Chapter 2

Topological, metric, and normed spaces

Definition 2.1 (Sequence)**.**
A sequence $(x_n) = (x_1, x_2, x_3, \ldots)$ in a set X is a map:

$$x : \mathbb{N} \to X, \qquad n \mapsto x_n$$

A sequence (x_n) in $\mathbb{R}$ *converges* to $x \in \mathbb{R}$, if

$$\forall \varepsilon > 0 \; \exists N_\varepsilon \in \mathbb{N} \; \forall n \geq N_\varepsilon : \quad |x_n - x| < \varepsilon$$

(if for every $\varepsilon > 0$ it holds 'eventually' that $|x_n - x| < \varepsilon$) and we write

$$\lim_{n\to\infty} x_n = x \quad \text{or simply} \quad x_n \to x\,.$$

Similarly, a sequence (x_n) in $\mathbb{R}^k$ converges to $x \in \mathbb{R}^k$, if

$$\forall \varepsilon > 0 \; \exists N_\varepsilon \in \mathbb{N} \; \forall n \geq N_\varepsilon : \quad \|x_n - x\|_2 < \varepsilon$$

where $\|\cdot\|_2$ denotes the usual Euclidean distance (see Example 2.3). We observe that in both examples the notion of *norm* is the key tool in order to define convergence of sequences.

Definition 2.2 (Norm and normed space)**.**
Let V be a vector space either over $\mathbb{R}$ or $\mathbb{C}$. A *norm* $\|\cdot\|$ on V is a map:

$$\|\cdot\| : V \to [0,\infty), \qquad x \mapsto \|x\|$$

with the properties:

1. $\|x\| = 0 \quad \Leftrightarrow \quad x = 0$
2. $\forall x \in V,\ \lambda \in \mathbb{K} : \qquad \|\lambda x\| = |\lambda| \cdot \|x\|$

3. $\forall x, y \in V: \quad \|x+y\| \leq \|x\| + \|y\|$

The pair $(V, \|\cdot\|)$ is called a *normed space*.

Example 2.3. 1. On $V = \mathbb{R}^n$ or $\mathbb{C}^n$ the following maps are norms:

$$\begin{aligned} \|x\|_2 &= \sqrt{|x_1|^2 + |x_2|^2 + \ldots + |x_n|^2} && \text{euclidean norm} \\ \|x\|_\infty &= \max\{|x_1|, \ldots, |x_n|\} && \text{maximum norm} \\ \|x\|_1 &= |x_1| + |x_2| + \ldots + |x_n| && \text{1-norm} \end{aligned}$$

or, more generally, for $p \in [1, \infty)$, we obtain:

$$\|x\|_p = \Big(\sum_{j=1}^{n} |x_j|^p\Big)^{\frac{1}{p}} \qquad \text{p-norm}$$

2. Let X a set, $(Y, \|\cdot\|_Y)$ a normed space, and

$$V := \{f : X \to Y \mid \sup_{x \in X} \|f(x)\|_Y < \infty\}.$$

Then $\|f\|_\infty = \sup_{x \in X} \|f(x)\|_Y$ is a norm on V.

Definition 2.4 (Convergence in normed spaces)**.**
A sequence (x_n) in a normed space $(V, \|\cdot\|)$ converges to $x \in V$ if

$$\forall \varepsilon > 0 \, \exists N_\varepsilon \in \mathbb{N} \, \forall n \geq N_\varepsilon : \quad \|x_n - x\| < \varepsilon \, .$$

A norm, however, can only be defined on a vector space and ideally we would like to forget about such structure. In view of the definition of convergence for normed spaces, a notion of "distance" between points should suffice.

Definition 2.5 (Metric and metric space)**.**
Let X be a set. A *metric* d on X is a map:

$$d : X \times X \to [0, \infty)$$

with the following properties:

1. $d(x, y) = 0 \quad \Leftrightarrow \quad x = y$
2. Symmetry: $\forall x, y \in X: \quad d(x, y) = d(y, x)$
3. Triangle inequality: $\forall x, y, z \in X: \quad d(x, z) \leq d(x, y) + d(y, z)$

The pair (X, d) is called a *metric space*

Example 2.6. 1. Let $(V, \|\cdot\|)$ be a normed space. Then $d : V \times V \to [0, \infty)$, $(x, y) \mapsto d(x, y) := \|x - y\|$ defines a metric on V.

2. Let X be as set. The discrete metric on X is defined by:

$$d(x,y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{otherwise} \end{cases}$$

3. The euclidean unit sphere $S^2 := \{x \in \mathbb{R}^3 \mid \|x\|_2 = 1\}$ with the metric:

$$d(x,y) := \arccos(\langle x, y \rangle)$$

is a metric space.

Definition 2.7 (Convergence in metric spaces)**.**
A sequence (x_n) in a metric space (X,d) converges to $x \in X$ if

$$\forall \varepsilon > 0\, \exists N_\varepsilon \in \mathbb{N}\, \forall n \geq N_\varepsilon : \quad d(x_n, x) < \varepsilon\,.$$

Definition 2.8 (Open sets in a metric space)**.**
Let (X,d) be a metric space.

