Rigorous Schrödinger Quantum Mechanics of Countably Many Degrees of Freedom

Quest: "$\lim_{n\to\infty} L^2(\mathbb{R}^n, d^n x)$"?

— Master's Thesis —

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1. PHYSICAL MOTIVATION : FIELD ONTOLOGY

2. THE DIFFICULTIES with "$d^\infty x$"

3. $d^n x$ WAS REPLACEABLE for QM

4\.
5\. REALIZE $\lim_{n \to \infty} L^2(\mathbb{R}^n, d^n x)$ as $L^2(\mathbb{R}^{|\mathbb{N}|}, d\mu)$?
as $\bigotimes_{k \in \mathbb{N}} L^2(\mathbb{R}, dx)$?

6. MIXING THE TWO $\Rightarrow$ TENTATIVE BORN RULE

7. THE JOINT SPECTRAL DIAGONALIZATION of ALL the POSITION OPERATORS IN $\bigotimes_{k \in \mathbb{N}} L^2(\mathbb{R}, dx)$

8. APPLICATIONS

9. CCR, FOCK SPACE etc. . .

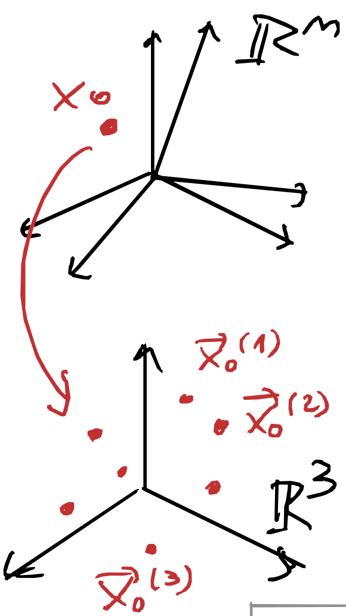
1. MOTIVATION

System with $m \in \mathbb{N}$ degrees of freedom

- Model config. space:

$$\mathbb{R}^m$$

e.g. N particles in 3D
$m = 3N$



- Wavefunction space:

$$L^2(\mathbb{R}^m, d^m x)$$

- Hamiltonian & SCOPUG:

$$(H, D(H)) \qquad u_t := e^{-iHt}$$

- **Ontology: Initial Condits.**

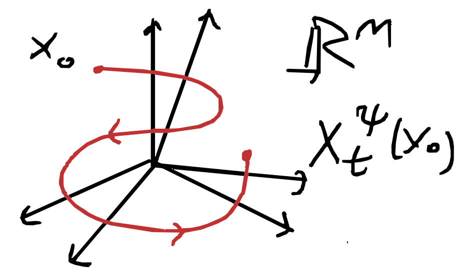
$$\psi_0 \in L^2(\mathbb{R}^n; d^n x) \qquad\qquad x_0 \in \mathbb{R}^n$$

(i) **Pilot Wave:**

$$t \in \mathbb{R} \longmapsto \psi_t := U_t \psi_0$$

(ii) **Trajectory of system:**

$$t \in \mathbb{R} \longmapsto X_t^\psi(x_0) \in \mathbb{R}^n$$



- **Constraint:** X^ψ is $|\psi|^2 d^n x$ -Equivariant

- **Corollary:** BORN RULE on $\mathbb{R}^n$.

- What if system has $n = |\mathbb{N}|$ dofs? 1. [redacted]
- Model config. space:

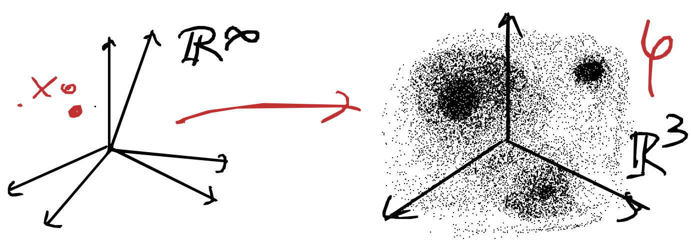
$$\mathbb{R}^\infty := \mathbb{R} \times \mathbb{R} \times \cdots \quad \text{(product topology)}$$

Ex.:

$$\varphi \in H^2(\mathbb{R}^3, \mathbb{R}) \Rightarrow \text{if ONB } (\phi_n)_{n \in \mathbb{N}}$$

$$\Rightarrow \exists! \, (\alpha_n)_{n \in \mathbb{N}} \in \ell^2(\mathbb{N}, \mathbb{R}) \subseteq \mathbb{R}^\infty$$

s.th. $\varphi = \sum_{n=1}^{\infty} \alpha_n \phi_n$



- Hindrance to copy strategy:

What are "$L^2(\mathbb{R}^\infty, d^\infty x)$" & "$|\psi_t|^2 d^\infty x$"?

2. INCONVENIENCES of "$d^\infty x$"

Ⓐ If understood as product measure

COR.1 & LEM.4: $\exists$ measure $d^\infty x$ on $\mathcal{B}(\mathbb{R}^\infty)$ s.th.

$$\forall m \in \mathbb{N}, \ \forall E_j \in \mathcal{B}(\mathbb{R})$$

$$d^\infty x(E_1 \times \cdots \times E_m \times \mathbb{R} \times \cdots) = \prod_{j=1}^{m} dx(E_j) \cdot \prod_{j=m+1}^{\infty} dx(\mathbb{R}).$$

- BUT any measure $d^\infty y$ with that prop. is s.th

$$d^\infty y(B) = +\infty \quad \forall B \text{ OPEN set of } \mathbb{R}^\infty.$$

B If using translation invariance   2.[redacted]

PROP. 5: For each $\mathbb{R}^m$, $m \in \mathbb{N}$, $\exists!\, \mu$:

(i) TRANSLAT. INVAR.: $\mu(B) = \mu(B+x)$ $\forall x \in \mathbb{R}^m$ $\forall B \in \mathcal{B}(\mathbb{R}^m)$

(ii) $\mu([0,1)^m) = 1$.

$\Longrightarrow$ the LEBESGUE MEASURE.

(OXTOBY 1946)

COR. 3: $\forall$ translation-invariant measure μ on $\mathcal{B}(\mathbb{R}^\infty)$, μ is NOT σ-FINITE &

$$\mu(B) = +\infty \quad \forall B \text{ OPEN set of } \mathbb{R}^\infty.$$

3. HOLD ON, $d^m x$ WAS REPLACEABLE!

BORN RULE & **EQUIVARIANCE** $\qquad |\psi|^2 d^m x$

(Experimental predictions) $\qquad$ (Ontological predictions)

- What if there is an equally "flexible" background measure?

PROP. 8: (X, Σ) msble sp. & $d\mu, d\nu$ σ-finite.

$$\left\{ |\psi|^2 d\mu \right\}_{\psi \in L^2(X, d\mu)} = \left\{ |\phi|^2 d\nu \right\}_{\phi \in L^2(X, d\nu)} \iff d\mu \sim d\nu$$

(mutually abs. conts.)

THM. 5: (X, Σ) & $d\mu, d\nu$ σ-finite.

$d\mu \sim d\nu$ (mutually abs. conts.) $\iff$ $\left\{ \begin{array}{l} \exists\, W : L^2(X, d\mu) \longrightarrow L^2(X, d\nu) \\ \quad \text{isomorphism:} \\ |\psi|^2 d\nu = |W\psi|^2 d\mu \quad \forall \psi \in L^2(X, d\nu) \end{array} \right.$

- W is unitary & "unique" (up to "gauge phase")

$$(W\psi)(x) = \frac{\psi(x)}{\rho(x)} e^{i\theta(x)} \qquad \left(\theta : X \to \mathbb{R} \text{ any measurable fct.} \right)$$

ρ^2 is the Rad-Nyk-der. $d\mu = \rho^2 \cdot d\nu$

$\Rightarrow \quad d^m\mu \sim d^m x$ iff

$\exists$ identification $\quad W: L^2(\mathbb{R}^m, d^m x) \longrightarrow L^2(\mathbb{R}^m, d^m\mu)$

$$\psi \longleftrightarrow \tilde{\psi} := W\psi$$

s.th.

(i) same empirical predictions (Born rule)

$$\mathbb{P}\begin{pmatrix} x \in B \\ \text{if } \psi, d^m x \end{pmatrix} = \int_{x\in B} |\psi|^2(x)\, d^m x = \int_{x\in B} |\tilde{\psi}|^2(x)\, d^m\mu = \mathbb{P}\begin{pmatrix} x \in B \\ \text{if } \tilde{\psi}, d^m\mu \end{pmatrix}$$

(ii) same ontological predictions (equivariance constraint)

$$X^\psi : \quad |\psi|^2 d^m x\text{-EQUIVAR.} \longleftrightarrow |\tilde{\psi}|^2 d^m\mu\text{-EQUIVAR.}$$

(iii) same dynamics for $\psi, \tilde{\psi}$ (a Schrödinger eqt.)

- " $\lim_{m\to\infty} d^m x$ " $\begin{cases} \text{Not } \sigma\text{-finite} \\ \text{Not unique} \end{cases}$ ...

- But IF $(d\mu_j)_{j\in\mathbb{N}}$ PROBAB. meas. on $\mathcal{B}(\mathbb{R})$

$\Rightarrow$ $\exists!$ product meas. $d^\infty\mu := \bigodot_{j\in\mathbb{N}} d\mu_j$.

& $d^\infty\mu$ is PROB. MEAS! (Kolmogorov Extension Thm.)

LEM. 9: $L^2(\mathbb{R}^\infty, d^\infty\mu)$ is SEPARABLE HILB. SP.

(Brezis 2011)

4. UNDERSTAND "$\lim_{n\to\infty} L^2(\mathbb{R}^n, d^n x)$" as $L^2(\mathbb{R}^\infty, d^\infty\mu)$?

Cor-7: Let $\{d\mu_j, d\nu_j\}_{j\in\mathbb{N}}$ prob. meas. on $\mathcal{B}(\mathbb{R})$ s.th. $dx \sim d\mu_j \sim d\nu_j$ $\forall j$. Then

$$\begin{tikzcd}
L^2(\mathbb{R}^n, d^n x) \arrow[d] \arrow[r, phantom, "\sim"] \arrow[dr, phantom, "\text{``}n\to\infty\text{''}"] & L^2(\mathbb{R}^n, d^n\mu) \arrow[d] \arrow[r, phantom, "\sim"] \arrow[dr, phantom, "\text{``}n\to\infty\text{''}"] & L^2(\mathbb{R}^n, d^n\nu) \arrow[d] \\
? & L^2(\mathbb{R}^\infty, d^\infty\mu) \arrow[r, phantom, "\overset{?}{\sim}"] & L^2(\mathbb{R}^\infty, d^\infty\nu)
\end{tikzcd}$$

(Kolmog. 2 series & Law of LN)

$\exists$ cases: $d^\infty\mu \not\sim d^\infty\nu$ ...

