mathbb{N},\ \#_j\in\{1,*\},\ f_j\in\mathcal{V}\right\} \tag{B.56}$$

is a *dense* subspace of $\mathscr{H}$.

(ii) $\mathscr{H}_\Omega$ is a core of $\widehat{C}(f)$ for all $f\in\mathcal{V}$.

The space $\mathscr{H}_\Omega$ shall be called *the space of states with a bounded number of bosons relative to the quantum vacuum* Ω. We classify the quantum vacuums Ω as follows:

(a) If Ω satisfies that $\widehat{C}(f)\Omega = 0$ for all $f\in\mathcal{V}$, we call it an *empty quantum vacuum.*

(b) If $\exists\widetilde{\Omega}\in\mathscr{H}_\Omega$ that is an empty quantum vacuum, we call Ω a *finite perturbation of an empty quantum vacuum.*

(c) In any other case, we call Ω a *myriotic quantum vacuum.*

When an empty vacuum exists, we say that $\left(\mathscr{H},\ \mathscr{D},\ \{\widehat{C}(f)\}_{f\in\mathcal{V}}\right)$ is a *representation of the (creation-annihilation) CCR with empty vacuum* Ω. If only myriotic quantum vacuums exist, we say that it is a *myriotic representation of the CCR.*

♦

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[12] Hence, such that $\Omega\in D\left(\widehat{C}(f_1)^{\#_1}\cdots\widehat{C}(f_n)^{\#_n}\right) \quad \forall f_j\in\mathcal{V},\ \ \forall\#_j\in\{*,\ 1\},\ \ \forall n\in\mathbb{N}.$

Lemma 59. In the notation of Def. 59, if Ω is an *empty* quantum vacuum, then

$$\mathscr{H}_\Omega = \operatorname{span}\Big\{\Omega,\ \widehat{C}(f_1)^* \cdots \widehat{C}(f_n)^*\Omega \ \Big|\ n \in \mathbb{N},\ f_j \in \mathcal{V}\ \Big\}$$

and thus,

$$\Big(\Omega \text{ is cyclic for } \{\widehat{C}(f),\ \widehat{C}(f)^*\}_{f\in\mathcal{W}}\Big) \iff \Big(\Omega \text{ is cyclic for } \{\widehat{C}(f)^*\}_{f\in\mathcal{W}}\Big).$$

(In words, *"it is enough to talk about the states generated from the quantum vacuum by addition of bosons —removal of bosons from the vacuum leaves nothing (it is empty)".*) ♦

Proof: The ($\Longleftarrow$) implication is trivial. We prove ($\Longrightarrow$).

Claim: Any $\widehat{C}(f_1)^{\#_1}\cdots\widehat{C}(f_n)^{\#_n}\Omega$ (with $\#_j \in \{1,*\}$, $f_j \in \mathcal{W}$) can be written as a linear combination of $\widehat{C}(f_{j_1})^*\cdots\widehat{C}(f_{j_m})^*\Omega$ for some $m \in \mathbb{N}, j_m \in \{1,...,n\}$.

Check: By using the CCR (Def. 48.(iii)), for every $f, g \in \mathcal{W}$, $\#, \#' \in \{1,*\}$, there is a $\beta \in \{\pm\langle f,g\rangle,\ 0\}$ such that $[\widehat{C}(f)^{\#}, \widehat{C}(g)^{\#'}] \restriction_{\mathscr{D}} = \beta\, Id$. Then, for arbitrary $f_j \in \mathcal{W}$, $N \in \mathbb{N}, \#_j \in \{1,*\}$,

$$\underbrace{\widehat{C}(f_0)\ \widehat{C}(f_1)^{\#_1}}_{\text{(apply CCR)}}\cdots\ \widehat{C}(f_n)^{\#_n}\ \Omega = \beta_1\ \widehat{C}(f_1)^{\#_1}\ \underbrace{\widehat{C}(f_0)\widehat{C}(f_2)^{\#_2}}_{\text{(apply CCR)}}\cdots\widehat{C}_n\Omega + \beta_2\widehat{C}(f_2)^{\#_2}\cdots\widehat{C}_n\Omega =$$

$$= \cdots = \sum_{j=1}^{N}\tilde{\beta}_j\ \widehat{C}(f_1)^{\#_1}\cdots\widehat{C}(f_{j-1})^{\#_{j-1}}\widehat{C}(f_{j+1})^{\#_{j+1}}\cdots\widehat{C}(f_N)^{\#_N}\Omega + \tilde{\beta}_0\widehat{C}(f_1)^{\#_1}\cdots\widehat{C}(f_N)^{\#_N}\ \underbrace{\widehat{C}(f_0)\Omega}_{0} =$$

$$= \sum_{j=1}^{N}\tilde{\beta}_j\ \widehat{C}(f_1)^{\#_1}\cdots\widehat{C}(f_{j-1})^{\#_{j-1}}\widehat{C}(f_{j+1})^{\#_{j+1}}\cdots\widehat{C}(f_N)^{\#_N}\Omega,$$

for some numbers $(\tilde{\beta}_j)_{j=0}^N \in \mathbb{R}$. Therefore, any annihilation operator $\hat{C}(f_0)$ can be "moved next to the Ω", in exchange generating linear combinations of "the remaining operators". Since in any $\widehat{C}(f_1)^{\#_1}\cdots\widehat{C}(f_n)^{\#_n}\Omega$ (as in the claim) there are finitely many annihilation operators, we can do this finitely many times and be left with a linear combination of $\widehat{C}(f_{j_1})^*\cdots\widehat{C}(f_{j_m})^*\Omega$.

- Therefore,

$$\operatorname{span}\Big\{\Omega,\ \widehat{C}(f_1)^{\#_1}\cdots\widehat{C}(f_n)^{\#_n}\Omega\ \Big|\ n\in\mathbb{N},\ \#_j\in\{1,*\},\ f_j\in\mathcal{V}\ \Big\} \subseteq$$

$$\subseteq \operatorname{span}\Big\{\Omega,\ \widehat{C}(f_1)^*\cdots\widehat{C}(f_n)^*\Omega\ \Big|\ n\in\mathbb{N},\ f_j\in\mathcal{V}\ \Big\}.$$

Since $\supseteq$ is trivially true, they are the same set (so, one is dense when the other one is dense).
o.ε.δ.

The key result we will use in this section is the following one from Arai (2018) (although adapted to our different definitions).