1. For $x_0 \in X$ and $r > 0$ the set:

$$B_r(x_0) := \{x \in X \mid d(x, x_0) < r\}$$

is called the *open ball*, with the radius r and the centre x_0.

2. A subset of $U \subset X$ is called a *neighbourhood* of $x_0 \in X$, if U contains an open ball around x_0, i.e.

$$\exists r > 0 : \quad B_r(x_0) \subset U.$$

Then x_0 is called an *interior point* of U.

3. A subset $U \subset X$ is called *open*, if it contains only interior points, i.e.

$$\forall x \in U \exists r > 0 : \quad B_r(x) \subset U.$$

Example 2.9. 1. Let (X,d) be a metric space. Then for any $x \in X$ and $r > 0$ the set $B_r(x)$ is open.

2. Let X be equipped with the discrete metric. Then any subset $U \subseteq X$ is open: $B_{\frac{1}{2}} = \{x\} \quad \forall x \in X$

Proposition 2.10. *Let (X,d) be a metric space. Then:*

1. $\emptyset$ and X are open.

2. If $U, V \subset X$ are open, then also $U \cap V$ is open.

3. *If $U_i \subset X$ is open for all $i \in I$, then also $\bigcup_{i \in I} U_i$ is open.*

The second property implies that intersections of finitely many open sets are open. However, this does not hold for infinite intersections: Let $U_n = (-\frac{1}{n}, \frac{1}{n}) \subset \mathbb{R}$, $n \in \mathbb{N}$. Then $\bigcap_{n \in \mathbb{N}} U_n = \{0\}$ is not open.

Definition 2.11 (Closed set)**.**
A subset $A \subset X$ of a metric space is *closed*, if its complement is open, i.e. $A^C = \{x \in X \mid x \notin A\}$ is open.

Example 2.12. 1. Let $X = \mathbb{R}$ and $a, b \in \mathbb{R}$ with $a < b$. Then $[a, b]$, $[a, \infty)$ are closed, but $[a, b)$ is neither open nor closed.

2. For any metric space (X, d) the sets $\emptyset$ and X are open and closed.

A metric endows a set not only with a notion of convergence, but also with a geometry (distance, angles, etc.), but actually, open sets turn out to be enough in order to define convergence.

Definition 2.13 (Topology and topological spaces)**.**
Let X be a set. A *topology* $\mathcal{T}$ on X is a collection $\mathcal{T} \subset \mathcal{P}(X)$ of subsets of X with the following properties:

1. $\emptyset, X \in \mathcal{T}$
2. $U, V \in \mathcal{T} \quad \Rightarrow \quad U \cap V \in \mathcal{T}$
3. $U_i \in \mathcal{T}$ for all $i \in I \quad \Rightarrow \quad \bigcup_{i \in I} U_i \in \mathcal{T}$

The sets $U \in \mathcal{T}$ are called the *open sets* and $(X, \mathcal{T})$ is called a *topological space*. $A \subset X$ is *closed*, if $A^C \in \mathcal{T}$. $U \subset X$ is called a *neighbourhood* of a $x_0 \in X$ (and x_0 is an *interior point* of U), if:

$$\exists O \in \mathcal{T} \quad \text{with} \quad x_0 \in O \subset U.$$

Remark 2.14. The property 2. implies that intersections of finitely many open sets are open. However, this does not hold for infinite intersections: Let $U_n = (-\frac{1}{n}, \frac{1}{n}) \subset \mathbb{R}$, $n \in \mathbb{N}$. Then $\bigcap_{n \in \mathbb{N}} U_n = \{0\}$ is not open.

Example 2.15. 1. $\mathcal{T} = \{\emptyset, X\}$ forms the so-called *trivial topology* on X.

2. $\mathcal{T} = \mathcal{P}(X)$ forms the so called *discrete topology* on X.

3. According to Proposition 2.10, the open sets in a metric space form a topology.

Definition 2.16 (Relative topology)**.**
$(X, \mathcal{T})$ a topological space and $Y \subset X$ a subset of X. Then $\mathcal{T}|_Y = \{O \cap Y \mid O \subset \mathcal{T}\}$ is a topology on Y, the *subspace* or *relative topology*. The elements $U \subset \mathcal{T}|_Y$ are called *relative open sets*.

Example 2.17.
1. $X = \mathbb{R}$, $Y = [0,1]$. Then $Y \in \mathcal{T}|_Y$, i.e. Y is relatively open in itself. Also $[0, \frac{1}{2}) \subset Y$ is relatively open, since $[0, \frac{1}{2}) = (-\frac{1}{2}, \frac{1}{2}) \cap Y$
2. (X, d) a metric space and $Y \subset X$ a subset. Then $(Y, d|_Y)$ is a metric space.
$$d|_Y : Y \times Y \to [0, \infty], \quad (y_1, y_2) \mapsto d|_Y(y_1, y_2) = d(y_1, y_2)$$
3. $(V, \|\cdot\|)$ normed space and $U \subset V$ a vector subspace. Then $(U, \|\cdot\||_U)$ is a normed space.