- So, limit of measure NOT nice.

$\exists$ Inequivalent choices! $\Rightarrow$ Next Strategy!

LEM. 11: $\exists!\ \mathcal{U} : \bigotimes_{j=1}^{m} L^2(\mathbb{R}, dx) \longrightarrow L^2(\mathbb{R}^m, d^m x)$

unitary s.th. $f_1 \otimes \cdots \otimes f_m \longmapsto \left[(x_1, \ldots, x_m) \longmapsto f_1(x_1) \cdots f_m(x_m)\right]$

$$\bigotimes_{j=1}^{m} L^2(\mathbb{R}, dx) \sim L^2(\mathbb{R}^m, d^m x).$$

"$m \longrightarrow \infty$"

$\exists!\ \bigotimes_{j=1}^{\infty} L^2(\mathbb{R}, dx)$ $\qquad$ $L^2(\mathbb{R}^\infty, d\mu_1) \ \ldots \ L^2(\mathbb{R}^\infty, d\mu_k)$

(von Neumann 39)

5. UNDERSTAND "$\lim_{n\to\infty} L^2(\mathbb{R}^n, d^n x)$" as $\bigotimes_{j\in\mathbb{N}} L^2(\mathbb{R}, dx)$?

- We want "$\|\psi_1 \otimes \psi_2 \otimes \cdots\|$" $= \prod_{j\in\mathbb{N}} \|\psi_j\|$

$\Rightarrow$ $$\mathcal{C} := \left\{ (\psi_j)_{j\in\mathbb{N}} \in \prod_{j\in\mathbb{N}} L^2(\mathbb{R}, dx) \;\middle|\; \prod_{j\in\mathbb{N}} \|\psi_j\| < +\infty \right\}$$

(v. Neumann 1939)

DEF. 12: $$\overline{\mathcal{L}_{\mathcal{C}}} := \left\{ \Phi : \mathcal{C} \to \mathbb{C} \;\middle|\; \Phi \text{ conjugate MULTILINEAR} \right\}$$

$\boxed{1}$ $(\psi_j)_{j\in\mathbb{N}} \in \mathcal{C} \Rightarrow \bigotimes_{j\in\mathbb{N}} \psi_j \in \overline{\mathcal{L}_{\mathcal{C}}}$ s.th.

$$\left(\bigotimes_{j\in\mathbb{N}} \psi_j\right)(\phi_1, \phi_2, \ldots) := \begin{cases} \prod_{j\in\mathbb{N}} \langle \psi_j, \phi_j \rangle & \text{if convergent} \\ 0 & \text{else} \end{cases}$$

$\hookrightarrow$ ELEMENTARY TENSORS

$\boxed{2}$ $V := \mathrm{span}\left\{ \bigotimes_{j\in\mathbb{N}} \psi_j \;\middle|\; (\psi_j)_{j\in\mathbb{N}} \in \mathcal{C} \right\} \subseteq \overline{\mathcal{L}}$

$\boxed{3}$ $\langle \cdot , \cdot \rangle : V \times V \longrightarrow \mathbb{C}$ SESQUILINEAR form:

$$\left\langle \bigotimes_{j\in\mathbb{N}} \psi_j , \bigotimes_{j\in\mathbb{N}} \phi_j \right\rangle := \begin{cases} \prod_{j\in\mathbb{N}} \langle \psi_j , \phi_j \rangle & \text{if } \exists \\ 0 & \text{else} \end{cases}$$

↳ $\langle \cdot , \cdot \rangle$ is inner pdct

↳ $\| \cdot \| := \sqrt{\langle \cdot , \cdot \rangle}$ NORM

$\boxed{4}$ $\bigotimes_{j\in\mathbb{N}} L^2(\mathbb{R}, dx) := \left\{ \begin{array}{l} \text{ptwise lims of} \\ \|\cdot\|\text{-Cauchy seqs. in } V \end{array} \right\} \Rightarrow$

$\boxed{4}$ $\Psi \in \bigotimes_{j\in\mathbb{N}} L^2(\mathbb{R}, dx)$ iff $\exists (\Psi_n)_{n\in\mathbb{N}} \subseteq V$:

CAUCHY

(i) $\lim_{\substack{n,m\to\infty \\ \text{unif.}}} \| \Psi_n - \Psi_m \| = 0$

POINTWISE LIMIT

(ii) $\lim_{n\to\infty} \Psi_n(f_1, f_2, \ldots) = \Psi(f_1, f_2, \ldots)$

$\boxed{5}$ Extend $\langle \cdot, \cdot \rangle : \bigotimes_{j\in\mathbb{N}} L^2(\mathbb{R}, dx) \times \bigotimes_{j\in\mathbb{N}} L^2(\mathbb{R}, dx) \longrightarrow \mathbb{C}$

$$\langle \Psi, \Phi \rangle := \lim_{n\to\infty} \langle \Psi_n, \Phi_n \rangle .$$

$\Longrightarrow$ $\left(\bigotimes_{j\in\mathbb{N}} L^2(\mathbb{R}, dx), \langle \cdot, \cdot \rangle \right)$ is a Hilbert space

the (PROPER) INFINITE TENSOR PRODUCT.

This Lim of $\otimes$ is unique:

(von Neumann 39)

THM. 24: If $\mathcal{H}$ is Hilbert sp. (ALTERNATIVE $\bigotimes_{n=1}^{\infty}$):

$\forall (\psi_j)_j, (\phi_j)_j \in \mathcal{C}$ can designate $\tilde{\bigotimes}_{j\in\mathbb{N}} \psi_j, \tilde{\bigotimes}_{j\in\mathbb{N}} \phi_j \in \mathcal{H}$:

(i) $$\left\langle \tilde{\bigotimes}_{j\in\mathbb{N}} \psi_j, \tilde{\bigotimes}_{j\in\mathbb{N}} \phi_j \right\rangle = \begin{cases} \prod_{j\in\mathbb{N}} \langle \psi_j, \phi_j \rangle & \text{if } \exists \\ 0 & \text{else} \end{cases}$$

(ii) $\operatorname{span}\left\{ \tilde{\bigotimes}_{j\in\mathbb{N}} \psi_j \right\}$ dense in $\mathcal{H}$

Then, $\exists!\ \mathcal{U} : \mathcal{H} \longrightarrow \bigotimes_{j\in\mathbb{N}} L^2(\mathbb{R}, dx)$ unitary

$$\tilde{\bigotimes}_{j\in\mathbb{N}} \psi_j \longleftrightarrow \bigotimes_{j\in\mathbb{N}} \psi_j .$$

A Convenient Dissection of ITP

REM.: $\bigotimes_{j\in\mathbb{N}} \psi_j , \ \bigotimes_{j\in\mathbb{N}} \phi_j \in \bigotimes_{j\in\mathbb{N}} L^2(\mathbb{R}, dx) \setminus \{\vec{0}\}$

$$
\begin{array}{ccccccccc}
\psi_1 & \otimes & \psi_2 & \otimes & \psi_3 & \otimes & \psi_4 & \otimes & \cdots \\
\updownarrow & & \updownarrow & & \updownarrow & & \updownarrow & & \langle \psi_j, \phi_j \rangle \longrightarrow 1 \\
\phi_1 & \otimes & \phi_2 & \otimes & \phi_3 & \otimes & \phi_4 & \otimes & \cdots
\end{array}
$$

A Convenient Dissection of ITP

REM. (vN39): $\bigotimes_{j\in\mathbb{N}}\psi_j,\ \bigotimes_{j\in\mathbb{N}}\phi_j \in \bigotimes_{j\in\mathbb{N}} L^2(\mathbb{R},dx)\setminus\{\vec{0}\}$

$$\left(\bigotimes_{j\in\mathbb{N}}\psi_j \approx \bigotimes_{j\in\mathbb{N}}\phi_j\right) :\Longleftrightarrow \left(\sum_{j\in\mathbb{N}} \left|\langle\psi_j,\phi_j\rangle - 1\right| < +\infty\right)$$

is EQUIVALENCE RELATION. → CLASS set Γ.

DEF. 14 (vN39): Given $\mathfrak{C}\in\Gamma$, sub-Hilbert space

$$\bigotimes_{j\in\mathbb{N}}^{\mathfrak{C}} L^2(\mathbb{R},dx) := \overline{\operatorname{span}\left\{\bigotimes_{j\in\mathbb{N}}\psi_j \in \mathfrak{C}\right\}}.$$

is the $\mathfrak{C}$-th LAYER of the ITP

THM. 9 (vN39):

(i) $\forall \mathfrak{C} \in \Gamma \;\; \exists \bigotimes_{j\in\mathbb{N}} \psi_j^\circ \in \mathfrak{C}: \quad \|\psi_j^\circ\| = 1 \;\; \forall j$

→ call it GENERATOR of $\mathfrak{C}$ because:

$$\text{span}\left\{\phi_1\otimes\cdots\otimes\phi_N\otimes\underbrace{\psi_{N+1}^\circ\otimes\psi_{N+2}^\circ\otimes\cdots}_{\text{Fixed tail}} \;\middle|\; \begin{array}{l} N\in\mathbb{N} \\ \phi_j\in L^2(\mathbb{R},dx)\end{array}\right\} = \bigotimes_{j\in\mathbb{N}}^{\mathfrak{C}} L^2(\mathbb{R},dx)$$

(States "finitely many changes" away fr. $\bigotimes_{j\in\mathbb{N}}\psi_j^\circ$)

(ii) Each $\bigotimes^{\mathfrak{C}}_{j\in\mathbb{N}} L^2(\mathbb{R},dx)$ is SEPARABLE.

• Obviously:

$$\bigotimes_{j\in\mathbb{N}} L^2(\mathbb{R}, dx) = \overline{\operatorname{span} \bigcup_{\mathfrak{C}\in\Gamma} \left(\bigotimes_{j\in\mathbb{N}}^{\mathfrak{C}} L^2(\mathbb{R}, dx) \right)}$$

LEM.: (vN39) $\mathfrak{C}, \mathfrak{D} \in \Gamma$: $\mathfrak{C} \neq \mathfrak{D} \Rightarrow \bigotimes_{j\in\mathbb{N}}^{\mathfrak{C}} L^2(\mathbb{R}, dx) \perp \bigotimes_{j\in\mathbb{N}}^{\mathfrak{D}} L^2(\mathbb{R}, dx)$

$$\Rightarrow \bigotimes_{j\in\mathbb{N}} L^2(\mathbb{R}, dx) \text{ "}=\text{" } \bigoplus_{\mathfrak{C}\in\Gamma} \left(\bigotimes_{j\in\mathbb{N}}^{\mathfrak{C}} L^2(\mathbb{R}, dx) \right) \text{ "} ?$$

• Well... PROP.: (vN39) $|\Gamma| > |\mathbb{N}|$ UNCOUNTABLE!

