Exercise Sheet 6: Measure Theory

1. Let $\mathcal{A}$ be a σ-algebra on X. Show that

(i) $X \in \mathcal{A}$

(ii) $A_k \in \mathcal{A}$ for $k \in \mathbb{N} \quad \Longrightarrow \quad \bigcap_{k\in\mathbb{N}} A_k \in \mathcal{A}$

(iii) $A, B \in \mathcal{A} \quad \Longrightarrow \quad A \cup B,\ A \cap B,\ A \setminus B \in \mathcal{A}$

2. Let μ be a measure on $(X, \mathcal{A})$ and $A, B \in \mathcal{A}$. Show that

(i) if $\mu(A \cup B) + \mu(A \cap B) = \mu(A) + \mu(B)$.

(ii) if $A \subset B$, then $\mu(A) \leq \mu(B)$.

(iii) For $A_j \in \mathcal{A},\ j \in \mathbb{N}$,

$$\mu\Big(\bigcup_{j=1}^{\infty} A_j\Big) \leq \sum_{j=1}^{\infty} \mu(A_j)\,.$$

(iv) If $A_j \subset A_{j+1}$ then

$$\lim_{j\to\infty} \mu(A_j) = \mu(\bigcup_{j=1}^{\infty} A_j)\,.$$

3. Let μ be a finite measure on $(X, \mathcal{A})$. Show that

$$L^p(X, \mu) \subset L^q(X, \mu)$$

for all $1 \leq q \leq p \leq \infty$.

4. Given $(\mathbb{R},\ \mathcal{B}(\mathbb{R}),\ \lambda)$ denotes the Lebesgue measure, prove that $\lambda(\mathbb{Q}) = 0$.