Theorem 39. Let $\Big(\mathscr{H},\ \mathscr{D},\ \{\widehat{C}(f)\}_{f\in\mathcal{V}}\ \Big)$ be a representation of the creation-annihilation CCR with an *empty* vacuum Ω. Then, the given representation is equivalent to the Fock representation $\Big(\ \mathcal{F}_b(\mathcal{H}),\ \mathcal{F}_b^{fin}(\mathcal{V}),\ \{A(f)\}_{f\in\mathcal{V}}\ \Big)$ and the equivalence identifies Ω with the Fock vacuum $(1,0,0,...)$. ♦

Proof: Combine Corollary 5.15 in (Arai, 2018) and Lemma 59. Mind the difference between his and our definition of quantum vacuum. ☐

Corollary 32. Let $R := \Big(\mathscr{H},\ \mathscr{D},\ \{\widehat{C}(f)\}_{f\in\mathcal{V}}\Big)$ be a representation of the creation-annihilation CCR. Then,

$$\begin{pmatrix} R \text{ has } \textit{an empty} \\ \text{quantum vacuum} \end{pmatrix} \iff \begin{pmatrix} R \text{ is } \textit{equivalent} \text{ to the} \\ \text{Fock representation} \\ \big(\mathcal{F}_b(\mathcal{H}),\ \mathcal{F}_b^{fin}(\mathcal{W}_{\mathbb{C}}),\ \{A(f)\}_{f\in\mathcal{W}_{\mathbb{C}}}\big) \end{pmatrix}.$$

Moreover,

$$\begin{pmatrix} \textit{Every} \text{ quantum vacuum} \\ \text{of } R \text{ (if there is any)} \\ \text{is } \textit{myriotic} \end{pmatrix} \iff \begin{pmatrix} R \text{ is } \textit{inequivalent} \text{ to the} \\ \text{Fock representation} \\ \big(\mathcal{F}_b(\mathcal{H}),\ \mathcal{F}_b^{fin}(\mathcal{W}_{\mathbb{C}}),\ \{A(f)\}_{f\in\mathcal{W}_{\mathbb{C}}}\big) \end{pmatrix}.$$

♦

Proof: The first claimed ($\iff$) is trivially given by Theorem 39. The second one is its negation after noting the following remark. By the classification of Def. 59, a quantum vacuum that is not empty is either a bounded perturbation of an empty vacuum or it is myriotic. But the first case implies the existence of an empty vacuum, so, a representation with no empty vacuum can only have (if any) a myriotic vacuum. ***o.ε.δ.***

B.6.2 Every Layer of the ITP has a Quantum Vacuum

Proposition 85. Let $\mathfrak{C} \in \Gamma$ be arbitrary and let $\otimes_{j\in\mathbb{N}}\rho_j^{\mathfrak{C}}$ be an arbitrary generator of the layer $\otimes_{j\in\mathbb{N}}^{\mathfrak{C}} L^2(\mathbb{R}, dx)$ with $\rho_j^{\mathfrak{C}} \in B_{Her}$ (i.e., such that $\otimes_{j\in\mathbb{N}}\rho_j^{\mathfrak{C}} \in \otimes_{j\in\mathbb{N}}^{\mathscr{F}} B_{Her}$) —which always exists by Prop. 82. Then,

- **(i)** $\otimes_{j\in\mathbb{N}}\rho_j^{\mathfrak{C}}$ is a cyclic vector of $\{\hat{q}(f)|_{\mathfrak{C}}, \hat{p}(f)|_{\mathfrak{C}}\}_{f\in\mathcal{W}}$.
- **(ii)** $\otimes_{j\in\mathbb{N}}\rho_j^{\mathfrak{C}}$ is a cyclic vector of $\{\hat{a}(f)|_{\mathfrak{C}}, \hat{a}(f)|_{\mathfrak{C}}^*\}_{f\in\mathcal{W}_{\mathbb{C}}}$. That is, the space $\mathscr{H}_{\otimes_{j\in\mathbb{N}}\rho_j^{\mathfrak{C}}}$ generated from $\otimes_{j\in\mathbb{N}}\rho_j^{\mathfrak{C}}$ as in (B.56) is *dense* in $\otimes_{k\in\mathbb{N}}^{\mathfrak{C}} L^2(\mathbb{R}, dx)$.
- **(iii)** $\mathscr{H}_{\otimes_{j\in\mathbb{N}}\rho_j^{\mathfrak{C}}}$ is a core of $\hat{a}(f)|_{\mathfrak{C}}$ for all $f \in \mathcal{W}_{\mathbb{C}}$.

Hence, $\otimes_{j\in\mathbb{N}}\rho_j^{\mathfrak{C}}$ is a *quantum vacuum* of the representation

$$\Big(\otimes_{k\in\mathbb{N}}^{\mathfrak{C}} L^2(\mathbb{R}, dx),\ \otimes_{k\in\mathbb{N}}^{\mathscr{F},\mathfrak{C}} B_{Her},\ \{\ \hat{a}(f)|_{\mathfrak{C}}\ \}_{f\in\mathcal{W}_{\mathbb{C}}}\Big).$$

(As such, it is justified to call a generator of a layer of the ITP a *vacuum*.) ♦

Proof: By Theorem 35, $\{\hat{q}(f)|_{\mathfrak{C}},\ \hat{p}(f)|_{\mathfrak{C}}\}_{f\in\mathcal{W}}$ and $\{\hat{a}(f)|_{\mathfrak{C}}\}_{f\in\mathcal{W}_{\mathbb{C}}}$ make CCR representations with dense domain $\otimes_{j\in\mathbb{N}}^{\mathscr{F},\mathfrak{C}} B_{Her}$. By property (i) of Defs. 46 and 48, this implies we can apply arbitrarily many mixed compositions of those operators to $\otimes_{j\in\mathbb{N}}\rho_j^{\mathfrak{C}}$ (which is in $\otimes_{j\in\mathbb{N}}^{\mathscr{F},\mathfrak{C}} B_{Her}$ by definition).

- **Item (i):** Consider the ONB $\mathscr{O}_{\mathfrak{C}}^{\otimes}$ of equation (B.55), associated to $\otimes_{j\in\mathbb{N}}\rho_j^{\mathfrak{C}}$. Using that $\otimes$ is linear (Prop. 53), every element of the ONB can be re-written as a (finite) linear combination of elements like:

$$(x^{n_1}\rho_1^{\mathfrak{C}}) \otimes \cdots \otimes (x^{n_N}\rho_N^{\mathfrak{C}}) \otimes \rho_{N+1}^{\mathfrak{C}} \otimes \cdots$$

for some $N \in \mathbb{N},\ n_j \in \mathbb{N}_0$. But,

$$\hat{q}(e_1)^{n_1} \cdots \hat{q}(e_N)^{n_N}\Big(\otimes_{j\in\mathbb{N}}\rho_j^{\mathfrak{C}}\Big) = (x^{n_1}\rho_1^{\mathfrak{C}}) \otimes \cdots \otimes (x^{n_N}\rho_N^{\mathfrak{C}}) \otimes \rho_{N+1}^{\mathfrak{C}} \otimes \cdots .$$

