Exercise Sheet 4: Differential Calculus

1. Let $f : \mathbb{R}^3 \setminus \{0\} \to \mathbb{R}$ and $F : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by

$$f(x) = \|x\|_2 \,, \quad F(x) = \lambda x - x_0$$

for some $\lambda \in \mathbb{R}$ and $x_0 \in \mathbb{R}^3$. Compute ∇f, $\nabla^2 f$, Δf, $\operatorname{div} F$ and $\operatorname{curl} F$.

2. Let $f : V \to W$ be a map between finite dimensional normed spaces and fix $v_0 \in V$. Show that there exists at most one linear map $L \in \mathcal{L}(V, W)$ such that

$$\lim_{v \to 0} \frac{f(v_0 + v) - f(v_0) - L(v)}{\|v\|_V} = 0 \,.$$

3. Let V, W be finite dimensional real normed vector spaces. For $f \in \mathcal{C}^1(G, W)$, prove that

$$\partial_v f(x) = Df|_x v \quad \forall v \in v, \ \forall x \in G.$$

4. Given X, Y are topological spaces with Y Hausdorff, let there be $f, g : X \to Y$ continuous. Prove that if $A \subseteq X$ is dense (in X), then

$$\Big(\ f|_A = g|_A \ \Big) \Longleftrightarrow \Big(\ f = g \ \Big).$$