But: $\mathcal{H} = \overline{\operatorname{span} \bigcup_{j\in I} \mathcal{H}_j}$ & $\mathcal{H}_j \perp \mathcal{H}_k$ & $\dim(\mathcal{H}_j) > 1$

$\Rightarrow \mathcal{H}$ is NON-SEPARABLE, e.g. $\bigotimes_{k\in\mathbb{N}} L^2(\mathbb{R}, dx)$!

- Couldn't reference of UNCOUNTABLE DIRECT SUM in literat $\rightsquigarrow$ possibly cause always non-separable & quantum community "dislike"
- But possible: [PARENTHESIS]

DEF 14: I arbitr., V topol. vect. sp., $(v_j)_{j\in I} \subseteq V$

$$\left(\sum_{j\in I} v_j = v \right) :\Longleftrightarrow \left(\left(\sum_{j\in J} v_j\right) \xrightarrow[J\in\{\text{finite} \subseteq I\}]{\text{NET}} v \right)$$

DEF 15: I, $\{\mathcal{H}_j\}_{j\in I}$ arbitrary (unctble, non-sep...)

$$\bigoplus_{j\in I} \mathcal{H}_j := \left\{ (f_j)_{j\in I} \in \prod_{j\in I} \mathcal{H}_j \;\middle|\; \sum_{j\in I} \|f_j\|^2 < +\infty \right\}$$

LEM. 14: (Relaxing lemma)

$$(f_j)_{j\in I} \in \bigoplus_{j\in I} \mathcal{H}_j \implies f_j = 0 \ \forall j \in I \text{ except (at most) countably many}$$

PROP. 12, 13: (i) $\bigoplus_{j\in I} \mathcal{H}_j$ complex vect.-space str.

$$\lambda (f_j)_{j\in I} + \mu (g_j)_{j\in I} := (\lambda f_j + \mu g_j)_{j\in I}$$

(ii) $\langle \cdot , \cdot \rangle : \bigoplus_{j\in I} \mathcal{H}_j \times \bigoplus_{j\in I} \mathcal{H}_j \longrightarrow \mathbb{C}$

$$\left\langle (f_j)_{j\in I}, (g_j)_{j\in I} \right\rangle := \sum_{j\in I} \langle f_j, g_j \rangle$$

is INNER PRODUCT & CAUCHY COMPLETE TOP.

Prop. 15: $\{\mathcal{H}_j\}_{j\in I}$ closed subspaces of $\mathcal{H}$

$$\mathcal{H}_j \perp \mathcal{H}_k \ \& \ \mathcal{H} = \overline{\operatorname{span} \bigcup_{j\in I} \mathcal{H}_j} \Rightarrow \exists! \text{ isomorphism}$$

$$\mathcal{U} : \bigoplus_{j\in I} \mathcal{H}_j \longrightarrow \mathcal{H} \quad : \quad \& \text{ it is}$$

$$(f_j)_{j\in I} \longmapsto \sum_{j\in I} f_j \qquad \text{UNITARY.}$$

Prop. 17: I, $\{\mathcal{H}_j\}_{j\in I}$ arb., $\{f_j^m\}_{m\in E_j} \subseteq \mathcal{H}_j$ ONB

$$\phi_{j_0}^{m_0}(j) := \begin{cases} f_{j_0}^{m_0} & \text{if } j = j_0 \\ 0 & \text{if } j \neq j_0 \end{cases}$$

(Embed them with 0 in other $\mathcal{H}_k$ in $\bigoplus_{j\in I} \mathcal{H}_j$)

$\{\{\phi_{j_0}^{m_0}\}_{m_0\in E_j}\}_{j_0\in I}$ is ONB of $\bigoplus_{j\in I} \mathcal{H}_j$.

Prop. 18: $(A_j, D(A_j))$ on $\mathcal{H}_j$, $j \in I$

If $\bigoplus_{j\in I} D(A_j) := \left\{ (\psi_j)_{j\in I} \in \bigoplus_j \mathcal{H}_j \;\middle|\; \begin{array}{l} \psi_j \in D_j \\ \|(A_j \psi_j)_{j\in I}\| < \infty \end{array} \right\}$

then, $\left(\bigoplus_{j\in I} A_j\right)\left((\psi_j)_{j\in I}\right) := (A_j \psi_j)_{j\in I}$ is densly defd op.

(i) If (A_j, D_j) closed $\Rightarrow \bigoplus_{j\in I} A_j$ closed

(ii) If (A_j, D_j) self-adjoint $\Longrightarrow \bigoplus_{j\in I} A_j$ self-adjoint

(Not all op. is like this !)

- All in all: $\bigotimes_{k\in\mathbb{N}} L^2(\mathbb{R}, dx) = \bigoplus_{\varepsilon\in\Gamma} \left(\bigotimes_{k\in\mathbb{N}}^{\varepsilon} L^2(\mathbb{R}, dx) \right)$

LEM. 13.: Given (ϕ_n) ONB of $L^2(\mathbb{R}, dx)$,

(i) $\left\{ \phi_{m_1} \otimes \phi_{m_2} \otimes \cdots \;\middle|\; m_j \in \mathbb{N} \right\}$

is an ORTHONORMAL FAMILY but

NOT an ONB of $\bigotimes_{j \in \mathbb{N}} L^2(\mathbb{R}, dx)$

(ii) (vN39) Given a generator* $\bigotimes_{j \in \mathbb{N}} \psi_j^0 \in \mathfrak{C}$

$$\left\{ \phi_{m_1} \otimes \cdots \otimes \phi_{m_N} \otimes \psi^0_{N+1} \otimes \psi^0_{N+2} \otimes \cdots \;\middle|\; N, m_j \in \mathbb{N} \right\}$$

IS an ONB of $\bigotimes_{j \in \mathbb{N}}^{\mathfrak{C}} L^2(\mathbb{R}, dx)$

Issue of $\bigotimes_{j\in\mathbb{N}} L^2(\mathbb{R}, dx)$ for a PWT

5. [redacted]

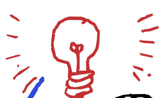
- $\psi_1 \otimes \psi_2 \otimes \ldots \ldots \rightarrow$ No expicit reference to $\mathbb{R}^\infty$

  $\hookrightarrow$ No clear "$|\Psi|^2 d^\infty x$"–measure

  $\hookrightarrow$ same issue in $\bigotimes_{j=1}^{N} L^2(\mathbb{R}, dx)$ BUT $\bigotimes_{j=1}^{m} \psi_j \rightsquigarrow \psi_1(x_1)\ldots\psi_m(x_m)$

- cannot do $(x_1, x_2, \ldots) \longmapsto \psi_1(x_1)\cdot\psi_2(x_2)\cdots$

  cause NO $d^\infty x$ measure to "host"!

$\hookrightarrow$ Blend  $\bigotimes_{j\in\mathbb{N}}$ idea with $d^\infty\mu$ idea!

6. Combine Approaches

Lem. 16: If $d\mu_j \sim dx \quad \forall j \in \{1, \dots, m\}$

$$d\mu_j = |\varphi_j|^2 \, dx$$

$\exists!$ $\mathcal{U} : \bigotimes_{k=1}^{m} L^2(\mathbb{R}, dx) \longrightarrow L^2(\mathbb{R}^m, d\vec{\mu})$ isomorph.

$$\psi_1 \otimes \dots \otimes \psi_m \longmapsto \left[(x_1, \dots, x_m) \longmapsto \frac{\psi_1(x_1) \cdots \psi_m(x_m)}{\varphi_1(x_1) \cdots \varphi_m(x_m)} \right]$$

& it is UNITARY.

THM. 10: If $(\varphi_j)_{j\in\mathbb{N}} \subset L^2(\mathbb{R}, dx)$:

(i) $\|\varphi_j\| = 1 \Longrightarrow |\varphi_j|^2 dx$ is PROB. MEAS

(ii) $\varphi_j(x) \neq 0$ a.e. $x \in \mathbb{R}$

↳ $|\varphi_j|^2(x) > 0$ a.e. $x \Longrightarrow |\varphi_j|^2 dx \sim dx$

$$\Rightarrow \exists!\ W_{[\otimes_k \varphi_k]} : \bigotimes_{j\in\mathbb{N}}^{[\otimes_k \varphi_k]} L^2(\mathbb{R}, dx) \longrightarrow L^2\Big(\mathbb{R}^\infty, \bigodot_{j\in\mathbb{N}} |\varphi_j|^2 dx\Big)$$

isomorph.

$$\psi_1 \otimes \cdots \otimes \psi_m \otimes \underbrace{\varphi_{m+1} \otimes \cdots}_{\text{Fixed tail}} \longmapsto \left[(x_1, x_2, \dots) \mapsto \frac{\psi_1(x_1)\cdots\psi_m(x_m)}{\varphi_1(x_1)\cdots\varphi_m(x_m)} \right]$$

& it is UNITARY.

- Innocent looking, but...

to prove it needed $L^2(\mathbb{R}^\infty, d^\infty\mu)$-theory.

PROP. 20: $d^I\mu := \bigodot_{j\in I} d\mu_j \Rightarrow \forall J = \{j_1, \dots, j_n\} \subset \mathbb{N}$

$$\int_{x\in\mathbb{R}^\infty} f(x_{j_1}, \dots, x_{j_n})\, d^{\mathbb{N}}\mu = \int_{x\in\mathbb{R}^n} f(x_{j_1}, \dots, x_{j_n})\, d^J\mu$$

LEM. 19: $\forall J = \{j_1 \dots j_n\} \subset \mathbb{N}$, $\forall (\phi_j)_{j\in J} \subset L^p(\mathbb{R}, d\mu_j)$

map $L^p(\mathbb{R}, d\mu_k) \longrightarrow L^p(\mathbb{R}^\infty, d^{\mathbb{N}}\mu)$ $p\in[1,+\infty)$.

$k \neq j$ $\quad \phi_k \longmapsto \left[x\in\mathbb{R}^\infty \longmapsto \prod_{j\in J} \phi_j(x_j)\cdot \phi_k(x_k)\right]$

is continuous.