Thus,

$$\operatorname{span}\mathscr{O}_{\mathfrak{C}}^{\otimes} \subseteq \operatorname{span}\Big\{ \bigotimes_{j\in\mathbb{N}}\rho_j^{\mathfrak{C}},\quad \hat{q}(f_1)\cdots\hat{q}(f_N)\Big(\bigotimes_{j\in\mathbb{N}}\rho_j^{\mathfrak{C}}\Big)\ \Big|\ N \in \mathbb{N},\ f_j \in \mathcal{W} \Big\} \subseteq$$

$$\subseteq \operatorname{span}\Big\{ \bigotimes_{j\in\mathbb{N}} \rho_j^{\mathfrak{c}},\quad \hat{h_1}(f_1)|_{\mathfrak{c}} \cdots \hat{h_n}(f_n)|_{\mathfrak{c}} \Big(\bigotimes_{j\in\mathbb{N}} \rho_j^{\mathfrak{c}}\Big) \,\Big|\, h_j\in\{q,p\},\ n\in\mathbb{N},\ f_j\in\mathcal{W} \Big\} =: S$$

By definition of ONB, $\operatorname{span}\mathscr{O}_{\mathfrak{c}}^{\otimes}$ is dense in $\otimes_{j\in\mathbb{N}}^{\mathfrak{c}} L^2(\mathbb{R},dx)$ and thus, S is dense as well, meaning that $\otimes_{j\in\mathbb{N}}\rho_j^{\mathfrak{c}}$ is a cyclic vector of $\{\hat{q}(f)|_{\mathfrak{c}},\ \hat{p}(f)|_{\mathfrak{c}}\}_{f\in\mathcal{W}}$.

- **Item (ii):** By Theorem 31, using the reducibility by $\mathfrak{C}$ of all the implied operators,

$$\hat{q}(f)|_{\mathfrak{c}} = \frac{1}{\sqrt{2}}\Big(\hat{a}(f)|_{\mathfrak{c}}^{*} + \hat{a}(f)|_{\mathfrak{c}}\Big) \restriction_{\otimes_{j\in\mathbb{N}}^{\mathscr{F},\mathfrak{c}} B_{Her}} \qquad \text{and}$$

$$\hat{p}(f)|_{\mathfrak{c}} = \frac{i}{\sqrt{2}}\Big(\hat{a}(f)|_{\mathfrak{c}}^{*} - \hat{a}(f)|_{\mathfrak{c}}\Big) \restriction_{\otimes_{j\in\mathbb{N}}^{\mathscr{F},\mathfrak{c}} B_{Her}} \qquad \forall f\in\mathcal{W}.$$

Hence, on $\otimes_{j\in\mathbb{N}}^{\mathscr{F},\mathfrak{c}} B_{Her}$ any $\hat{h_1}(g_1)|_{\mathfrak{c}} \cdots \hat{h_n}(g_n)|_{\mathfrak{c}}$ with $h\in\{p,q\}$ can be written as a linear combination of $\hat{a}(f_1)|_{\mathfrak{c}}^{\#_1}\cdots\hat{a}(f_N)|_{\mathfrak{c}}^{\#_N}$'s. And thus,

$$S \subseteq \operatorname{span}\Big\{ \bigotimes_{j\in\mathbb{N}} \rho_j^{\mathfrak{c}},\quad \hat{a}(f_1)|_{\mathfrak{c}}^{\#_1}\cdots\hat{a}(f_N)|_{\mathfrak{c}}^{\#_N} \Big(\bigotimes_{j\in\mathbb{N}} \rho_j^{\mathfrak{c}}\Big) \,\Big|\, \#_j\in\{1,*\},\ f_j\in\mathcal{W}_{\mathbb{C}} \Big\} =: \mathscr{H}_{\otimes_{j\in\mathbb{N}}\rho_j^{\mathfrak{c}}}.$$

Since S was proven to be dense in $\otimes_{j\in\mathbb{N}}^{\mathfrak{c}} L^2(\mathbb{R},dx)$, then $\mathscr{H}_{\otimes_{j\in\mathbb{N}}\rho_j^{\mathfrak{c}}}$ must be so. And therefore, by definition, $\bigotimes_{j\in\mathbb{N}}\rho_j^{\mathfrak{c}}$ is cyclic for $\{\hat{a}(f)|_{\mathfrak{c}},\ \hat{a}(f)|_{\mathfrak{c}}^{*}\}_{f\in\mathcal{W}_{\mathbb{C}}}$.

- **Item (iii): Claim 1:** If $(\phi_n)_{n\in\mathbb{N}} \subseteq B_{Her}$ is the harmonic oscillator ONB of Lemma 43 with $\omega = 1$ and

$$H := \Big\{ \phi_{n_1}\otimes\cdots\otimes\phi_{n_M}\otimes\rho_{M+1}^{\mathfrak{c}}\otimes\rho_{M+2}^{\mathfrak{c}}\otimes\cdots \,\Big|\, M\in\mathbb{N},\ n_j\in\mathbb{N}_0 \Big\} \subseteq \mathscr{H}_{\otimes_{j\in\mathbb{N}}\rho_j^{\mathfrak{c}}}, \tag{B.57}$$

then, $\operatorname{span}(H) \subseteq \mathscr{H}_{\otimes_{j\in\mathbb{N}}\rho_j^{\mathfrak{c}}}$.

Check: By definition, each $\rho_j^{\mathfrak{c}} \in B_{Her} = \operatorname{span}\{\phi_n\}_{n\in\mathbb{N}}$, so, $\rho_j^{\mathfrak{c}}(x) = \beta_{N_j}\phi_{N_j} + \cdots + \beta_0\phi_0$ for some $N_j\in\mathbb{N}_0$, $(\beta_\ell)_{\ell=1}^{N_j} \subseteq \mathbb{C}$ with leading coefficient $\beta_{N_j} \neq 0$. Then, by Lemma 43, $(\hat{a})^{N_j}\rho_j^{\mathfrak{c}} = \sqrt{N_j!}\,\beta_{N_j}\ \phi_0$, such that, $\phi_0 = \frac{1}{\sqrt{N_j!\beta_{N_j}}}(\hat{a})^{N_j}\rho_j^{\mathfrak{c}}$. Hence, by Lemma 43, for each $n\in\mathbb{N}_0$, $\phi_n = \frac{1}{N_j!\beta_{N_j}}(\hat{a}^{\dagger})^n(\hat{a})^{N_j}\rho_j^{\mathfrak{c}}$. Therefore, using Proposition 79 and the linearity of $\otimes$ (Prop. 53) in $(\star)$:

$$\frac{1}{N_j!\beta_{N_j}}(\hat{a}(e_j))^n(\hat{a}(e_j))^{N_j}\Big(\bigotimes_{k\in\mathbb{N}}\rho_k^{\mathfrak{c}}\Big) \overset{(\star)}{=} \rho_1^{\mathfrak{c}}\otimes\cdots\otimes\rho_{j-1}^{\mathfrak{c}}\otimes\Big(\frac{1}{N_j!\beta_{N_j}}(\hat{a}^{\dagger})^n(\hat{a})^{N_j}\rho_j^{\mathfrak{c}}\Big)\otimes\rho_{j+1}^{\mathfrak{c}}\otimes\cdots =$$

$$= \rho_1^{\mathfrak{c}}\otimes\cdots\otimes\rho_{j-1}^{\mathfrak{c}}\otimes\phi_n\otimes\rho_{j+1}^{\mathfrak{c}}\otimes\cdots,$$

and thus, for any $M\in\mathbb{N}$, $n_1,\dots,n_M\in\mathbb{N}_0$,

$$\frac{1}{N_1\cdots N_M!\beta_{N_1}\cdots\beta_{N_M}}\ (\hat{a}(e_1))^{n_1}(\hat{a}(e_1))^{N_1}\cdots(\hat{a}(e_M))^{n_M}(\hat{a}(e_M))^{N_M}\Big(\bigotimes_{k\in\mathbb{N}}\rho_k^{\mathfrak{c}}\Big) =$$

$$= \phi_{n_1}\otimes\cdots\otimes\phi_{n_M}\otimes\rho_{M+1}^{\mathfrak{c}}\otimes\rho_{M+2}^{\mathfrak{c}}\otimes\cdots.$$

