# Exercise Sheet 5: ODEs and Diffeomorphisms

**1.** Determine and draw some integral curves for the vector fields

$$v : \mathbb{R}^2 \to \mathbb{R}^2, \quad (x, y) \mapsto v(x, y) = \begin{pmatrix} -x \\ y \end{pmatrix},$$
$$w : \mathbb{R}^2 \to \mathbb{R}^2, \quad (x, y) \mapsto w(x, y) = \begin{pmatrix} x \\ y \end{pmatrix}.$$

(*Hint: Draw the velocity fields*)

**2.** Show that every autonomous ODE of order $m$ can be reduced to a system of $m$ first order autonomous ODEs.

**3.** Let $A, B > 0$. Then,

(i) Find a solution to the ODE: $\frac{d\gamma(t)}{dt} = A(\gamma(t) + B)$ with generic initial condition $\gamma(0) = x_0 \in \mathbb{R}$. (*Hint: it is an exponential.*)

(ii) Let $f : [0, +\infty) \to \mathbb{R}$ be a differentiable function with

$$\frac{d}{dt} f(t) \leq A(f(t) + B) \quad \forall t \in [0, \infty).$$

Prove that for all $t \in [0, +\infty)$, $f(t) \leq \gamma(t)$, where $\gamma$ is the solution of (i) with $\gamma(0) = f(0)$.

(This is called the *Grönwall lemma.*)

**To do in class after the "inverse function theorem lecture":**

**4.** Show that if $f : \mathbb{R}^n \supset \Omega \to \Omega' \subset \mathbb{R}^n$ is a diffeomorphism, its differential $Df|_x : \mathbb{R}^n \to \mathbb{R}^n$ is an isomorphism for every $x \in \Omega$.

**5.** Find an example of a continuously differentiable bijection that is not a diffeomorphism.