COR. 13.:

$$\operatorname{span}\left\{ \Psi \in L^p(\mathbb{R}^\infty, d\tilde{\mu}) \;\middle|\; \begin{array}{l} \Psi(x) = \phi_1(x_1)\cdots\phi_N(x_N) \\ N \in \mathbb{N} \;\&\; \phi_j \in L^p(\mathbb{R}, d\mu_j) \end{array} \right\}$$

is DENSE in L^p.

PROP. 21: IF $(\phi_j^m)_{m\in\mathbb{N}} \subset L^2(\mathbb{R}, d\mu_j)$ ONB: $\phi_j^1 \equiv 1$

$$\left\{ x \in \mathbb{R}^\infty \longmapsto \phi_1^{m_1}(x_1)\cdots\phi_N^{m_N}(x_N) \;\middle|\; \begin{array}{l} N, m_j \in \mathbb{N} \\ m_N \neq 1 \end{array} \right\}$$

is an ONB of $L^2(\mathbb{R}^\infty, \bigodot_{j\in\mathbb{N}} d\mu_j)$.

6.[redacted]

PROP. 22: For every $\mathcal{E} \in \Gamma$ $\exists$ such generator $\bigotimes_{j\in\mathbb{N}} \varphi_j^{\mathcal{E}}$.

$\Rightarrow$ ALL LAYERS admit $\rightsquigarrow L^2(\mathbb{R}^\infty, d\mu_{\mathcal{E}})$

THM. 11.(i) $\forall \bigotimes_{j\in\mathbb{N}} \varphi_j, \bigotimes_{j\in\mathbb{N}} \eta_j \in \mathcal{E}$ such generators

(Kakutani 1948)

$$\bigodot_{j\in\mathbb{N}} |\varphi_j|^2 dx \sim \bigodot_{j\in\mathbb{N}} |\eta_j|^2 dx.$$

- Thus,

$$L^2\Big(\mathbb{R}^\infty, \bigodot_{j\in\mathbb{N}} |\varphi_j|^2 dx\Big) \cong L^2\Big(\mathbb{R}^\infty, \bigodot_{j\in\mathbb{N}} |\eta_j|^2 dx\Big)$$

"SAME QM"

THM. 11: (ii) $\bigotimes_{j\in\mathbb{N}} \varphi_j ,\ \bigotimes_{j\in\mathbb{N}} \eta_j \in \mathcal{C}$

GENERATORS

$$\left(\bigodot_{j\in\mathbb{N}} |\eta_j|^2 dx\right) = \left| W_{[\otimes\varphi_j]}(\eta_1\otimes\eta_2\otimes\cdots)\right|^2 \left(\bigodot_{j\in\mathbb{N}} |\varphi_j|^2 dx\right)$$

RADON-NIKODYM der.

$$W_{[\otimes\varphi_j]}(\eta_1\otimes\eta_2\otimes\cdots) = \overset{L^2(\mathbb{R}^\infty,\, \bigodot_j |\varphi_j|^2 dx)}{\lim_{N\to\infty}} \frac{\eta_1\cdots\eta_N}{\varphi_1\cdots\varphi_N}$$

PROP. 24: Arbitrary $(\mathcal{H}_j)_{j\in\mathbb{N}}$, $\mathcal{C}\in\Gamma$

$$\bigotimes_{j\in\mathbb{N}} f_j ,\ \bigotimes_{j\in\mathbb{N}} \varphi_j \in \mathcal{C} \ : \ \|\varphi_j\| = 1 \implies$$

$$(f_1\otimes\cdots\otimes f_n\otimes\varphi_{n+1}\otimes\varphi_{n+2}\otimes\cdots) \xrightarrow[n\to\infty]{} \bigotimes_{j\in\mathbb{N}} f_j$$

A Naive Solution?

$$\bigotimes_{k=1}^{n} L^2(\mathbb{R}, dx) \cong L^2(\mathbb{R}^n, d^n x)$$

$n \to \infty$ $\qquad\qquad$ $n \to \infty$

$$\bigotimes_{k\in\mathbb{N}} L^2(\mathbb{R}, dx)$$

$$\bigotimes_{k\in\mathbb{N}}^{\varepsilon_1} L^2(\mathbb{R}, dx) \cong L^2(\mathbb{R}^\infty, d\mu^\infty_{\varepsilon_1})$$

$$\bigotimes_{k\in\mathbb{N}}^{\varepsilon_2} L^2(\mathbb{R}, dx) \cong L^2(\mathbb{R}^\infty, d\mu^\infty_{\varepsilon_2})$$

$$\bigotimes_{k\in\mathbb{N}}^{\varepsilon_3} L^2(\mathbb{R}, dx) \cong L^2(\mathbb{R}^\infty, d\mu^\infty_{\varepsilon_3})$$

$$\vdots \qquad\qquad \vdots$$

$$\left(\bigotimes_{j\in\mathbb{N}} e_j^{\varepsilon}\right)_{\varepsilon\in\Pi}$$

Build big unitary!

(choice of $\mathcal{R} := \left(\bigotimes_{j\in\mathbb{N}} e_j^{\,\mathcal{C}} \right)_{\mathcal{C}\in\Gamma}$)

$$\bigotimes_{j\in\mathbb{N}} L^2(\mathbb{R}, dx) = \bigoplus_{\mathcal{C}\in\Gamma} \left(\bigotimes_{j\in\mathbb{N}}^{\mathcal{C}} L^2(\mathbb{R}, dx) \right) \overset{\downarrow}{\cong} \bigoplus_{\mathcal{C}\in\Gamma} L^2(\mathbb{R}^\infty, d\tilde{\mu}_{\mathcal{C}})$$

$$\Psi = \left(\Psi^{\mathcal{C}} \right)_{\mathcal{C}\in\Gamma} \longleftrightarrow \left(\tilde{\Psi}^{\mathcal{C}}_{\mathcal{R}} \right)_{\mathcal{C}\in\Gamma}$$

ABSTRACT Forms

$\mathbb{R}^\infty \longrightarrow \mathbb{C}$

WAVEFUNCTIONS

- As in SPINOR wavefuncts.

$$\text{“} |\Psi|^2 d^\infty x \text{”} := \sum_{\mathcal{C}\in\Gamma} |\tilde{\Psi}^{\mathcal{C}}_{\mathcal{R}}|^2 d\tilde{\mu}_{\mathcal{C}}$$

$\mathcal{R}$ dependence?

THM. 13: $\mathcal{R} := \left(\bigotimes_{j\in\mathbb{N}} \varphi_j^{\varepsilon}\right)_{\varepsilon\in\Gamma}$ $\left(\varphi_j^{\varepsilon}(x)\neq 0 \ \text{a.e. } x\right)$

Backgr. meas. $d\mu^{\infty}_{\varepsilon} := \bigodot_{j\in\mathbb{N}} |\varphi_j^{\varepsilon}|^2 dx$, BLOCK UNITARY

$$\mathcal{W}_{\mathcal{R}} := \bigoplus_{\varepsilon\in\Gamma} W_{[\otimes\varphi_j^{\varepsilon}]} : \bigotimes_{j\in\mathbb{N}} L^2(\mathbb{R}, dx) \longrightarrow \bigoplus_{\varepsilon\in\Gamma} L^2(\mathbb{R}^{\infty}, d\mu^{\infty}_{\varepsilon})$$

Wavefunction Representation (WR)

$$\Psi \longmapsto \left(\Psi^{\varepsilon}_{\mathcal{R}}\right)_{\varepsilon\in\Gamma}$$

(i) If $\tilde{\mathcal{R}} := \left(\bigotimes_{j\in\mathbb{N}} \eta_j^{\varepsilon}\right)_{\varepsilon\in\Gamma}$, change of WR

$$\mathcal{W}_{\mathcal{R}} = \left(\bigoplus_{\varepsilon\in\Gamma} \mathcal{U}_{\varphi^{\varepsilon}\leftarrow\eta^{\varepsilon}}\right) \circ \mathcal{W}_{\tilde{\mathcal{R}}}$$

THM. 13: Change layer's rep.

$$\mathcal{U}_{\varrho^\varepsilon \leftarrow \eta^\varepsilon} : L^2(\mathbb{R}^\infty, d\mu_{\eta^\varepsilon}) \longrightarrow L^2(\mathbb{R}^\infty, d\hat{\mu}_{\varrho^\varepsilon})$$

$$f \longmapsto \mathcal{M}_{W_{[\varrho^\varepsilon]}\left(\bigotimes_{j\in\mathbb{N}} \eta_j^\varepsilon\right)}(f)$$

(ii) $\forall\, \Psi \in \bigotimes_{j\in\mathbb{N}} L^2(\mathbb{R}, dx)$ $\forall\, B \in \mathcal{B}(\mathbb{R}^\infty)$,

$$\mathbb{P}^{\Psi}(B) := \sum_{\varepsilon\in\Gamma} \int_{(x_1,x_2,\dots)\in B} |\Psi_R^\varepsilon|^2(x_1,x_2,\dots)\, d\hat{\mu}_\varepsilon$$

is

FINITE BOREL MEASURE INDEPENDENT of R.

$$\mathbb{P}^{\Psi}(\mathbb{R}^\infty) = \|\Psi\|^2_{\otimes L^2(dx)} = \sum_{\varepsilon\in\Gamma} \|\Psi_R^\varepsilon\|^2 .$$

- $\mathbb{P}^{\Psi}(\mathbb{R}^{\infty}) = \|\Psi\|^2$

↪ if $\|\Psi\|^2 = 1 \rightarrow \left\{ \begin{array}{l} \mathbb{P}^{\Psi} \text{ PROBAB MEAS.} \\ \text{on } \mathbb{R}^{\infty} \end{array} \right.$

↪ Usable as BORN RULE MEASURE & Trajectory Equivariance in $\mathbb{R}^{\infty}$!

↪ Can build Pilot Wave Theories over $\mathbb{R}^{\infty}$!

$\boxed{\text{But what if just LUCK? • NEED +++!}}$

7. THIS WAS EXACTLY the JOINT SPECTRAL DIAGONALIZATION of ALL POSITION OPS.

Step Back: WHAT WOULD a PHYSICIST DO if given an abstract Hilbert space $\mathcal{H}$ and asked for wavefunctions over configuration?