This proves the claim that $H \subseteq \mathscr{H}_{\otimes_{j\in\mathbb{N}}\rho_j^{\mathfrak{c}}}$. Since the l.h.s is a vector space, this implies that $\operatorname{span}(H) \subseteq \mathscr{H}_{\otimes_{j\in\mathbb{N}}\rho_j^{\mathfrak{c}}}$.

- **Claim 2:** $\operatorname{span}(H) = \mathscr{H}_{\otimes_{j\in\mathbb{N}}\rho_j^{\mathfrak{c}}}$.

(As a side-note, by Lemma 13.(ii), H is an ONB of $\otimes^{\mathfrak{C}}_{j\in\mathbb{N}} L^2(\mathbb{R}, dx)$, so Claim 2 constitutes an alternative proof of item (ii).)

Check: Each $\rho^{\mathfrak{C}}_j$ is a linear combination of ϕ_n and the repeated applications of $\hat{a}$ and $\hat{a}^\dagger$ on them (by Lemma 43) only change which vectors of the ONB $\{\phi_n\}_{n\in\mathbb{N}}$ describe this linear combination. As such, by the linearity of $\otimes$, $\otimes_{j\in\mathbb{N}}\rho^{\mathfrak{C}}_j$ and any application of $\hat{a}_n$ and $\hat{a}^*_n$ on $\otimes_{j\in\mathbb{N}}\rho^{\mathfrak{C}}_j$ are in $\operatorname{span}(H)$ —together with their linear combinations, which include all the compositions of $\hat{a}(f)|_{\mathfrak{C}}$, $\hat{a}(f)|^*_{\mathfrak{C}}$. Hence, $\operatorname{span}(H) \supseteq \mathscr{H}_{\otimes_{j\in\mathbb{N}}\rho^{\mathfrak{C}}_j}$, which, together with Claim 1, proves Claim 2.

- Now, by definition, for every $f \in \mathcal{W}_{\mathbb{C}}$, $\otimes^{\mathscr{F}}_{j\in\mathbb{N}} B_{Her}$ is a core of $\hat{a}(f)$, so, by Thm. 1.47.(iv) in (Arai, 2018), $P^{\mathfrak{C}} \otimes^{\mathscr{F}}_{j\in\mathbb{N}} B_{Her} = \otimes^{\mathscr{F},\mathfrak{C}}_{j\in\mathbb{N}} B_{Her}$ is a core of $\hat{a}(f)|_{\mathfrak{C}}$. As such, for an arbitrary $\Psi \in D(\hat{a}(f)|_{\mathfrak{C}})$, there exists a sequence $(\Psi_k)_{k\in\mathbb{N}} \subseteq \otimes^{\mathscr{F},\mathfrak{C}}_{j\in\mathbb{N}} B_{Her}$ converging to Ψ such that $\|\hat{a}(f)\Psi - \hat{a}(f)\Psi_k\| \xrightarrow[k\to\infty]{} 0$. But by definition, each $\Psi_k = \sum_{\ell=1}^{M_k} c^\ell_k (\otimes_{j\in\mathbb{N}} \psi^\ell_{j,k})$ for some $M_k \in \mathbb{N}$, $\psi^\ell_{j,k} \in B_{Her}$ and thus, for arbitrary $f = \sum_{n=1}^N \beta^f_n e_n$,

$$\hat{a}(f)\Psi_k = \sum_{\ell=1}^{M_k} \underbrace{\sum_{n=1}^{N} c^\ell_k \beta^f \ \psi^\ell_{1,k} \otimes \cdots \otimes (\hat{a}\psi^\ell_{n,k}) \otimes \cdots \otimes \psi^\ell_{N,k}}_{=:\hat{A}_N(\eta^\ell_k)} \otimes \psi^\ell_{N+1,\ k} \otimes \cdots \overset{(\star)}{=}$$

$$= \sum_{\ell=1}^{M_k} \left(\hat{A}_N(\eta^\ell_k)\right) \otimes \psi^\ell_{N+1,\ k} \otimes \psi^\ell_{N+2,\ k} \otimes \cdots .$$

In $(\star)$ we used the linearity of $\otimes$ (Prop. 53) and its associativity (Thm. 26), together with $\eta^\ell_k := \psi^\ell_{1,k} \otimes \cdots \otimes \psi^\ell_{N,k}$. Now, using twice Proposition 24, we get the following two norm-convergences of sequences in m (with $m \in \{N+1,\ N+2, \ldots\}$)

$$\bigotimes_{j\in\mathbb{N}} \psi^\ell_{j,k} = \lim_{m\to\infty} \psi^\ell_{1,k} \otimes \cdots \otimes \psi^\ell_{m,k} \otimes \rho^{\mathfrak{C}}_{m+1} \otimes \rho^{\mathfrak{C}}_{m+2} \otimes \cdots \quad \text{and} \tag{B.58}$$

$$\left(\hat{A}_N(\eta^\ell_k)\right) \otimes \psi^\ell_{N+1,\ k} \otimes \psi^\ell_{N+2,\ k} \otimes \cdots = \lim_{m\to\infty} \left(\hat{A}_N(\eta^\ell_k)\right) \otimes \psi^\ell_{N+1,\ k} \otimes \cdots \otimes \psi^\ell_{m,\ k} \otimes \rho^{\mathfrak{C}}_{m+1} \otimes \cdots =$$
$$\overset{\text{(by def.)}}{=} \lim_{m\to\infty} \hat{a}(f)\Big(\psi^\ell_{1,k} \otimes \cdots \otimes \psi^\ell_{m,\ k} \otimes \rho^{\mathfrak{C}}_{m+1} \otimes \cdots\Big). \tag{B.59}$$