Definition 2.18 (Interior, closure, boundary)**.**
Let $(X, \mathcal{T})$ be a topological space and $Y \subset X$.

1. The set $\overset{\circ}{Y} = \bigcup\limits_{\substack{U \in \mathcal{T} \\ U \subset Y}} U$ is called the *interior* of Y.
2. The set $\overline{Y} = \bigcap\limits_{\substack{U \in \mathcal{T} \\ U \subset Y^C}} U^C$ is called the *closure* of Y.
3. The set $\partial Y = \overline{Y} \setminus \overset{\circ}{Y}$ is called the *boundary* of Y.

Proposition 2.19.
1. *$\overset{\circ}{Y} \subset Y \subset \overline{Y}$.*
2. *$\overset{\circ}{Y}$ is the largest open set contained in Y.*
3. *Y is open $\Leftrightarrow Y = \overset{\circ}{Y}$.*
4. *$\overline{Y}$ is the smallest closed set containing Y.*
5. *Y is closed $\Leftrightarrow Y = \overline{Y}$.*
6. *$(\overset{\circ}{Y})^C = \overline{(Y^C)}$ and $\left(Y^C\right)^\circ = \left(\overline{Y}\right)^C$.*[1]

Proposition 2.20. *Let $(X, \mathcal{T})$ be a topological space and $Y \subset X$. Then:*

1. *$\overset{\circ}{Y}$ is the set of interior points of Y.*

[1]This can be proven using de Morgan's laws: For a family of sets $(A_i)_{i \in I}$ it holds

(a) $\left(\bigcap\limits_{i \in I} A_i\right)^C = \bigcup\limits_{i \in I} A_i^C$,

(b) $\left(\bigcup\limits_{i \in I} A_i\right)^C = \bigcap\limits_{i \in I} A_i^C$.

2. $x \in \partial Y \quad \Leftrightarrow \quad$ *for any neighbourhood* U *of* x $U \cap Y \neq \emptyset$ *and* $U \cap Y^C \neq \emptyset$.

3. $\overset{\circ}{Y} = Y \backslash \partial Y$ *and* $\overline{Y} = Y \cup \partial Y$

Example 2.21. 1. For $Y = [a, b) \subset \mathbb{R}$ we have $\overset{\circ}{Y} = (a, b)$, $\overline{Y} = [a, b]$, $\partial Y = \{a, b\}$.

2. For $\mathbb{Q} \subset \mathbb{R}$ we have $\overset{\circ}{\mathbb{Q}} = \emptyset$, $\overline{\mathbb{Q}} = \mathbb{R}$, $\partial\mathbb{Q} = \mathbb{R}$.

Definition 2.22 (Convergence in topological spaces)**.**
Let X be a topological space. A sequence (x_n) in X *converges* to $a \in X$ and we write:

$$\lim_{n\to\infty} x_n = a,$$

if for any neighbourhood U of the point a there exists $N \in \mathbb{N}$ such that $x_n \in U$ for all $n \geq N$.

Remark 2.23. In general convergence points are not unique: On any set X with the trivial topology any sequence converges to every point in X!

Definition 2.24 (Hausdorff spaces)**.**
A topological space $(X, \mathcal{T})$ is *Hausdorff*, if:

$$\forall x, y \in X, x \neq y, \exists U, V \in \mathcal{T} : \quad x \in U, y \in V, \quad U \cap Y = \emptyset\,.$$

Proposition 2.25. *1. In Hausdorff spaces, sequences have at most one limit.*

2. All metric spaces are Hausdorff.

Definition 2.26 (Cluster point)**.**
A point $a \in X$ is called a *cluster point* of a sequence (x_n) if any neighbourhood U of a contains infinitely many elements of (x_n).

Thus far, we have encountered spaces with different level of abstraction,

$$\textit{normed spaces} \quad \subset \quad \textit{metric spaces} \quad \subset \quad \textit{Hausdorff spaces}\,,$$

in which we call always define a notion of convergence and they indeed coincide. Both in physics and mathematics, the analysis is often carried out in so-called *Banach spaces.*

Definition 2.27 (Cauchy sequence)**.**
A sequence (x_n) in a metric space X is called *Cauchy sequence*, if

$$\forall \varepsilon > 0 \exists N \in \mathbb{N} \forall n, m \geq N : \quad d(x_n, x_m) < \varepsilon$$

Proposition 2.28. *Every convergent sequence in a metric space is also a Cauchy sequence.*

Definition 2.29 (Complete metric space, Banach space)**.**

1. A metric space X is called *complete* if every Cauchy sequence in X converges.
2. A complete normed space (the norm induces the metric) is called a *Banach space*.

Example 2.30. 1. $\mathbb{R}, \mathbb{C}, \mathbb{R}^n, \mathbb{C}^n$ (with e.g. the Euclidean norm) are Banach spaces.

2. $(\mathbb{Q}, |\cdot|)$ is not complete, as there exists a Cauchy sequence $(x_n) \subset \mathbb{Q}$ with $\lim_{n\to\infty} x_n = \sqrt{2}$ but $\sqrt{2} \notin \mathbb{Q}$.