JOINTLY DIAGONALIZE POSITION OPS. $\hat{x}_1, \dots, \hat{x}_m$ $\longrightarrow$ Find $|x_1, \dots, x_m\rangle$:

$$\hat{x}_j = \int_{x \in \mathbb{R}^m} x_j \, |x_1, \dots, x_m\rangle\langle x_1, \dots, x_m| \, d^m x$$

- Then $|\psi\rangle \in \mathcal{H}$ $\longrightarrow$

This is it!

$$|\psi\rangle = \int_{x \in \mathbb{R}^m} \psi(x_1, \dots, x_m) \, |x_1, \dots, x_m\rangle \, d^m x$$

- Physicists forget: ONLY IF spectrum SIMPLE get 1 wavefunction!
  ↪ Else TUPLE $(\psi_e)_{e \in \Gamma} \dots$

PLAN: Given $(\hat{q}\psi)(x) = x\psi(x)$, $\psi \in L^2(\mathbb{R}, dx)$

LIFT $\hat{q}$ to ITP: $\hat{q}_j := \underbrace{\mathrm{Id} \otimes \dots \otimes \mathrm{Id} \otimes \hat{q}}_{j} \otimes \mathrm{Id} \otimes \dots$

j-th POSITION OP. on "ABSTRACT" $\bigotimes_{j \in \mathbb{N}} L^2(\mathbb{R}, dx)$.

Find JOINT SPECTRAL DIAGONALISAT.

SPACE OF $\{\hat{q}_j\}_{j \in \mathbb{N}}$

PVMs on Arbitrary Meas. Spaces

DEF. 17: Σ σ-alg on X, $P : \Sigma \to \mathcal{L}_e(\mathcal{H})$

(i) $P(B)$ orthg. proj (ii) $P(X) = Id$

(iii) $\{B_m\}_{m\in\mathbb{N}} \subseteq \Sigma$ disj: $\operatorname{s-lim}_{N\to\infty} \sum_{m=1}^{N} P(B_m) = P\left(\bigcup_{m=1}^{\infty} B_m\right)$.

is a PVM. $\rightsquigarrow$ If Σ Boolean alg. $\Rightarrow$ PVpreM

(Schmüdgen 2012) lift usual results: functional calc. etc.

Denote Φ_P

KEY PROP. 31 (Schm. 2012 + Hahn Kolmog.):

$(\mathcal{H}, (X, \Sigma_0), P_0)$ PVpreM $\Rightarrow \exists! (\mathcal{H}, (X, \sigma(\Sigma_0)), P)$ PVM

Extension

THM 15: $I, \mathcal{H}$ arbit., X_j, $j \in I$ locally compact, secd. countable & Hausdorff (e.g. $\mathbb{R}$)

- If $(\mathcal{H}, (X_j, \mathcal{B}(X_j)), P_j)_{j\in I}$ COMMUTING PVMs

(i) $\exists!$ PVM $\left(\mathcal{H}, \left(\prod_{j\in I} X_j, \bigodot_{j\in I} \mathcal{B}(X_j)\right), P_I\right)$ s.th.

$$P_I\left(E_{j_1}\times\cdots\times E_{j_m}\times \prod_{j\in I\setminus J} X_j\right) = P_{j_1}(E_{j_1})\cdots P_{j_m}(E_{j_m})$$

$\forall$ finite $J = \{j_1,\dots,j_m\} \subset I$, $E_j \in \mathcal{B}(X_j)$.

call it JOINT / PRODUCT PVM

(Convenient Boolean algebra + Kolmogorov Extens + Key Prop. + Finite I case Schm.)

(ii) If $f: \prod_{j\in I} X_j \longrightarrow \mathbb{C}$ meas.

$$f((x_j)_{j\in I}) \equiv g(x_k) \quad \text{for some } g: X_k \longrightarrow \mathbb{C}$$

$$\Longrightarrow \quad \Phi^{P_I}(f) = \Phi^{P_k}(g)$$

- Enough for basic joint spectral thm. but need <u>MULTIPLICATION OP. VERSION!</u>

<u>THM. 16</u>: $\mathcal{H}, (X,\Sigma), P$ arbitrary. (i) $\forall \psi \in \mathcal{H}$

$$\mathcal{H}^\psi := \left\{ \Phi^P(f)\psi \mid f \in L^2(X, d\mu) \right\} = \overline{\operatorname{span}\left\{ P(B)\psi \right\}_{B\in\Sigma}}$$

<u>closed</u> <u>SPECTRAL SUBSPACE</u> cycled by ψ

$$\left(d\mu_\varphi \ll d\mu_\psi \quad \forall \varphi \in \mathcal{H}^\psi \right)$$

(ii) $\varphi \perp \mathcal{H}^{\psi} \Rightarrow \mathcal{H}^{\varphi} \perp \mathcal{H}^{\psi}$

(iii) $\exists$ ON fam. $(\psi_\kappa)_{\kappa \in \mathcal{K}}$: $(|\mathcal{K}| < \dim \mathcal{H})$

$$\mathcal{H} = \bigoplus_{\kappa \in \mathcal{K}} \mathcal{H}^{\psi_\kappa}.$$

a SPECTRAL BASIS of P

(iv) $\forall \psi \in \mathcal{H}$ well-def, unitary &

$$\mathcal{U} : \mathcal{H}^{\psi} \longrightarrow L^2(X, d\mu_\psi)$$
$$\varphi = \Phi^P(f)\psi \longmapsto f$$

Note: $d\mu_\varphi = |f|^2 d\mu_\psi$

(v) $\forall f : X \to \mathbb{C}$ meas. $\Rightarrow$ $\mathcal{H}^{\psi}$ REDUCES $\Phi^P(f)$

$$\Rightarrow \Phi^P(f)_{|\mathcal{H}^\psi} = \mathcal{U}^{-1} \circ M_f \circ \mathcal{U}_{\restriction D_f^\psi}.$$

THM. 19 (JOINT SPECTRAL DIAG. on Steroids!) 7.

Let I arbitrary (e.g., UNCOUNTABLE)

$\{(A_j, D(A_j))\}_{j\in I}$ self-adjoint ops. on (NON-SEPARABLE) $\mathcal{H}$ & A_j STRONGLY COMMUTING

(i.e., spectral PVMs commute)

$\Rightarrow$ $\exists!$ PRODUCT PVM $\{P_I(B)\}_{B\in \bigodot_{j\in I}\mathcal{B}(\mathbb{R})}$

$\forall \{j_1,\dots,j_N\}\subset I$, $B_{j_\ell}\in\mathcal{B}(\mathbb{R})$,

$$P_I\Big(B_{j_1}\times\dots\times B_{j_N}\times \prod_{j\in I\setminus\{j_1\dots j_N\}}\mathbb{R}\Big) = P_{j_1}(B_{j_1})\cdots P_{j_N}(B_{j_N})$$

THM. 19: (JOINT SPECTRAL DIAG. on Steroids!) 7.

- P_I satisfies: (i) $\forall f: \prod_{j\in I} \mathbb{R} \to \mathbb{C}$ s.th.

$\exists k \in I$, $g: \mathbb{R} \to \mathbb{C}$ measbl. : $f((x_j)_{j\in I}) \equiv g(x_k)$

$$\Rightarrow \quad \underline{\Phi^{P_I}(f) = \Phi^{P_k}(g).}$$

$$\Rightarrow \quad A_k = \Phi^{P_I}(\pi_k).$$

(ii) $\exists$ ON. $(\psi_n)_{n\in \mathcal{X}} \subseteq \mathcal{H}$ SPECTRAL BASIS

$$\mathcal{H}^{\psi_n} := \left\{ \Phi^{P_I}(f)\psi_n \;\middle|\; f \in L^2\Big(\prod_{j\in I}\mathbb{R}, d\mu_{\psi_n}\Big)\right\}$$

CYCLIC SUBSPACES: $\quad \mathcal{H} = \bigoplus_{n\in\mathcal{X}} \mathcal{H}^{\psi_n}.$

THM. 19 (JOINT SPECTRAL DIAG. on Steroids!) 7. [redacted]

(iii) $\exists$ unitary

$$\|P_I(\cdot)\psi_n\|^2$$

$$\mathcal{U}: \mathcal{H} = \bigoplus_{n \in \mathcal{X}} \mathcal{H}^{\psi_n} \longrightarrow \bigoplus_{n \in \mathcal{X}} L^2\Big(\prod_{j \in I} \mathbb{R}, d\mu_{\psi_n}\Big)$$

$$\varphi = \Big(\Phi^{P_I}(f_n)\psi_n\Big)_{n \in \mathcal{X}} \longmapsto (f_n)_{n \in \mathcal{X}}$$

s.th. $\forall$ measurable $f: \prod_{j \in I} \mathbb{R} \longrightarrow \mathbb{C}$,

$$\Phi(f) = \mathcal{U}^{-1} \circ \bigoplus_{n \in \mathcal{X}} M_f \circ \mathcal{U}.$$

e.g. $$A_k = \mathcal{U} \circ \bigoplus_{n \in \mathcal{X}} \hat{x}_k \circ \mathcal{U}^{-1}$$ $$(\hat{x}_k f)(x) \equiv x_k \cdot f(x)$$

$\Longrightarrow$ a JOINT DIAGONALIZATION SPACE $(A_k)_{k \in I}$.

• A physicist calls the tuple

$$\left(f_\pi\right)_{\pi\in\mathcal{X}} \in \bigoplus_{\pi\in\mathcal{X}} L^2\left(\prod_{j\in I}\mathbb{R},\ d\mu_{\psi_\pi}\right),$$

the JOINT $(A_j)_{j\in I}$-REPRESENTATION of

abstract vector $\mathcal{U}^{-1}\left(\left(f_\pi\right)_{\pi\in\mathcal{X}}\right) =: \Psi$

$\rightsquigarrow$ if A_j QUANTIZED DOFs $\Rightarrow$ | CONFIGURATION SPACE REPRESENT.!

• "Genrlzd Born Rule": "$\mathbb{P}^{\Psi}(a\in B)$" $B\in\bigodot_{j\in I}\mathcal{B}(\mathbb{R})$

$$d\mu_\Psi(B) = \left\|P^I(B)\Psi\right\|^2 \underset{(\mathcal{U}\text{ UNITARY})}{=} \sum_{\pi\in\mathcal{X}}\int_{a\in B}\left|f_\pi\right|^2\left((a_j)_{j\in I}\right) d\mu_{\psi_\pi}$$

- IDEA: Check if lifted position ops. in $\bigotimes_{k\in\mathbb{N}} L^2(\mathbb{R}, dx)$ strongly commute (find PVMs etc.) $\Longrightarrow$ Find Functional Calculus $\Longrightarrow$ check if diagonal rep. is "naive guess"!