Note that by linearity of $\otimes$, the two involved sequences are in $\operatorname{span}(H)$: after all, by definition, $\psi^\ell_{j,k} \in B_{Her} \equiv \operatorname{span}\{\phi_n\}_{n\in\mathbb{N}}$. And here comes the relevance of the above claims: the sequences are thus, sequences in $\mathscr{H}_{\otimes_{j\in\mathbb{N}}\rho^{\mathfrak{C}}_j}$. In particular, their linear combination $\Phi^m_k := \sum_{\ell=1}^{M_k} \psi^\ell_{1,k} \otimes \cdots \otimes \psi^\ell_{m,k} \otimes \rho^{\mathfrak{C}}_{m+1} \otimes \rho^{\mathfrak{C}}_{m+2} \otimes \cdots$ is an element of $\mathscr{H}_{\otimes_{j\in\mathbb{N}}\rho^{\mathfrak{C}}_j}$. Then, using that the sum in a vector space is continuous,

$$\Psi_k \overset{\text{(B.58)}}{=} \lim_{m\to\infty} \sum_{\ell=1}^{M_k} \psi^\ell_{1,k} \otimes \cdots \otimes \psi^\ell_{m,k} \otimes \rho^{\mathfrak{C}}_{m+1} \otimes \rho^{\mathfrak{C}}_{m+2} \otimes \cdots \overset{\text{(by def.)}}{=} \lim_{m\to\infty} \Phi^m_k. \tag{B.60}$$

$$\hat{a}(f)\Psi_k \overset{\text{(B.59)}}{=} \lim_{m\to\infty} \sum_{\ell=1}^{M} \hat{a}(f)\Big(\psi^\ell_{1,k} \otimes \cdots \otimes \psi^\ell_{m,k} \otimes \rho^{\mathfrak{C}}_{m+1} \otimes \cdots\Big) \overset{(\hat{a}(f)\text{ is linear})}{=}$$
$$= \lim_{m\to\infty} \hat{a}(f)\Big(\sum_{\ell=1}^{M} \psi^\ell_{1,k} \otimes \cdots \otimes \psi^\ell_{m,k} \otimes \rho^{\mathfrak{C}}_{m+1} \otimes \cdots\Big) = \lim_{m\to\infty} \hat{a}(f)\Phi^m_k. \tag{B.61}$$

To finalize the proof, we just need a diagonal sequence argument. By equations (B.60) and (B.61), for any $\varepsilon > 0$ there exist $N_1^\varepsilon,\ N_2^\varepsilon \in \mathbb{N}$ such that $\|\Psi_k - \Phi_k^m\| \leq \varepsilon\ \forall m \geq N_1^\varepsilon$ and $\|\hat{a}(f)\Psi_k - \hat{a}(f)\Phi_k^m\| \leq \varepsilon\ \forall m \geq N_2^\varepsilon$. Then, taking for each $k \in \mathbb{N}$, $N_{1/k} := \max\{N_1^{1/k},\ N_2^{1/k}\}$, the sequence $(\Phi_k^{N_{1/k}})_{k\in\mathbb{N}} \subseteq \operatorname{span}(H)$ satisfies:

$$\left\|\Psi - \Phi_k^{N_{1/k}}\right\| \leq \|\Psi - \Psi_k\| + \left\|\Psi_k - \Phi_k^{N_{1/k}}\right\| \leq \|\Psi - \Psi_k\| + \frac{1}{k} \xrightarrow[k\to\infty]{} 0 \qquad \text{and}$$

$$\left\|\hat{a}(f)\Psi - \hat{a}(f)\Phi_k^{N_{1/k}}\right\| \leq \|\hat{a}(f)\Psi - \hat{a}(f)\Psi_k\| + \left\|\hat{a}(f)\Psi_k - \hat{a}(f)\Phi_k^{N_{1/k}}\right\| \leq \|\hat{a}(f)\Psi - \hat{a}(f)\Psi_k\| + \frac{1}{k} \xrightarrow[k\to\infty]{} 0.$$

Therefore, $\Psi \in D\Big(\overline{\hat{a}(f) \restriction_{\operatorname{span}(H)}}\Big)$ and $\overline{\hat{a}(f) \restriction_{\operatorname{span}(H)}}\Psi = \hat{a}(f)\Psi$, i.e., $\hat{a}(f)|_{\mathfrak{C}} \subseteq \overline{\hat{a}(f)|_{\mathfrak{C}} \restriction_{\operatorname{span}(H)}}$. Since $\supseteq$ holds trivially,[a] $\operatorname{span}(H)$ (so, by the claims above, $\mathscr{H}_{\otimes_{j\in\mathbb{N}}\rho_j^{\mathfrak{C}}}$) is a core of $\hat{a}(f)|_{\mathfrak{C}}$ $\boldsymbol{o.\varepsilon.\delta.}$

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[a] Every $\psi \in \operatorname{span}(H)$ is obviously in $\otimes_{j\in\mathbb{N}}^{\mathscr{F},\mathfrak{C}} B_{Her}$ which is a core of $\hat{a}(f)|_{\mathfrak{C}}$ by definition.

Definition 60. Define the unit vector $\phi_0(x) := \pi^{-1/4}\ e^{-x^2/2}$ —it is the "ground state of the harmonic oscillator" (see Lemma 43). The elementary tensor product

$$\Psi_G := \phi_0 \otimes \phi_0 \otimes \cdots$$

will be called *"the ground state of infinitely many oscillators"*. We will use $\mathfrak{C}_G \in \Gamma$ to denote the class of equivalence of $(\phi_0)_{j\in\mathbb{N}}$. Lastly, for an arbitrary $\mathfrak{C} \in \Gamma$, we will denote $R(\mathfrak{C}) := \Big(\otimes_{k\in\mathbb{N}}^{\mathfrak{C}} L^2(\mathbb{R}, dx),\ \otimes_{k\in\mathbb{N}}^{\mathscr{F},\mathfrak{C}} B_{Her},\ \{\ \hat{a}(f)|_{\mathfrak{C}}\ \}_{f\in\mathcal{W}_{\mathbb{C}}}\Big)$ (the creation-annihilation CCR irrep of layer $\mathfrak{C}$ as in Thm. 35).

♦

Lemma 60. Ψ_G is an *empty vacuum* of the CCR representation $R(\mathfrak{C}_G)$.

♦

Proof: • Ψ_G is a quantum vacuum by Prop. 85.

• We only miss to prove emptiness. Let $(f, g) \equiv f + ig \in \mathcal{W}_{\mathbb{C}}$ arbitrary. Then $\exists N \in \mathbb{N} :\ f = \sum_{n=1}^N \langle e_n, f\rangle e_n,\ g = \sum_{n=1}^N \langle e_n, g\rangle e_n$. By Prop. 79 and the conjugate linearity of an annihilation operator (see Def. 48.(ii)), for any $\Psi := \otimes_{j\in\mathbb{N}}\psi_j \in \otimes_{j\in\mathbb{N}}^{\mathscr{F}} B_{Her}$,

$$\hat{a}(f + ig)\Psi = \sum_{n=1}^N \underbrace{(\langle e_n, f\rangle - i\langle e_n, g\rangle)}_{=:d_n^{(f,g)}}\ \hat{a}(e_n)\Psi = \sum_{n=1}^N d_n^{(f,g)}\ \underbrace{\psi_1 \otimes \cdots \otimes (\hat{a}\psi_n)}_{n} \otimes \psi_{n+1} \otimes \cdots .$$

Then, plugging $\Psi = \Psi_G$ and using that $\hat{a}\phi_0 = 0$ (see Lemma 43), we get immediately that $\hat{a}(f + ig)\Big(\Psi_G\Big) = 0$. $\boldsymbol{o.\varepsilon.\delta.}$

B.6.3 Conclusion: Most ITP Layers Convey "Myriotic" Vacuums

Theorem 40. In the notation of Definition 60, every $R(\mathfrak{C})$ is a *cyclic irreducible representation of the CCR* for which *any* layer-generator $\otimes_{j\in\mathbb{N}}\rho_j^{\mathfrak{C}} \in \otimes_{j\in\mathbb{N}}^{\mathscr{F}} B_{Her}$ is a *quantum vacuum.* Moreover,

$$\left(\begin{array}{c} \otimes_{j\in\mathbb{N}}^{\mathfrak{C}} L^2(\mathbb{R}, dx) \text{ has } \textit{at least one} \\ \text{layer-generator that is an} \\ \textit{empty} \text{ quantum vacuum of } R(\mathfrak{C}) \end{array}\right) \iff \left(\mathfrak{C} \overset{q}{\approx} \mathfrak{C}_G\right) \iff \left(\begin{array}{c} R(\mathfrak{C}) \text{ is } \textit{equivalent} \text{ to the} \\ \text{Fock representation} \\ \Big(\mathcal{F}_b(\mathcal{H}),\ \mathcal{F}_b^{fin}(\mathcal{W}_{\mathbb{C}}),\ \{A(f)\}_{f\in\mathcal{W}_{\mathbb{C}}}\Big) \end{array}\right).$$

In particular, the equivalence with the Fock representation maps the *empty vacuum of the ITP to the Fock vacuum.* ♦

Proof: • We name the statements as L (left one), M (middle one) and R (right one).

$(L) \implies (R)$: By Theorem 39, if a layer $\otimes^{\mathfrak{C}}_{j\in\mathbb{N}} L^2(\mathbb{R}, dx)$, $\mathfrak{C} \in \Gamma$, has a layer-generator that is an empty vacuum, then $R(\mathfrak{C})$ is equivalent to the Fock representation $\Big(\mathcal{F}_b(\mathcal{H}), \mathcal{F}_b^{fin}(\mathcal{W}_\mathbb{C}), \{A(f)\}_{f\in\mathcal{W}_\mathbb{C}} \Big)$.

$(M) \iff (R)$: By Lemma 60, $\mathfrak{C}_G$ has an empty vacuum that is a layer-generator, so by the above it is equivalent to the Fock representation. But by Theorem 35, for some $\mathfrak{C} \in \Gamma$, $R(\mathfrak{C})$ is equivalent to $R(\mathfrak{C}_G)$, and hence to the Fock representation, *if and only if* $\mathfrak{C} \overset{q}{\approx} \mathfrak{C}_G$.

$(L) \impliedby (M)$: If $\mathfrak{C} \overset{q}{\approx} \mathfrak{C}_G$, as we found in the proof of Theorem 35, the unitary $U : \otimes^{\mathfrak{C}_G}_{j\in\mathbb{N}} L^2(\mathbb{R}, dx) \to \otimes^{\mathfrak{C}}_{j\in\mathbb{N}} L^2(\mathbb{R}, dx)$ giving the equivalence of $R(\mathfrak{C})$ and $R(\mathfrak{C}_G)$, can be chosen such that if $\otimes_{j\in\mathbb{N}}\psi_j \in \otimes^{\mathfrak{C}_G}_{j\in\mathbb{N}} L^2(\mathbb{R}, dx)$, then $U(\otimes_{j\in\mathbb{N}}\psi_j) = \otimes_{j\in\mathbb{N}}(e^{i\theta_j}\psi_j)$ for some constants $(\theta_j)_{j\in\mathbb{N}} \subseteq [-\pi, \pi)$. In particular, $U(\phi_0 \otimes \phi_0 \otimes \cdots) = (e^{i\theta_1}\phi_0) \otimes (e^{i\theta_2}\phi_0) \otimes \cdots$. Since all its factors are unit norm elements of B_{Her}, it is still a layer-generator in $\otimes^{\mathscr{F}}_{j\in\mathbb{N}} B_{Her}$ (now for $\mathfrak{C}$). Thus, by Prop. 85, it is a quantum vacuum (of $R(\mathfrak{C})$). Finally, for an arbitrary $f \in \mathcal{W}$, using in $(\star)$ that by Lemma 60, $\Psi_G = \phi_0 \otimes \phi_0 \otimes \cdots$ is empty for $R(\mathfrak{C}_G)$,

$$\hat{a}(f)|_{\mathfrak{C}}\Big((e^{i\theta_1}\phi_0) \otimes (e^{i\theta_2}\phi_0) \otimes \cdots\Big) = U\hat{a}(f)|_{\mathfrak{C}_G}U^{-1}\Big((e^{i\theta_1}\phi_0) \otimes (e^{i\theta_2}\phi_0) \otimes \cdots\Big) = U\hat{a}(f)|_{\mathfrak{C}_G}(\phi_0 \otimes \phi_0 \otimes \cdots) \overset{(\star)}{=} 0.$$

Therefore, $(e^{i\theta_1}\phi_0) \otimes (e^{i\theta_2}\phi_0) \otimes \cdots$ is empty for $R(\mathfrak{C})$. ***ο.ε.δ.***

Corollary 33. In the setting of Theorem 40,

$$\begin{pmatrix} \textit{Every}\text{ quantum vacuum} \\ \text{of } R(\mathfrak{C}) \text{ is } \textit{myriotic} \end{pmatrix} \iff \left(\mathfrak{C} \overset{q}{\not\approx} \mathfrak{C}_G \right) \iff \begin{pmatrix} R(\mathfrak{C}) \text{ is } \textit{inequivalent} \text{ to the} \\ \text{Fock representation} \\ \Big(\mathcal{F}_b(\mathcal{H}),\ \mathcal{F}_b^{fin}(\mathcal{W}_\mathbb{C}),\ \{A(f)\}_{f\in\mathcal{W}_\mathbb{C}} \Big) \end{pmatrix}.$$

♦

Proof: One just needs to blend the negation of Theorem 40 with Corollary 32. ☐

With all, any layer that has no generator close to the product of harmonic oscillator's ground state has *only* myriotic vacuums. Let us understand why this perfectly fits the physics behind.