EXAMPLE OF PROP. 32 $\left(\hat{q}_k = Id \otimes \cdots \otimes \hat{q} \otimes Id \otimes \cdots\right)$:

Def. $D_0(\hat{q}_k) := \operatorname{span}\left\{ \bigotimes_{j\in\mathbb{N}} \psi_j \;\middle|\; \psi_k \in D(\hat{q}) \right\}$

$$\hat{q}_k \left(\bigotimes_{j\in\mathbb{N}} \psi_j \right) := \psi_1 \otimes \cdots \otimes (\hat{q}\,\psi_k) \otimes \psi_{k+1} \otimes \cdots$$

$\Longrightarrow$ $(\hat{q}_k, D_0(\hat{q}_k))$ densely def., essentially self-adjoint & reduced by $\bigotimes_{j\in\mathbb{N}}^{e} L^2(\mathbb{R}, dx)$.

STRATEGY: For $L^2(\mathbb{R}^\infty, d^\infty\mu)$

$$(\hat{x}_k \psi)(x_1, x_2, \dots) := x_k \psi(x_1, x_2, \dots)$$

multipl. op.'s PVM & FC & JOINT STUFF easy

$\Rightarrow$ Pullback by $\mathcal{W}_R$ $\Rightarrow$ check if result is PVM & FC of $\overline{\hat{q}_k}$.

Prop. 36: For $B_k \in \mathcal{B}(\mathbb{R})$, $g : \mathbb{R} \to \mathbb{C}$ meas

$$P_k(B_k) := \mathcal{M}_{\mathbb{1}_{B_k} \circ \pi_k}, \qquad \Phi_k(g) := \mathcal{M}_{g \circ \pi_k}$$

are the PVM & FC of $(\hat{x}_k, D(\hat{x}_k))$.

Prop. 35: $B \in \mathcal{B}(\mathbb{R}^\infty)$, $f : \mathbb{R}^\infty \to \mathbb{C}$ measbl.

$$P(B) := M_{\mathbb{1}_B} \quad \& \quad \hat{\Phi}(f) := M_f$$

are joint PVM & FC of $\{\hat{x}_k\}_{k \in \mathbb{N}}$

Technical checks for pull-back

Lem. 24. $U : \mathcal{H} \to \mathcal{K}$ unitary, $(\mathcal{H}, (X, \Sigma), P^{\mathcal{H}})$ PVM

$P^{\mathcal{K}}(\cdot) := U P^{\mathcal{H}}(\cdot) U^{-1}$ is a PVM &

$$(\Phi^{\mathcal{K}}(f), D_f^{\mathcal{K}}) = (U \Phi^{\mathcal{H}}(f) U^{-1}, U D_f^{\mathcal{H}})$$

& if $\{P_j^{\mathcal{H}}\}_{j \in I}$ commuting &

$$\bigodot_{j \in I} P_j^{\mathcal{K}} = U \left(\bigodot_{j \in I} P_j^{\mathcal{H}} \right) U^{-1}.$$

PROP. 37: If $\mathcal{H} = \bigoplus_{x \in \mathcal{X}} \mathcal{H}_x$ arbitr. [redacted]

& $(\mathcal{H}_x, (X, \Sigma), \mathbb{P}_x)_{x \in \mathcal{X}}$ PVMs

(i) $\mathbb{P}(\cdot) := \bigoplus_{x \in \mathcal{X}} \mathbb{P}_x(\cdot)$ is a PVM

(ii)
$$\left(\Phi^{\mathbb{P}}(f), D_f \right) = \left(\bigoplus_{x \in \mathcal{X}} \Phi^{\mathbb{P}_x}(f), \bigoplus_{x \in \mathcal{X}} D^x_f \right)$$

(iii) spectral measure of $\Psi = (\psi_x)_{x \in \mathcal{X}} \in \mathcal{H}$

$$d\mu_{\Psi}(B) = \sum_{x \in \mathcal{X}} d\mu^x_{\psi_x}(B)$$

LEM. 25:

$$\bigodot_{j \in I} \left(\bigoplus_{x \in \mathcal{X}} \mathbb{P}^x_j \right) = \bigoplus_{x \in \mathcal{X}} \left(\bigodot_{j \in I} \mathbb{P}^x_j \right)$$

Prop. 38: Let $R := \left(\bigotimes_{j\in\mathbb{N}} \varphi_j^{\varepsilon}\right)_{\varepsilon\in\Gamma}$ 7.

$$\mathcal{W}_R = \bigoplus_{\varepsilon\in\Gamma} W_\varepsilon \quad \text{"naive guess"} \qquad W_\varepsilon\left(\bigotimes_j \psi_j\right)(x) = \prod_j \frac{\psi_j(x_j)}{\rho_j(x_j)}$$

(i) SPECTRAL PVM of $\overline{\hat{q}_k}$, $k\in\mathbb{N}$, is:

$$Q_k(B_k) = \bigoplus_{\varepsilon\in\Gamma}\left(W_\varepsilon^{-1} M_{1_{B_k}\circ\pi_k} W_\varepsilon\right) \quad \forall B_k\in\mathcal{B}(\mathbb{R})$$

(ii) $\left(\overline{\hat{q}_k}\right)_{k\in\mathbb{N}}$ COMMUTE STRONGLY $\Rightarrow$ $\exists$ JOINT PVM

& it is:

$$Q(B) = \bigoplus_{\varepsilon\in\Gamma}\left(W_\varepsilon^{-1} M_{1_B} W_\varepsilon\right)$$

(iii) FUNCTIONAL CALCULUS of Q is: $f:\mathbb{R}^\infty \to \mathbb{C}$ measble,

$$\Phi^Q(f) = \left(\bigoplus_{\varepsilon\in\Gamma} W_\varepsilon^{-1} M_f W_\varepsilon \,,\, \bigoplus_{\varepsilon\in\Gamma} W_\varepsilon^{-1} D(M_f)\right).$$

PROP. 38:

(iv) For $\Psi = \psi_1 \otimes \psi_2 \otimes \cdots \in \bigotimes_{j\in\mathbb{N}} L^2(\mathbb{R}, dx)$

$\|\psi_j\| = 1 \quad \forall j \in \mathbb{N} \Rightarrow$ spectral measure is

$$d\mu_\Psi = \bigodot_{j\in\mathbb{N}} |\psi_j|^2 dx_j .$$

THM. 20: $\mathcal{W}_{\mathbb{R}} : \bigotimes_{j\in\mathbb{N}} L^2(\mathbb{R}, dx) \longrightarrow \bigoplus_{C\in\Gamma} \left(\bigotimes^{C}_{j\in\mathbb{N}} L^2(\mathbb{R}, dx) \right)$

"naive guess" is EXACTLY joint diagonalizat.

of position ops. $(\hat{q}_k)_{k\in\mathbb{N}} \Rightarrow$ for each

$\Psi \in \bigotimes_{j\in\mathbb{N}} L^2(\mathbb{R}, dx)$, tuple of $\mathbb{R}^\infty \to \mathbb{C}$ wavefcts

$\mathcal{W}_{\mathbb{R}} \Psi$ is its CONFIGURATION REPR.

THM. 20: (i) The choice

$R = (\bigotimes_{j \in \mathbb{N}} \varphi_j^{\mathcal{E}})_{\mathcal{E} \in \Gamma}$ is EXACTLY a choice of SPECTRAL BASIS for joint PVM

$d^\infty \mu_{\mathcal{E}}$ is the SPECTRAL MEASURE of $\bigotimes_{j \in \mathbb{N}} \varphi_j^{\mathcal{E}}$.

(Such R exhaust spectral bases made of elemt. tens. pds.)

(ii) Q distinguish meas on $\mathbb{R}^\infty$ per $\Psi \in \bigotimes_{k \in \mathbb{N}} L^2(\mathbb{R}, dx)$

$$d\mu_\Psi(B) = \|Q(B)\Psi\|^2 = \sum_{\mathcal{E} \in \Gamma} \int_{(x_1, x_2, \dots) \in B} |W_{\mathcal{E}} \Psi^{\mathcal{E}}|^2_{(x_1, x_2, \dots)} \, d\mu_{\mathcal{E}}$$

→ It's the "naive" guess!

⟹ the BORN RULE for CONFIG.

8. APPLICATIONS

A. |N| Non-Relativistic Dist. Particles in $\mathbb{R}^3$

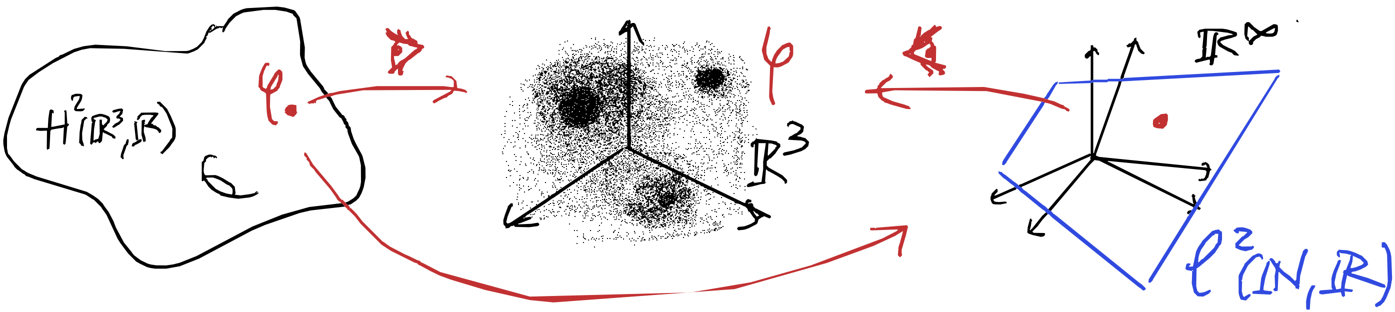
- $L^2(\mathbb{R}^{3N}, d^{3N}x) \rightsquigarrow \bigotimes_{k\in\mathbb{N}} L^2(\mathbb{R}, dx) = \bigoplus_{C\in\Gamma}\left(\bigotimes_{k\in\mathbb{N}}^{C} L^2(\mathbb{R}, dx)\right)$

- $|\psi|^2 d^{3M}x \rightsquigarrow \|Q(\cdot)\,\Psi\|^2 = \sum_{C\in\Gamma} |\Psi^C|^2 \, d\mu_C$

- $X_t^{\psi}(\vec{x}_1^{\,0}, \dots, \vec{x}_M^{\,0}) \rightsquigarrow X_t^{\Psi}(\vec{x}_1^{\,0}, \vec{x}_2^{\,0}, \dots)$

- Find $(H, D(H))$ (see thesis) $\Rightarrow$ check if new preds.! etc.