Given $(\phi_n)_{n\in\mathbb{N}} \subseteq L^2(\mathbb{R}, dx)$ is the ONB of the harmonic oscillator (see Lemma 43), consider the elementary tensor product $\phi_{n_1} \otimes \phi_{n_2} \otimes \cdots$ for some $n_j \in \mathbb{N}_0$, $j \in \mathbb{N}$. All its factors have unit norm, so, it is a generator of its layer —which we denote by $\mathfrak{C}_{n_1,n_2,\ldots} \in \Gamma$. Following the terminology introduced in §1.3, $\phi_{n_1} \otimes \phi_{n_2} \otimes \cdots$ describes n_1 bosons of mode 1, n_2 bosons of mode 2 etc. In particular, the operators $\hat{a}(e_j)$ and $\hat{a}(e_j)^*$ act by adding and subtracting these bosons from the j-th mode. Thus, we are exactly in the situation of §1.3, now rigorously generalized to infinitely many modes. In fact, this gives a precise physical meaning to a "layer of the ITP": by Theorem 85, the closure of the space that is generated from $\phi_{n_1} \otimes \phi_{n_2} \otimes \cdots$ by adding and subtracting finitely many bosons is *exactly* the layer of the ITP generated by $\phi_{n_1} \otimes \phi_{n_2} \otimes \cdots$! So, each layer[13] is the space describing bounded perturbations in the number of bosons around some reference arrangement.

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[13] Technically, by Prop. 55, there are more layers than those generated by states like $\phi_{n_1} \otimes \phi_{n_2} \otimes \cdots$, $n_j \in \mathbb{N}_0$. However, by Prop. 82, every layer has a generating vector $\varphi_1 \otimes \varphi_2 \otimes \cdots$ with $\varphi_j \in \operatorname{span}(\phi_n)_{n\in\mathbb{N}}$. Since the (unfortunately hand-wavy) "boson jargon" allows one to say that $c_1\phi_0 + \cdots + c_n\phi_n$ represents a state possessing between zero and n bosons, our arguments remain unaltered after the technicality.

Importantly, note that the ITP poses no restriction on the number of (virtual) particles that can be described, e.g., $\phi_1 \otimes \phi_1 \otimes \cdots$ has 1 boson of each of the infinitely many modes. Interestingly, whenever there is an infinite difference of the number of bosons held by two states, the layers[14] they generate will be different, i.e.,

$$\mathfrak{C}_{n_1,n_2,\ldots} \neq \mathfrak{C}_{m_1,m_2,\ldots} \quad \textit{if and only if} \quad n_j \neq m_j \text{ for infinitely many } j \in \mathbb{N}.$$

The reason is obvious:

$$\sum_{j=1}^{\infty} |\langle \phi_{n_j},\ \phi_{m_j}\rangle - 1| \overset{(\phi_n \text{ is an ONB})}{=} \sum_{j\in\mathbb{N}:\ n_j \neq m_j} 1,$$

which, if finite, implies by definition $\mathfrak{C}_{n_1,n_2,..} = \mathfrak{C}_{m_1,m_2,\ldots}$ and otherwise, $\mathfrak{C}_{n_1,n_2,..} \neq \mathfrak{C}_{m_1,m_2,\ldots}$.

Now, as explained in Lemma 58, the Fock space allows the description of unboundedly many bosons, but *not* of infinitely many. It is this why one can always annihilate all the bosons of any state of the Fock space by applying finitely many annihilation operators. Likewise, it is possible to fully annihilate every state of the layer generated by $\Psi_G := \phi_0 \otimes \phi_0 \otimes \cdots$ (heuristically, because they are states forced to have a tail close to $\phi_0 \otimes \phi_0 \otimes \cdots$ and $\hat{a}\phi_0 = 0$). Ψ_G itself describes a state with zero bosons in every mode. That is why we called it an *empty* vacuum. On the other hand, a state $\phi_{n_1} \otimes \phi_{n_2} \otimes \cdots$ where $\sum_{j=1}^{\infty} n_j = +\infty$ will not be annihilable by finitely many applications of the annihilation operators. Simply because it contains "myriads" of bosons. That is why we called it a *myriotic* vacuum —term originally employed in a similar spirit by Friedrichs (1953). Note that such a state with infinitely many bosons should not be a problem within a field ontology because bosons are *virtual* particles there, namely, apparent but *fictitious* entities (not part of the primitive ontology).

Finally, one could ask: why call myriotic layer-generators "vacuums" if they are (by far) not empty? The reason is historical and is rooted in the particle interpretation of bosons. Myriotic vacuums play a similar role to the "Dirac sea" of fermionic QFTs —we will explain their relation in detail elsewhere. As explained in Chapter 7, the myriotic vacuums are the states to which the ("asymptotically free") empty vacuum evolves when an interaction dynamics (say, with an external field) generates infinitely many bosons. Since the myriotic state is the "new reference state" in the sense that it generates the layer that is important for those times in the evolution —analogous to the empty vacuum during free time evolution—, one still calls this reference state a "vacuum". It is remarkable that from this perspective, Haag's theorem is not mysterious at all: we have proven that all the layers describing infinitely many bosons (i.e., those layers with no empty vacuum available), give CCR representations inequivalent to the Fock space. If one insists in using only *irreducible* representations of the CCR to describe dynamics, one will feel the urge to restrict their state-space to a particular layer of the ITP (or an equivalent space, like the Fock space). But then, if interactions generate infinitely many (virtual) particles, one will be forced to dynamically change the state-space itself (namely, the layer). Only in this way will the interaction be "implementable". On the other hand, if we used a more "honest" space like the proper ITP, no such "mysterious" change would be necessary. All in all, the only mystery of the Haag theorem seems to be the (pathological) resolve of the mainstream quantum community to make things incomprehensible, when in reality they are reasonably explainable.

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[14] Or what is more relevant: their *quasi*-layers will also be different.

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NOTATION

ACRONYMS AND ABBREVIATIONS

ITP	=	infinite tensor product	**wrt**	=	with respect to
QM	=	quantum mechanics	**iff**	=	if and only if
QFT	=	quantum field theory	**sym.**	=	symmetric
EM	=	electromagnetic	**conj.**	=	conjugate
ONB	=	orthonormal basis	**mont. conv.**	=	monotone convergence
PWT	=	pilot-wave theory	**conv.**	=	convergence
irrep	=	irreducible representation	**op.**	=	operator
l.h.s.	=	left-hand side	**by def.**	=	by definition
r.h.s.	=	right-hand side	**Prop.**	=	Proposition
s.th.	=	such that	**Thm.**	=	Theorem
triang. ineq.	=	triangle inequality	**Cor.**	=	Corollary
conts.	=	continuous	**Lem.**	=	Lemma
a.e.	=	almost everywhere	**isom**	=	isometry
ess sup	=	essential supremum	**mut. a.c.**	=	mutually absolutely conts.

CONVENTIONS

- Quotation marks “ · ” around a mathematical symbol mean that we only mean it heuristically —possibly having provided no proper definition.
- In this work, every Hilbert space has *arbitrary Hilbert dimension* unless the contrary is explicitly stated.
- The index set I has an arbitrary cardinality unless the contrary is explicitly stated.
- The d in front of a measure, such as $d\mu$ has no special meaning. That is, $d\mu \equiv \mu$.
- *Countable* means *countable infinity*, excluding the finite case.
- The *inner products* $\langle \cdot, \cdot \rangle$ are conjugate-linear in the first slot and linear in the second slot.
- The context will make clear which is the space where an inner product $\langle \cdot, \cdot \rangle$ acts. Likewise for a norm $\|\cdot\|$.
- The gray boxes are either well-known results and/or not essential for the reading of the main text.
- All, complex conjugation and the topological closure of a set or a closable operator will be denoted by an “overline”. The context will make clear which one we mean.

MATHEMATICAL SYMBOLS

Let X, X_j (with $j \in I$) denote arbitrary sets. Let Σ, Σ_j denote the σ-algebras of X, X_j respectively and $d\mu, d\mu_j$ some measures on them.

- I is an arbitrary set unless otherwise specified.
- J is a finite subset of I unless otherwise specified.
- $\mathcal{H}$ and $\mathcal{H}_j$ are Hilbert spaces of arbitrary Hilbert dimension unless their dimension is explicitly specified.
- Id denotes the identity operator.
- $\mathbb{1}_B$ for some $B \subseteq X$ denotes the indicator function of B, namely, $\mathbb{1}_B(x) := 1$ if $x \in B$ and 0 otherwise.
- $\mathcal{P}(X)$ is the power set of X (i.e., the family of all subsets of X).
- $\sigma(\mathcal{A})$ with $\mathcal{A} \subseteq \mathcal{P}(X)$ is the σ-algebra generated by $\mathcal{A}$ (i.e., the smallest σ-algebra on X containing all sets in $\mathcal{A}$).
- $\mathfrak{B}(X)$ is the Borel σ-algebra of the topology assumed for X.
- $\overline{A}$ for a set $A \subseteq X$ denotes the topological closure of A in the topology assumed for X.
- A^c denotes the complement of A in its obvious ambient space.
- $\prod_{j\in I} X_j$ is the Cartesian product of Def. 2.
- $\mathbb{R}^\infty := \prod_{j\in\mathbb{N}} \mathbb{R}$. If a topology is not specified, it is the product topology (Def. 6).
- π_k is the projection to the k-th coordinate of a product set, as in Def. 2.
- $\pi_{J\leftarrow I}$ is the partial projection map of Def. 8.
- $E \times \prod_{j\in I\setminus J} X_j$ for $E \in \odot_{j\in J}\Sigma_j$ denotes $\pi^{-1}_{J\leftarrow I}(E)$ as in Def. 8.
- $d\mu_j \sim d\mu_k$, $d\mu_j \ll d\mu_k$ and $d\mu_j \perp d\mu_k$ are given in Def. 10.
- $\frac{d\mu_j}{d\mu_k}$ is the Radon-Nikodym derivative of $d\mu_j, d\mu_k$. See Thm. 4.
- $\odot_{j\in I}\Sigma_j$ is the product σ-algebra of $\{\Sigma_j\}_{j\in I}$. See Def. 7.
- $\odot_{j\in I} d\mu_j$ is the product measure of $\{d\mu_j\}_{j\in I}$. For a finite I see Def. 2, for infinite I see Cor. 5.
- $d^\infty\mu$ denotes the infinite product measure of $\{d\mu_j\}_{j\in I}$ as in Cor. 5.
- $\mathbb{P}$("statement") means the probability of "statement".
- $\mathfrak{c}_0$ is the set of finite cylinders of measurable sets defined in Def. 7.
- C_{meas} is the Boolean algebra of Lem. 18.
- $\mathfrak{G}_0$ is the set of finite cylinders of open sets defined in Def. 6.
- $\mathfrak{A}_0$ is the Boolean algebra of Def. 6.
- $\sum_{j\in I}$ and $\prod_{j\in I}$, unless otherwise stated, are to be understood as in Definitions 30 and 6, respectively.
- $\mathcal{L}(V, W)$ with V, W normed vector space, is the space of bounded linear operators from V to W.
- $\mathcal{L}(\mathcal{H}) := \mathcal{L}(\mathcal{H}, \mathcal{H})$.
- $\|A\|_{op}$ for a linear operator $A : V \to W$ denotes the operator norm of A.
- $L^p(X, \mathbb{K}, d\mu)$ with $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ and $p \in [1, +\infty)$ denotes the space of functions $f : X \to \mathbb{K}$ such that $\|f\|_{L^p} := (\int_{x\in X} |f|^p(x) d\mu)^{1/p} < +\infty$, that are identified when they differ in a $d\mu$-null set.
- $\ell^p(I, \mathbb{K})$ for $p \in [1, \infty)$ is the space $L^p(I, \mathbb{K}, d\nu)$ where $d\nu$ is the counting measure on I (discrete σ-algebra).

- $\widetilde{\mathcal{L}}$ denotes the conjugate multilinear forms of Definitions 12 and 38.
- $\mathscr{C}$ are the $\mathscr{C}$-sequences of Definitions 34 and 11.
- $\mathscr{C}_0$ are the $\mathscr{C}_0$-sequences of Definitions 36 and 13.
- $\approx$ is the equivalence relation of Theorems 23 and 9.
- Γ is the set of $\approx$-equivalence classes as in Theorems 23 and 9.
- $\mathfrak{C}$ always denotes an element of Γ.
- $\overset{q}{\approx}$ is the equivalence relation of Prop. 57.
- Γ^q is the set of $\overset{q}{\approx}$-equivalence classes as in Prop. 57
- $\mathfrak{C}^q$ always denotes an element of Γ^q.
- $\bigotimes_{j\in I} \mathcal{H}_j$ denotes the proper ITP as in Definitions 38 and 12.
- $\bigotimes^{\mathfrak{C}}_{j\in I} \mathcal{H}_j$ denotes the improper ITP as in Definitions 39 and 14.
- $v \perp u$ for two vectors u, v of an inner product vector space means they are orthogonal.
- $\mathcal{H}_j \perp \mathcal{H}_k$ for two subspaces of $\mathcal{H}$ mean all their vectors are pairwise orthogonal.
- $\bigotimes_{j\in I} \rho_j^{\mathfrak{C}}$ and $\bigotimes_{j\in I} \eta_j^{\mathfrak{C}}$ always denote a generator of the $\mathfrak{C}$-th layer of the ITP, unless otherwise stated. See Theorems 25 and 9.
- $W_{\mathfrak{C}}$ and $W_{[\rho^{\mathfrak{C}}]}$ denote the unitary operator of the $\mathfrak{C}$-th layer, generated by $\bigotimes_{j\in I} \rho_j^{\mathfrak{C}}$ as in Thm. 10.
- $\mathfrak{R}$ denotes a representation basis as in Thm. 13.
- $\mathscr{W}_{\mathfrak{R}}$ denotes the unitary of Thm. 13.
- $(\hat{x}_k,\ D(\hat{x}_k))$ denotes the operators of Def. 19.
- $(\widehat{q}_k,\ D(\widehat{q}_k))$ denotes the operators defined in Cor. 17.

INDEX OF RESULTS

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