$\boxed{B}$. Field Configuration Spaces given by Real Separable Hilbert Spaces K



- ONB $(e_n)_{n\in\mathbb{N}} \subset K$

$$\left.\begin{array}{l} \\ \end{array}\right\} \quad U: K \longrightarrow \ell^2(\mathbb{N},\mathbb{R}) \subset \mathbb{R}^\infty$$

$$\sum_{n\in\mathbb{N}} \alpha_n e_n \longmapsto (\alpha_n)_{n\in\mathbb{N}}$$

- Can we do: "$L^2(\cancel{\mathbb{R}^\infty}, d^\infty x)$" ?

$$\ell^2(\mathbb{N},\mathbb{R})$$

$$\bigotimes_{k\in\mathbb{N}} L^2(\mathbb{R}, dx) \overset{R}{=} \bigoplus_{\varepsilon\in\Gamma} L^2(\mathbb{R}^\infty, d\mu_\varepsilon^R)$$

- If $d\mu_\varepsilon^R(\ell^2) = d\mu_\varepsilon^R(\mathbb{R}^\infty) \Longrightarrow$

  "canonically" $L^2(\mathbb{R}^\infty, d\mu_\varepsilon^R) = L^2(\ell^2, d\mu_\varepsilon^R)$

- If $d\mu_\varepsilon^R(\ell^2) = 0 \Longrightarrow L^2(\ell^2, d\mu_\varepsilon^R) = \{0\}$

- All R agree! $d\mu_\varepsilon^R \sim d\mu_\varepsilon^{\tilde{R}}$ so,

$$\left(\begin{array}{c} d\mu_\varepsilon^R(\ell^2) = d\mu_\varepsilon^R(\mathbb{R}^\infty) \\ = 0 \end{array}\right) \Longleftrightarrow \left(\begin{array}{c} d\mu_\varepsilon^{\tilde{R}}(\ell^2) = d\mu_\varepsilon^{\tilde{R}}(\mathbb{R}^\infty) \\ = 0 \end{array}\right)$$

- But if $d\mu_\varepsilon^R(\ell^2) \in \left(0, d\mu_\varepsilon^R(\mathbb{R}^\infty)\right)$

  unclear & no agreement...

- Fortunately:

  LEM. 27: $\ell^2(\mathbb{N}, \mathbb{R}) \in \mathcal{B}(\mathbb{R}^\infty)$ ✓

COR. 17: $(d\mu_j)_{j \in \mathbb{N}}$ pblty meas. $\mathcal{B}(\mathbb{R})$

$$\Big(\bigodot_{j \in \mathbb{N}} d\mu_j\Big)(\ell^2) = \begin{cases} \text{Either } 1 = \Big(\bigodot_{j \in \mathbb{N}} d\mu_j\Big)(\mathbb{R}^\infty) \\ \text{or } \quad 0 \end{cases}$$

(Kolmogorov's 1-0 Law & ℓ^2_ε "tail event")

WHAT MEANS RESTRICT L^2?

- **Original:** $(\mathbb{R}^\infty, \sigma(\tau^{prod}), d\mu)$
- **Restricted:** $(\ell^2, \sigma(\tau^{prod})|_{\ell^2}, d\mu|_{\ell^2})$

$$\sigma(\tau^{prod})|_{\ell^2} := \{E \cap \ell^2 \mid E \in \sigma(\tau^{prod})\}$$

LEM. 28: $\sigma(\tau^{prod})|_{\ell^2} = \sigma(\tau^{prod}|_{\ell^2})$

PROP. 40: $\tau^{prod}|_{\ell^2} \subsetneq \tau^{norm}_{\ell^2}$

But natural topol. of $\ell 2$ is NORM's!

Yet: $\sigma(\tau^{prod}|_{\ell^2}) = \sigma(\tau^{norm}_{\ell^2})$

Def 21.:

$$\Gamma_1 := \left\{ \mathcal{E} \in \Gamma \;\middle|\; d^{\infty}\mu_{\mathcal{E}}\left(\ell^2(\mathbb{N},\mathbb{R})\right) = 1 \text{ for } \underline{\text{one \& hence any}}\ R \right\}$$

- "Restriction $\mathbb{R}^{\infty} \rightsquigarrow \ell^2$" is:

$$\text{"}L^2(\ell^2, d^{\infty}x)\text{"} := W_R^{-1}\left(\bigoplus_{\mathcal{E}\in\Gamma_1} L^2(\ell^2, d^{\infty}\mu_{\mathcal{E}}) \right) =$$

$$= \bigoplus_{\mathcal{E}\in\Gamma_1} \left(\bigotimes_{k\in\mathbb{N}}^{\mathcal{E}} L^2(\mathbb{R}, dx) \right)$$

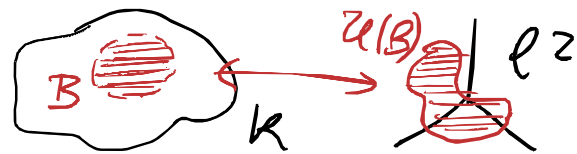
- It is a definition INDEPENDENT of the choice of R

THE PULLBACK: "$L^2(K, d^\infty x)$" 8 [redacted]

LEM. 30: Fix $\mathcal{C} \in \Gamma_1$, pullback by $\mathcal{U} : K \to \ell^2$

$$(\ell^2, \mathcal{B}(\ell^2), d\mu_{\mathcal{C}}) \hookrightarrow (K, \mathcal{B}(K), d\mu^K_{\mathcal{C}}),$$

$$d\mu^K_{\mathcal{C}} := d\mu_{\mathcal{C}} \circ \mathcal{U}.$$



- $$V_{\mathcal{U}} : L^2(\ell^2(\mathbb{N}, \mathbb{R}), d\mu_{\mathcal{C}}) \to L^2(K, d\mu^K_{\mathcal{C}})$$
$$\Psi \longmapsto \Psi \circ \mathcal{U} \qquad \Psi(\psi)$$

is UNITARY

$$(V_{\mathcal{U}}\Psi)(\phi) := \Psi(x_1 = \langle e_1, \phi \rangle, x_2 = \langle e_2, \phi \rangle, \dots)$$

WHY THIS SEEMS RIGHT?

$$\sum_{n=1}^{\infty} \langle e_n, \psi \rangle \hat{X}_n \iff \hat{\Psi}_x \Psi(\psi) = \psi(x) \cdot \Psi(\psi)$$

POSITION OPS. LITERAL FIELD OPS.

PROP. 41:

$$\beta \in \ell^2 \quad \left\{ \begin{array}{l} X_\beta : \ell^2 \to \mathbb{R} \\ \alpha \mapsto \langle \beta, \alpha \rangle \end{array} \right.$$

is FINITE & Measurable

- **Def.** MULTIPLICATION OPERATOR:

$$\Psi \in L^2(\ell^2, d\mu_e^\infty) \Rightarrow (\hat{X}_\beta \Psi)(\alpha) := \langle \beta, \alpha \rangle \Psi(\alpha).$$

- It is SELF-ADJOINT & $\tilde{e}_k := (0, \ldots, 0, 1, 0, \ldots)$

$$\hat{X} \tilde{e}_k = \hat{X}_k.$$

PROP. 41: $\hat{X}_\beta$ in $L^2(\mathcal{K}, d\mu_C^{\mathcal{K}})$ is: [redacted]

$$\hat{\Psi}_\varphi := \mathcal{V}_u \, \hat{X}_{u\varphi} \, \mathcal{V}_u^{-1} \qquad \forall \varphi \in \mathcal{K}$$

$\Rightarrow$ for $\Psi \in L^2(\mathcal{K}, d\mu_C^{\mathcal{K}})$ as

$$(\hat{\Psi}_\varphi \Psi)(\phi) \equiv \underline{\langle \varphi, \phi \rangle} \, \Psi(\phi)$$

call $\{\hat{\Psi}_\varphi\}_{\varphi \in \mathcal{K}}$ the **FIELD OPs**.

Example 1: If $\mathcal{K} = L^2(\mathbb{Z}^3, \mathbb{R}, d\nu)$

$\forall x \in \mathbb{Z}^3 \Rightarrow \delta_x \in \mathcal{K} \Rightarrow (\hat{\Psi}_{\delta_x} \Psi)(\psi) = \psi(x) \cdot \Psi(\psi)$.

DIRAC Delta

Example 2: If $\mathcal{K} = L^2(\mathbb{R}^3, \mathbb{R}, d^3x)$

$$\left(\hat{\Phi}_\varphi \Psi\right)(\psi) = \int_{x\in\mathbb{R}^3} \varphi(x)\cdot \underbrace{\psi(x)\,\Psi(\psi)}_{=:\left(\hat{\Phi}_x \Psi\right)(\psi)}\, d^3x =$$

$$=: \left(\int_{x\in\mathbb{R}^3} \varphi(x)\cdot \hat{\Phi}_x \, d^3x\right) \Psi \; (\psi)$$

OPERATOR-VALUED DISTRIBUTION!

- Similarly with lifted MOMENTUM OPS. $\Rightarrow$ Rigorous CAN. MOMENT. FIELD OPERATORS $\hat{\pi}_\varphi$ acting as FUNCTNL DERIVTVS! (And CCR with $\hat{\Phi}_\varphi$).

To be continued...

9. The Obvious CCR Representation of $\bigotimes_{j\in\mathbb{N}} L^2(\mathbb{R},dx)$

DEF. 56: $\tilde{e}_m := (\underbrace{0,\dots,0,1}_{m},0,\dots) \in \ell^2(\mathbb{N},\mathbb{R})$.

$\forall \beta \in \operatorname{span}(\tilde{e}_m)_{m\in\mathbb{N}} = \mathcal{W}$, $\beta = \sum_{m=1}^{N} \beta_m \tilde{e}_m$,

$$\boxed{\hat{q}(\beta) := \sum_{m=1}^{N} \beta_m \hat{q}_m \quad \& \quad \hat{p}(\beta) := \sum_{m=1}^{N} \beta_m \hat{p}_m}$$

$$\begin{cases} \hat{q}_m(\psi_1 \otimes \psi_2 \otimes \cdots) = \psi_1 \otimes \cdots \otimes (\hat{q}\psi_m) \otimes \psi_{m+1} \otimes \cdots \\ \hat{p}_m(\psi_1 \otimes \psi_2 \otimes \cdots) = \psi_1 \otimes \cdots \otimes (\hat{p}\psi_m) \otimes \psi_{m+1} \otimes \cdots \end{cases}$$

and $(\hat{q}\psi)(x) = x\psi(x)$, $(\hat{p}\psi)(x) = -i\frac{d}{dx}\psi(x)$

• $\phi_m(x) := c_m \cdot h_m(x)\, e^{-x^2/2}$

Hermite function ONB – Eigenfcts. of $\left(-\frac{d^2}{dx^2}+x^2\right)$.

$\Rightarrow$ $B_{Her} := \operatorname{span}\{\phi_m\}_{m\in\mathbb{N}}$. $\rightsquigarrow$ Dense in $L^2(\mathbb{R}, dx)$

$\Rightarrow$ by Prop. 70,

$$\bigotimes^{F}_{j\in\mathbb{N}} B_{Her} := \operatorname{span}\left\{ \bigotimes_{j\in\mathbb{N}} \psi_j \;\middle|\; \psi_j \in B_{Her}\ \forall j\in\mathbb{N} \right\}$$

is DENSE subsp. of $\bigotimes_{j\in\mathbb{N}} L^2(\mathbb{R}, dx)$.

↳ use as common domain of $\{\hat{q}^j(\beta), \hat{p}^j(\beta)\}$.

THMs. 29, 30, 31: $\hat{q}(\beta), \hat{p}(\beta)$ ess. s.a. &

(i) $\left(\{ \hat{q}(\beta), \hat{p}(\beta) \}_{\beta \in W}, \bigotimes^{F}_{j \in \mathbb{N}} B_{Her} \right)$ are HEISENB CCR rep

— Arai (2019)'s Def —

(ii) $\{ \overline{\hat{q}(\beta)}, \overline{\hat{p}(\beta)} \}_{\beta \in W}$ are WEYL CCR rep.

(iii) Def "ITP Segal Field operators"

Strategy of Arai (2019)

$$\hat{\Phi}(\gamma) := \hat{q}(\mathrm{Re}\,\gamma) + \hat{p}(\mathrm{Im}\,\gamma), \quad \gamma \in W_{\mathbb{C}}.$$

& $$\hat{a}(\gamma) := \frac{1}{\sqrt{2}} \overline{\left(\hat{\Phi}(\gamma) + i \, \hat{\Phi}(i\gamma) \right)}. \quad \Longrightarrow$$

$\{ \hat{a}(\gamma), \hat{a}(\gamma)^{*} \}_{\gamma \in W_{\mathbb{C}}}$ is CREATION-ANNIHILAT. CCR rep.

THM. 35: ALL REPS. REDUCED by $\bigotimes_{j\in\mathbb{N}}^{\mathcal{C}} L^2(\mathbb{R})$

& $\left\{\overline{\hat{q}(\beta)|_{\mathcal{C}}}, \overline{\hat{p}(\beta)|_{\mathcal{C}}}\right\}_{\beta\in W}, \left\{\hat{a}(\beta)|_{\mathcal{C}}, \hat{a}(\beta)^*|_{\mathcal{C}}\right\}_{\beta\in W_{\mathbb{C}}}$

still Heisenberg, Weyl and creat.-annih. CCR

But now IRREDUCIBLE! (Streit 67)

• $\mathcal{C}, \mathcal{D} \in \Pi$,

(REDUCTIONS EQUIVALENT IRREPS) $\Longleftrightarrow$ ($\mathcal{C} \overset{q}{\approx} \mathcal{D}$ i.e.,

$\exists\, (\theta_j)_{j\in\mathbb{N}} \subset [-\pi, \pi)$:

for one (& hence any)

$\otimes\phi_j \in \mathcal{C}$, $\otimes\psi_j \in \mathcal{D}$.

$$\bigotimes_{j\in\mathbb{N}} \phi_j \approx \bigotimes_{j\in\mathbb{N}} \left(e^{i\theta_j}\psi_j\right)$$)

(Klauder et al. 1966
Streit 1967)

Prop. 78: Ladder operators

$$\hat{a} := \frac{1}{\sqrt{2}}(\hat{q} + i\hat{p}), \quad \hat{a}^{+} := \frac{1}{\sqrt{2}}(\hat{q} - i\hat{p})$$

$$\Rightarrow \begin{cases} \hat{a}(\tilde{e}_m) = \overline{\underbrace{Id \otimes \cdots \otimes}_{m} \hat{a} \otimes Id \otimes \cdots} \\ \hat{a}(\tilde{e}_m)^* = \overline{Id \otimes \cdots \otimes \hat{a}^{+} \otimes Id \otimes \cdots} \end{cases}$$

(Lift as before)

Prop. 81: $\exists \; \mathcal{R} := \left(\bigotimes_{j \in \mathbb{N}} \rho_j^{\varepsilon} \right)_{\varepsilon \in \Gamma}$ s.th.

$$\rho_j^{\varepsilon} \in \underline{B_{Her}} \quad \forall j \in \mathbb{N} \quad \forall \varepsilon \in \Gamma$$

i.e.,

$$\bigotimes_{j \in \mathbb{N}} \rho_j^{\varepsilon} \in \bigotimes_{j \in \mathbb{N}}^{F} B_{Her}$$

CONNECT. BTW. FOCK SPACE, the ITP, the CCR REPS. and the VACUUM

DEF. 59: $(\mathcal{H}, \mathcal{D}, \{\hat{c}(f)\}_{f\in\mathcal{V}})$ creation-annihilation CCR rep.

$\Longrightarrow$ QUANTUM VACUUM is UNIT $\Omega \in \mathcal{D}$

CYCLIC for $\{\hat{c}(f), \hat{c}(f)^*\}_{f\in\mathcal{V}}$, i.e.,

$$\mathcal{H}_\Omega := \operatorname{span}\left\{ \Omega,\ \hat{c}(f_1)^{\#_1} \cdots \hat{c}(f_n)^{\#_n}\Omega \;\middle|\; \begin{array}{l} n\in\mathbb{N}, \\ \#_j \in \{1,*\} \\ f_j \in \mathcal{V} \end{array} \right\}$$

is DENSE. (Finite Bosons away from Ω)

- If $\hat{c}(f)\Omega = 0 \quad \forall f \in \mathcal{V}$ EMPTY QV

  else a MYRIOTIC QV.

  (Friedrichs 1953)

THM. 40: $\forall\, \mathfrak{C} \in \Gamma$, EVERY generator

$$\bigotimes_{j\in\mathbb{N}} e_j^{\mathfrak{C}} \in \bigotimes^{F}_{j\in\mathbb{N}} B_{Her}$$ is QUANTUM VACUUM for

IRREP. $$\left(\{\hat{a}(\beta)|_{\mathfrak{C}}\}_{\beta\in W_{\mathbb{C}}},\ \bigotimes^{\mathfrak{C},F}_{j\in\mathbb{N}} B_{Her} \right) =: \mathfrak{R}(\mathfrak{C}).$$

- $\phi_0(x) := \pi^{-1/4} e^{-x^2/2}$, $\Psi_G := \phi_0 \otimes \phi_0 \otimes \cdots \in \mathfrak{C}_G$

$(\mathfrak{C} \overset{q}{\approx} \mathfrak{C}_G)$ $\Longleftrightarrow$ ($\bigotimes^{\mathfrak{C}}_{j\in\mathbb{N}} L^2(\mathbb{R}, dx)$ has generator that is an EMPTY vacuum of $\mathfrak{R}(\mathfrak{C})$.) $\Longleftrightarrow$ ($\mathfrak{R}(\mathfrak{C})$ is EQUIVALENT to the FOCK CCR REPRESENTATION $\mathcal{F}_b^{fin}(W_{\mathbb{C}}), \{\hat{A}(\beta)\}_{\beta\in W_{\mathbb{C}}}$)

(Haag, Segal, Chaiken knew in 60s)

And equivalence is s.th.

9.

$\phi_0 \otimes \phi_0 \otimes \cdots \longleftrightarrow \Omega_{Fock} = (1, 0, 0, \ldots)$

up to const.

$\phi_1 \otimes \phi_0 \otimes \phi_0 \otimes \cdots \longleftrightarrow \hat{A}(\tilde{e}_1)^* \Omega_{Fock} = (0, \tilde{e}_1, 0, \ldots)$

up to cons

$\phi_0 \otimes \phi_3 \otimes \phi_0 \otimes \cdots \longleftrightarrow \hat{A}(\tilde{e}_2)^* \hat{A}(\tilde{e}_2)^* \hat{A}(\tilde{e}_2)^* \Omega_{Fock} \doteq$

$= (0, 0, 0, \tilde{e}_2 \otimes \tilde{e}_2 \otimes \tilde{e}_2, 0, \ldots)$

$\hat{q}(\beta), \hat{p}(\beta) \longleftrightarrow$ Canonical conjugate field operators

- So every quasi-layer except $\mathfrak{S} \overset{\circ}{\cong} \mathfrak{S}_G$ have: "all" generators are MYRIOTIC vacuums!

WHY WAS THIS OBVIOUS?

9.[redacted]

- $\phi_{m_1} \otimes \phi_{m_2} \otimes \phi_{m_3} \otimes \cdots$ describes:

m_1 bosons in mode 1, m_2 bosons in mode 2, etc.

$$\begin{cases} \phi_0 \otimes \phi_0 \otimes \cdots \rightsquigarrow 0 \text{ bosons.} \rightsquigarrow \underline{\text{EMPTY}}\ \underline{\text{VACUUM}} \\ \phi_1 \otimes \phi_1 \otimes \cdots \rightsquigarrow 1 \text{ boson / mode} = \underline{\infty \text{ bosons}} \end{cases}$$

- If $\sum_{j=1}^{\infty} m_j = +\infty$ <u>cannot be annihilated</u> by finite $\hat{a}(\beta)$ $\rightsquigarrow$ <u>MYRIADS of BOSONS</u>

↳ <u>Myriotic vacuum.</u>

- $\left.\begin{array}{l}\text{If } m_j \neq m_j' \text{ for} \\ \infty\text{ly many } j\end{array}\right\}$ $\left[\bigotimes_{j \in \mathbb{N}} \phi_{m_j}\right] \neq \left[\bigotimes_{j \in \mathbb{N}} \phi_{m_j'}\right]$

HAAG'S THM. NO MYSTERY 9.[redacted]

If as $t \to -\infty$ finitely many or No bosons,

$\Rightarrow$ $\phi_0 \otimes \phi_0 \otimes \cdots$'s layer (or equiv. Fock sp.) describes state. But if interaction generates ∞ly many bosons by $t=0$

↳ Need other layers

↳ i.e. irreps. inequivalent to Fock

↳ Either change state-space dynamically

↳ or use full ITP as state-space!!!

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(van Hove 1952, Segal 1963, Reed 1970 knew this was the issue!)

To Be
Continued

Thanks for your attention!